A TEXT-BOOK OF MATHEMATICS

VOL. I

M. K. SINGAL & A. R. SINGAL

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OF
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SRINIVASA RAMANUJAN (1887-1920)

Srinivasa Ramanujan, the greatest mathematical genius produced in India, was born on the 22nd December, 1887 in Tamil Nadu. He belonged to a poor Brahmin family. He earned a name for extraordinary mathematical ability even as a child. In 1913, Ramanujan joined the University of Madras as the first research scholar of the University. In 1914, he went to England where he collaborated with Hardy and Littlewood to produce some of the most outstanding work. In 1918, he was elected a Fellow of the Royal Society, In 1917, Ramanujan fell ill in England and returned back to Madras in 1919. He passed away on the 26th April, 1920. Even on his deathbed, he produced research work of the highest order. Ramanujan used to write on notebooks. His notebooks contain more than three thousand important theorems.

Ramanujan will be remembered not only because his work has kept first rate mathematicians busy for nearly seventy years even after his death, but also because he was able to do so without any formal training and without any means of support.

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A TEXT-BOOK OF MATHEMATICS

VOL. I

(FOR CLASS XI)

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Dedicated

To

The Future Mathematicians

And

Users of Mathematics

PREFACE TO THE FIRST EDITION

The book has been specially designed as a text for use in class XI of Senior Secondary Schools (under the 10+2 pattern of education). In respect of subject matter content, it strictly covers the syllabus prescribed by the Central Board of Secondary Education, New Delhi.

In the preparation of the book, the authors have kept in view the idea of an Integrated Approach to Mathematics which has now been universally accepted as a sound pedagogical principle in Mathematics Education. Wherever possible, a new mathematical concept has been introduced in the setting of real life situations, as an abstracting model, rather than an abstraction in itself. The concepts and techniques learnt have been sought to be applied to practical problems from various co-curricular subjects like Physics, Chemistry, Biology, Economics etc. An attempt has been made to present mathematics as a single entity.

The exposition is simple, yet rigorous. The language is such as a student at this level can easily follow. Since sets provide the most convenient medium in which mathematical ideas find their simplest expression, therefore, the language of sets has been used throughout the book. While no formal detailed discussion of sets has been included in the book, the book opens with a revision of the basic concepts of sets, relations and functions learnt in classes IX and X. A proper balance between the learning of concepts and proofs, and the mastery of skills has been sought to be achieved throughout the book.

Short biographical notes have been added at appropriate places to give the student some idea about the Makers of Mathematics. Full page photographs of such mathematical giants as Niels Henrik Abel, Jakob Bernoulli, Georg Cantor, Rene Descartes, Leonhard Euler, Pierre de Fermat, Ronald A. Fisher, Carl Friedrich Gauss, W.R. Hamilton, Johann Kepler, G.W. Leibnitz, Isaac Newton, Guiseppe Peano, Srinivasa Ramanujan, and John von Neumann have been included in the book to add to the historical perspective and to enhance the aesthetic appeal. Historical notes have been given wherever necessary.

The book may be conveniently divided into four parts. The *first* part, comprising Chapters 1 to 8 is on Algebra. The *second* part comprising Chapters 9 to 12 is on Co-ordinate Geometry. The *third* part comprising Chapters 13 to 16 deals with Trigonometry. The *fourth* part comprising Chapters 17 and 18 deals with Statistics and Linear Programming.

Throughout the book a large number of examples have been solved to illustrate the various concepts and techniques. The problems have been carefully selected and properly graded and the answers have been thoroughly checked. They have been given in the form of problem-sets at the end of each section, and their number is just the right one for having a proper understanding of the subject as well as for acquiring the necessary computational skills. A serious effort has been made to keep the book free from mistakes.

At the end of each chapter a brief summary of the chapter, a set of objective type questions, and a set of review exercises has been given. Trigonometric and logarithmic tables have been given at the end of the book.

It is hoped that the book will be found useful by all those for whom it is meant. Suggestions for the improvement of the book will be gratefully received and acknowledged.

Meerut, M. K. SINGAL
August 15, 1989. ASHA RANI SINGAL

PREFACE TO THE SECOND EDITION

The authors are grateful to the students and teachers for the overwhelming demand for the book.

This edition differs from the first edition in several respects. The book has been subjected to a thorough revision. The first, third and seventeenth chapters have been rewritten and a lot of new material has been added in these chapters. A set of assorted problems consisting mostly of questions set at Roorkee Entrance/IIT JEE has been added. Answers have been thoroughly checked. It is hoped that the book in its present form will be more useful than before.

Suggestions for the improvement of the book will be gratefully received and acknowledged.

Meerut M. K. SINGAL
August 15, 1990. ASHA RANI SINGAL

CONTENTS

PART I : ALGEBRA

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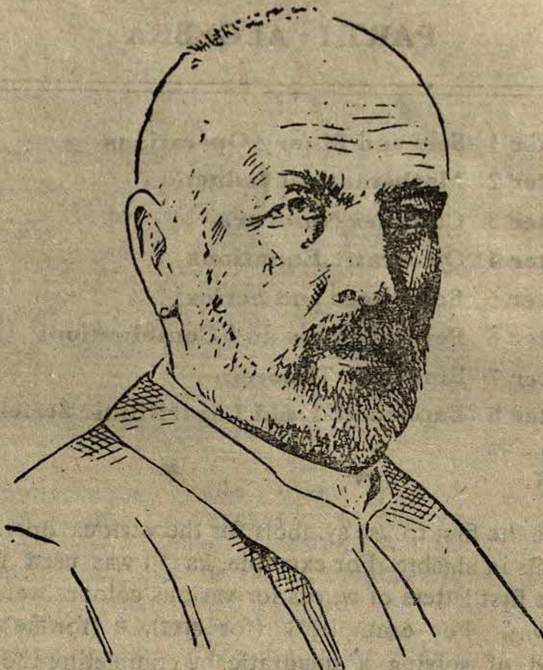
Hindus were the first to use symbols for the various operations and the unknowns in algebra. For example, ka (क) was used for square-root and the first letters of words for various colours were used for the unknowns. For example, ह (for हरित); न (for नील) and so on. The method of solving a quadratic by completing the square was also given to the world by Hindus. Today the solution of polynomial equations like $ax+b=0$, $ax^2+bx+c=0$ is regarded trivial. But once upon a time when people had no symbols to write an equation, the solution of even particular linear and quadratic equations was considered a great achievement. People usually guarded the solutions and posed these as challenging problems. Hence the importance of symbolism.

As the sun eclipses the stars by its brilliance, so the man of knowledge will eclipse the fame of others in assemblies of people if he proposes algebraic problems, and still more if he solves them.

—*Brahmagupta*



GEORG CANTOR (1845-1918)

Georg Ferdinand Ludwing Philipp Cantor was born on the 3rd March, 1845, at St. Petersburg. His father Georg Woldemar Cantor was a merchant by profession. He was the eldest of six brothers.

Georg had his elementary school education at St. Petersburg. In 1856 his parents moved to Wiesbaden, and he was admitted to Wiesbaden Gymnasium. In 1859 he went to the Grand-Ducal Realschule in Darmstat, and in the following year he attended the Hoheren Gewerbschute (business school), where he studied till 1862. In 1862 he began his higher education at the polytechnikum in Zurich, but the following year his father died. He then moved over to the University of Berlin where he had the opportunity of having Weierstrass, Kummer and Kronecker as his teachers.

Cantor will always be remembered for his theory of sets, and his theory of real numbers. His first revolutionary discovery was to show that there are as many points in the whole of the plane as there are in a line.

During his later life he suffered from mental depression, and spent much of his time in hospital, where he died on the 6th January, 1918.

CHAPTER 1

Sets and Binary Operations

1·1. SETS

We often come across such phrases as a bunch of keys, a pack of wolves, a class of students, a deck of cards, a team of players etc. The words *bunch*, *pack*, *class*, *deck* and *team*, all denote collections. In mathematics too, we have to deal with collections. Mathematicians use the word set for a well-defined collection of objects.

A **set** is a well-defined collection of objects. Each object belonging to a set is called an **element** of the set. We generally use capital letters A, B, C etc., to denote sets and lower case letters *a*, *b*, *c*, *x* etc., to denote elements of a set. The symbol $\in$ is used to indicate 'belongs to' (or 'is an element of'). Thus, '*p* is an element of S' is written as

$$p \in S.$$

The symbol $\notin$ is used to indicate 'does not belong to'. Thus '$p \notin S$' means that '*p* does not belong to S'.

Given a set S and an object *p*, exactly one of the following statements should be true :

(*i*) $p \in S$; (*ii*) $p \notin S$.

Illustrations :

(*a*) Let S be the set of all Prime Ministers of India. Then Smt. Indira Gandhi $\in$ S but Dr. Rajendra Prasad $\notin$ S.

(*b*) Let V be the set of vowels in the English alphabet. Then

$$a \in S, d \notin S.$$

(*c*) The collection **N** of all natural numbers is a set.

(*d*) The collection **Z** of all integers is a set.

(*e*) The collection **Q** of all rational numbers is a set.

(*f*) The collection **R** of all real numbers is a set.

(*g*) The collection **C** of all complex numbers is a set.

The sets described in the illustrations (*c*)—(*g*) above are some of the most useful sets. We shall come across these sets throughout the book. The symbols **N**, **Z**, **Q**, **R**, **C** will be used freely throughout the book for the various sets as described above.

1·1·1. Two Methods for Describing Sets

The following are the most common methods of writing sets :

(*i*) *Roster Method.* A set may be described by listing all its elements. Thus, for example, the set V of vowels in the English alphabet is

$$V=\{a, e, i, o, u\}.$$

This method is called the *roster method.* Sometimes it is not possible to list all the elements, but after knowing a few elements we can easily see as to what the other elements are. Thus the set **N** of natural numbers may be written as

$$\mathbf{N}=\{1, 2, 3, \ldots\ldots\}.$$

The dots indicate that the set contains all the natural numbers.

(*ii*) *Set Selector Method* (*Property Method*). Sometimes a set is described by means of some property which characterises all the elements of the set (that is, the property is shared by all the elements of the set and if a certain object has the property, then it belongs to the set). A set S characterised by a property p about an object x may be written as

$$S=\{x : p(x)\},$$

where $p(x)$ means that x has property p. (We read the expression within braces as 'x such that $p(x)$ is true'.)

Illustrations :

(*a*) $\mathbf{N}=\{n : n$ is a natural number$\}$.

(*b*) $P=\{p : p$ is a prime$\}$.

(*c*) $E=\{n : n$ is an even number$\}$.

EXERCISE 1 (*a*)

1. If $A=\{0, 1, 2, 3\}$, then which of the following statements are true :

   (*a*) $0 \in A$; (*b*) $1 \notin A$; (*c*) $2 \notin A$?

2. If $B=\{a, b, c, d\}$, then which of the following statements are true :

   (*a*) $a \in B$; (*b*) $b \notin B$?

3. Describe the following sets by the roster method :

   (*a*) $A=\{x : x$ is an integer between 5 and 10$\}$;

   (*b*) $B=\{x : x^2-5x-24=0\}$;

   (*c*) $C=\{x : x$ is a positive integer and a multiple of 5$\}$;

   (*d*) $D=\{x : x$ is a natural number and $3x-4 < 7\}$.

4. Describe the following sets by the set selector method :

   (*a*) $A=\{2, 4, 6, 8, 10\}$;

   (*b*) $B=\{5\}$;

   (*c*) $C=\{101, 102, 103, \ldots\ldots, 999\}$.

1·1·2. Equality of Sets

Two sets are said to be **equal** if they contain the same elements. Thus two sets A and B are equal if every element of A is also an element of B, and every element of B is also an element of A. That is,

$A=B$ if and only if $(x \in A \Rightarrow x \in B) \wedge (x \in B \Rightarrow x \in A)$.

For example,

(*a*) $\{3, 5, 9\}=\{9, 5, 3\}$.

(*b*) $\{2, 3, 4\} \neq \{1, 3, 5\}$.

1·1·3. Subsets

Let A and B be two sets. If every element of A is also an element of B, then we say that A **is contained in** B, or that A **is a subset of** B. In symbols, we then write '$A \subset B$'. If A is contained in B, then we also say that B contains A. We express it in symbols as '$B \supset A$'.

From the above we see that $A \subset B$ and $B \supset A$ are two different ways of expressing the same fact.

If $A \subset B$ but $A \neq B$, we say that A is a **proper subset** of B.

Illustration. If $A=\{1, 2, 3\}$, $B=\{1, 2, 3, 4, 5, 6\}$, and $C=\{1, 3, 5\}$, then $A \subset B$ and $B \supset C$.

Theorem 1·1. *Two sets A and B are equal if and only if* $A \subset B$ and $B \subset A$. In symbols,

$$A=B \Leftrightarrow (A \subset B) \wedge (B \subset A).$$

Proof. First let $A=B$. Then $x \in A \Rightarrow x \in B$, so that $A \subset B$.

Similarly, $\quad B \subset A$.

Thus $\quad A=B \Rightarrow (A \subset B) \wedge (B \subset A)$.

Conversely, let $(A \subset B) \wedge (B \subset A)$.

Then $\quad x \in A \Rightarrow x \in B$ and $y \in B \Rightarrow y \in A$.

Thus each element of A is an element of B, and each element of B is an element of A, that is, $A=B$.

Therefore, $\quad (A \subset B) \wedge (B \subset A) \Rightarrow A=B$.

Hence the theorem.

1·1·4. The Empty Set

Definition 1·1. The set having no element is called the **empty set** (or the **null set** or the **void set**).

We denote the empty set by the symbol 'ϕ'. If we were to describe the empty set by the listing method, we would write

$$\phi=\{\ \}.$$

A description of the empty set by the property method (by no means unique) would be

$$\phi=\{x : x \text{ is a negative integer whose square is } -1\}.$$

Theorem 1·2. *The empty set is a subset of every set.*

Proof. Given any set A,

$$\phi \subset A \text{ iff } x \in \phi \Rightarrow x \in A.$$

But ϕ has no element. Therefore, the latter implication is true (though trivially).

Hence $\phi \subset A$ for every set A.

1·1·5. Universal Set

In any mathematical discussion, we usually consider all the sets to be subsets of a fixed set, called the **universal set** or the **universe.** The universe is generally denoted by X. For example, while discussing properties of triangles, we may consider the universe to be the set X of all triangles in a plane Σ. The set of all obtuse-angled triangles in Σ, the set of all right-angled triangles in Σ, the set of all equilateral triangles in Σ, the set of all isosceles triangles in Σ, the set of all scalene triangles in Σ, are all subsets of X.

The relations between sets can be conveniently illustrated by certain diagrams called Venn diagrams. In a Venn diagram, we denote the universe X by the region enclosed within a rectangle and any subset of X by the region enclosed within a closed curve lying in the region enclosed by the rectangle (see Fig. 1·1).

EXERCISE 1 (*b*)

1. Which of the following sets are equal ?
   $A=\{1, 2, 3\}$, $B=\{1, \{2, 3\}\}$, C = the set of all positive integers not exceeding 3.
2. Which of the following sets are equal ?
   $A=\{1, 2\}$, $B=\{5, \sqrt{9}\}$, $C=\{\sqrt{2}, 2\sqrt{3}\}$.
3. Show that the following pairs of sets are equal :
   (*a*) $A=\{x : x^2-1=0 \text{ and } x \in \mathbf{Z}\}$,
   $B=\{x : x^4-1=0 \text{ and } x \in \mathbf{Z}\}$;
   (*b*) $A=\{x : x^3-1=0 \text{ and } x \in \mathbf{R}\}$,
   $B=\{x : x^5-1=0 \text{ and } x \in \mathbf{R}\}$.

4. Consider the following sets of geometrical figures in a certain plane :

X = the set of all triangles, E = the set of all equilateral triangles, I = the set of all isosceles triangles, S = the set of all scalene triangles, R = the set of all right-angled triangles, A = the set of all acute-angled triangles.

Which of the following statements are true and which are false ?

(*a*) $X \supset I \supset E$; (*b*) $S \subset E$; (*c*) $R \subset A$.

5. Find at least one set S which is such that :

(*a*) $\{1, 2\} \subset S \subset \{1, 2, 3, 4\}$;

(*b*) $\{a, b, c\} \subset S \subset \{a, b, c, d, e\}$.

6. Which of the following sets are equal to the empty set ?

(*a*) $\{x : x > 4 \text{ and } x < 3\}$;

(*b*) $\{x : x^2-3x+2=0 \text{ and } x < 2\}$;

(*c*) $\{x : x^2-5x+6=0 \text{ and } x > 6)$;

(*d*) $\{x : 2x=3 \text{ and } x \text{ is an integer}\}$;

(*e*) $\{x : x^2+1=0 \text{ and } x \in \mathbf{R}\}$;

(*f*) $\{x : x^2-1=0 \text{ and } x \in \mathbf{Q}\}$;

(*g*) $\{x : x^2+x+1=0 \text{ and } x \in \mathbf{R}\}$.

1·2. OPERATIONS ON SETS

Sets can be combined in several ways so as to produce new sets. We shall now describe some of these ways.

1·2·1. Union of Two Sets

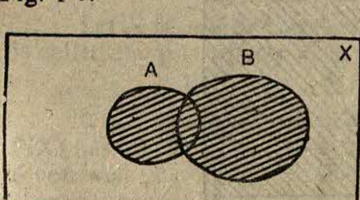
Definition 1·2. *If* A *and* B *be two given sets, then their union is the set of all those elements that belong to either* A *or* B *(or both).*

The union of two sets A and B is denoted by the symbol '$A \cup B$' which is read as

'A union B' or 'A cup B'.

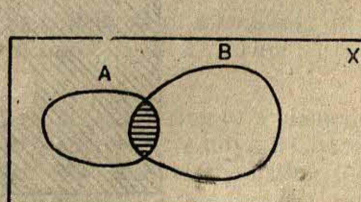
In symbols, $A \cup B = \{x : x \in A \vee x \in B\}$.

The union of A and B is indicated by the shaded region in Fig. 1·1.



$A \cup B$

Fig. 1·1.



$A \cap B$

Fig. 1·2.

Illustration. Let A$=\{a, c\}$, B$=\{b, c, e\}$. Then

$$A \cup B=\{a, b, c, e\}.$$

1·2·2. Intersection of Two Sets

Definition 1·3. *If* A *and* B *be two given sets, their intersection is the set of all those elements that belong to both* A *and* B.

The intersection of two sets A and B is denoted by the symbol '$A \cap B$' which is read as

'A intersection B' or 'A cap B'.

In symbols, $A \cap B=\{x : x \in A \wedge x \in B\}$.

The intersection of A and B is indicated by the shaded region in Fig. 1.2.

Two sets are said to be **disjoint** if their intersection is empty.

Illustration. Let X$=\{1, 2, 3, 4, 5, 6\}$, A$=\{1, 3, 5\}$, B$=\{1, 2, 5, 6\}$, C$=\{2, 3, 6\}$, D$=\{2, 4, 6\}$. Then

$A \cap B=\{1, 5\}$, $A \cap C=\{3\}$, $B \cap C=\{2, 6\}$ and $A \cap D=\phi$. The sets A and D are disjoint.

1·2·3. Difference of Sets

Definition 1·4. *If* A *and* B *be two given sets, then the set of all those members of* A *which do not belong to* B *is called the complement of* B *relative to* A.

The complement of B relative to A is denoted by $A \sim B$ which is read as 'A difference B'.

In symbols, $A \sim B=\{x : x \in A \wedge x \notin B\}$.

Also, $\quad B \sim A=\{x : x \in B \wedge x \notin A\}$.

The set $X \sim A$ is called the complement of A. Thus, the complement of A is the set of all those elements of X which are not in A. In symbols,

$$X \sim A=\{x : x \in X \wedge x \notin A\}.$$

The complement of A is denoted by $\sim A$ or A' also.

In Fig. 1·3, the shaded region indicates $X \sim A$.

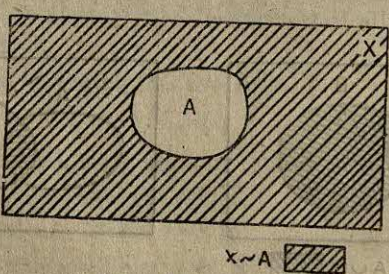


Fig. 1·3.

Illustration. Let $X=\{3, 7, 13, 19, 21\}$, $A=\{3, 13, 19\}$, and $B=\{13, 19, 21\}$.

Then $A\sim B=\{3\}$, $B\sim A=\{21\}$, $X\sim A=\{7, 21\}$, and $X\sim B=\{3, 7\}$.

The following theorem gives some important properties of the operations of union, intersection and complementation.

Theorem 1·3. *Let* A, B, C *be any subsets of a set* X. *Then*

(*a*) $A\cup A=A$, $A\cap A=A$;

(*b*) $A\cup B=B\cup A$, $A\cap B=B\cap A$;

(*c*) $(A\cup B)\cup C=A\cup(B\cup C)$, $(A\cap B)\cap C=A\cap(B\cap C)$;

(*d*) $A\cap(B\cap C)=(A\cap B)\cap(A\cap C)$;

$A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$;

(*e*) $X\sim(A\cup B)=(X\sim A)\cap(X\sim B)$.

$X\sim(A\cap B)=(X\sim A)\cup(X\sim B)$.

The above properties (*a*)—(*d*) are generally referred to as the idempotent, commutative, associative and distributive properties respectively. Property (*e*) is known as De Morgan's rule.

The proofs of the above results are simple and may be constructed by the reader himself. We give below, as a sample, the proof of the first of the relations (*e*).

To show that

$$X\sim(A\cup B)=(X\sim A)\cap(X\sim B),$$

we shall show that x is an element of the set on the left hand side of the above relation if and only if it is an element of the set on the right-hand side.

$$\begin{aligned}\text{Now } x\in X\sim(A\cup B) &\Leftrightarrow x\notin A\cup B,\\ &\Leftrightarrow x\notin A \wedge x\notin B,\\ &\Leftrightarrow x\in X\sim A \wedge x\in X\sim B,\\ &\Leftrightarrow x\in(X\sim A)\cap(X\sim B).\end{aligned}$$

Hence $X\sim(A\cup B)=(X\sim A)\cap(X\sim B)$.

1·2·4. Power Set of a Set

Let X be a non-empty set. The collection of all subsets of X is called the **power set** of X and is denoted by P(X). Thus

$$P(X)=\{A : A\subset X\}.$$

It can be easily seen that if X be a set consisting of n elements, P(X) has 2^n elements. As a matter of fact, after studying Chapter 6 you well be able to see that X has ${}^nC_r$ subsets consisting of r elements each. Thus the number of all subsets of X is ${}^nC_0+{}^nC_1+\cdots+{}^nC_r+\ldots+{}^nC_n=(1+1)^n$ etc.

(A proof is also given in Chapter 2.)

EXERCISE 1 (*c*)

1. Let X be the set of all letters of the English alphabet, and let $A=\{a, b, c, d\}$, $B=\{c, d, e\}$ and $C=\{a, e\}$.

   Determine :

   (*a*) $A \cup B$; (*b*) $A \cap B$;
   (*c*) $B \cup C$; (*d*) $B \cap C$;
   (*e*) $A \cup C$; (*f*) $A \cap C$.

2. Let $A=\{1, 2, 3, 4, 5\}$, $B=\{1, 2, 3, 4\}$, and $C=\{3, 4, 5\}$,

   Determine :

   (*a*) $A \cap (B \cap C)$; (*b*) $A \cup (B \cap C)$;
   (*c*) $(A \cup B) \cup C$; (*d*) $(A \cap B) \cup C$;
   (*e*) $B \cap (A \cup C)$; (*f*) $B \cup (A \cap C)$.

3. Let $A=\{2, 4, 6\}$, $B=\{3, 4, 5, 7\}$, and $C=\{3, 6, 7, 8\}$.

   Determine :

   (*a*) $A \cap (B \cup C)$; (*b*) $A \cup (B \cup C)$;
   (*c*) $(A \cup B) \cap C$; (*d*) $(A \cap B) \cap C$;
   (*e*) $(A \cap B) \cup C$; (*f*) $A \cap (B \cap C)$.

4. Consider the following sets of geometrical figures in a certain plane :

   X = the set of all quadrilaterals, T = the set of all trapezia,

   S = the set of all squares, P = the set of all parallelograms,

   R = the set of all rhombuses, and R* = the set of all rectangles.

   Which of the following statements are true ?

   (*a*) $S \subset R^* \subset P \subset T \subset X$; (*b*) $R \subset P \subset T \subset X$;
   (*c*) $R \cap R^* = S$; (*d*) $T \cap R^* \supset S$;
   (*e*) $S \cup P \subset R \cup R^*$.

5. If A and B be subsets of a set X, show that :

   (*a*) $A \cup X = X$; (*b*) $A \cap X = A$;
   (*c*) $A \cup \phi = A$; (*d*) $A \cap \phi = \phi$;
   (*e*) $A \subset A \cup B$; (*f*) $B \subset A \cup B$;
   (*g*) $A \cap B \subset A$; (*h*) $A \cap B \subset B$;
   (*i*) $A \subset B \Leftrightarrow A \cup B = B$; (*j*) $A \subset B \Leftrightarrow A \cap B = A$;
   (*k*) $A \cup (A \cap B) = A$; (*l*) $A \cap (A \cup B) = A$.

6. Which of the following statements are true ?

   (*a*) $x \notin A \cap B \Rightarrow x \notin A$ or $x \notin B$;
   (*b*) $x \notin A \cap B \Rightarrow x \notin A$ and $x \notin B$;
   (*c*) $x \notin A \cap B \Rightarrow x \notin A$ or $x \in B$;

(*d*) $x \notin A \cap B \Rightarrow x \notin A$ or $x \notin B$;
(*e*) $A \subset B \Rightarrow [x \in A \Rightarrow x \in B]$;
(*f*) $A \subset B \Rightarrow [x \notin A \Rightarrow x \notin B]$;
(*g*) $A \subset B \Rightarrow [x \in \sim A \Rightarrow x \notin B]$;
(*h*) $A \subset B \Rightarrow [x \notin B \Rightarrow x \notin A]$.

7. Let A, B, C be subsets of a set X. Use Venn diagrams to discover all possible equalities between the following sets :
$A \cap B$, $A \cap (B \cap C)$, $B \cap A$, $(A \cap C) \cap B$, $A \sim (A \sim B)$, $B \cap (A \cap C)$, $B - (B - A)$, $(C \cap B) \cap A$, $A \cap (B \cup C)$, $(A \cup B) \cup (A \cup C)$, $B \cap (A \cup B)$, $A \cap (A \cup B)$, $(A \cap B) \cup (A - B)$, and $(A \cap B) \cup (B - A)$.

8. If A, B, C be subsets of a set X, show that :
(*a*) $A \cap (B \cap C) = (A \cap B) \cap C$;
(*b*) $A \cup (B \cup C) = C \cup (B \cup A)$;
(*c*) $A \subset C \wedge B \subset C \Rightarrow A \cup B \subset C$;
(*d*) $A \supset C \wedge B \supset C \Rightarrow A \cap B \supset C$;
(*e*) $A \cap (B \cup C)$ need not always be equal to $B \cap (A \cup C)$.

9. Give an example of sets A, B, C for which $A \cap B \neq \phi$, $A \cap C \neq \phi$, but $A \cap B \cap C = \phi$.

10. If $A \cup B = A \cup C$, $A \cap B = A \cap C$, show that $B = C$.
Also show by examples that
(*i*) $A \cup B = A \cup C$ alone need not imply $B = C$, and
(*ii*) $A \cap B = A \cap C$ alone need not imply $B = C$.

11. Let A, B be subsets of a set X. Show that :
(*a*) $A \subset B \Rightarrow \sim B \subset \sim A$;
(*b*) $\sim (\sim A) = A$, $\sim \phi = X$, $\sim X = \phi$;
(*c*) $A \cap (\sim A) = \phi$; $A \cup (\sim A) = X$;
(*d*) $\sim (A \cap B) = (\sim A) \cup (\sim B)$;
(*e*) $\sim (A \cup B) = (\sim A) \cap (\sim B)$;
(*f*) $A - B = A \cap (\sim B)$;
(*g*) $A \sim \phi = A$; $A - A = \phi$; $\phi \sim A = \phi$;
(*h*) $A \sim B = A \sim (A \cap B) = (A \cup B) \sim B$;
(*i*) $(A - B) \cap (B \sim A) = \phi$;
(*j*) $(A \cap B) \cap (A \sim B) = \phi$;
(*k*) $A \sim (A \sim B) = B \sim (B \sim A) = A \cap B$;
(*l*) $A \sim B = B - A$ iff $A = B$.

12. Let A, B, C be subsets of a set X. Show that:
(*a*) $(A - B) \sim C = A \sim (B \cup C)$;
(*b*) $A \sim (B \sim C) = (A \sim B) \cap (A \sim C)$;

(*c*) $A \cup (B \sim C) = (A \cup B) \sim (C \sim A)$;

(*d*) $A \cap (B \sim C) = (A \cap B) \sim (A \cap C)$;

(*e*) $A \cap (B \sim C) = (A \cap B) \sim C$;

(*f*) $(A \sim B) \cup (A \sim C) = A \sim (B \cup C)$.

1·3. NUMBER OF ELEMENTS IN A FINITE SET

Let A be a finite set. The number of elements in A is usually denoted by n (A). The following theorem regarding the number of elements in a set is interesting as well as useful.

Theorem 1·4. *If A and B be finite subsets of a non-empty set X, then*

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

Proof. It is clear that if A and B are disjoint sets, then

$$n(A \cup B) = n(A) + n(B).$$

Since we can write A as the union of the pairwise disjoint sets $A \cap B$ and $A \cap B'$, therefore,

$$n(A) = n(A \cap B) + n(A \cap B'). \qquad \ldots(i)$$

Again, since we can write B as the union of the pairwise disjoint sets $A \cap B$ and $A' \cap B$, therefore,

$$n(B) = n(A \cap B) + n(A' \cap B). \qquad \ldots(ii)$$

Also, since the set $A \cup B$ can be expressed as the union of pairwise disjoint sets $A \cap B$, $A \cap B'$ and $A' \cap B$, therefore,

$$n(A \cup B) = n(A \cap B) + n(A \cap B') + n(A' \cap B). \qquad \ldots(iii)$$

Adding corresponding sides of (*i*) and (*ii*), and using (*iii*), we have

$$n(A) + n(B) = n(A \cap B) + n(A \cup B),$$

or

$$\boxed{n(A \cup B) = n(A) + n(B) - n(A \cap B).}$$

Corollary. If A, B, C be three finite sets, then

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(B \cap C) - n(C \cap A)$$
$$- n(A \cap B) + n(A \cap B \cap C).$$

Proof.

$$n(A \cup B \cup C) = n(A \cup B) + n(C) - n[(A \cup B) \cap C],$$
$$= [n(A) + n(B) - n(A \cap B)] + n(C) - n[(A \cap C) \cap (B \cap C)],$$
$$- [n(A \cap C) + n(B \cap C)$$
$$= n(A) + n(B) + n(C) - n(B \cap C) - n(C \cap A)$$
$$- n(A \cap B) + n(A \cap B \cap C).$$

1·3·1. An Application of Venn Diagrams

Let X be a non-empty set, and let A be a non-empty proper subset of X. The sets A and A′ are non-empty disjoint subsets of X whose union is X.

Thus $$X = A \cup A'$$

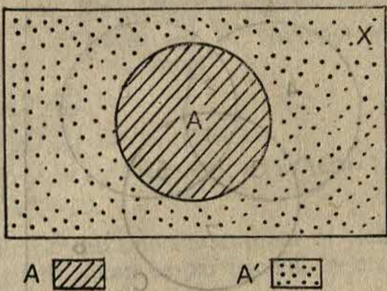


Fig. 1·4.

Again, let A and B be non-empty proper subsets of X. The sets $A \cap B$, $A \cap B'$, $A' \cap B$ and $A' \cap B'$ are pairwise disjoint sets whose union is X. That is,

$$X = (A \cap B) \cup (A \cap B') \cup (A' \cap B) \cup (A' \cap B').$$

To obtain the above expression, we simply write

$$X = (A \cup A') \cap (B \cup B'),$$

and apply the distributive law.

In Fig. 1.5, we have drawn the corresponding Venn diagram.

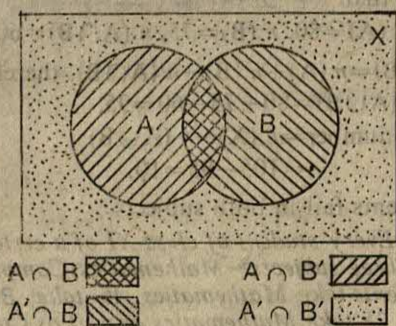


Fig. 1·5.

Let us now consider three non-empty proper subsets A, B, C of X. We can express X as the union of eight pairwise disjoint sets :

(1) $A \cap B \cap C$
(2) $A \cap B \cap C'$
(3) $A \cap B' \cap C$
(4) $A \cap B' \cap C'$
(5) $A' \cap B \cap C$
(6) $A' \cap B \cap C'$
(7) $A' \cap B' \cap C$
(8) $A' \cap B' \cap C'$

These sets are shown in the Venn diagram below :

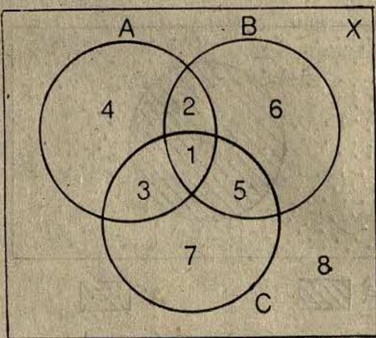


Fig. 1·6.

Example 1. *105 students take an examination out of whom 80 students pass in English, 75 students pass in Mathematics, and 60 students pass in both subjects. How many students fail in both subjects ?*

Solution. Let

X = the set of students who take the examination,

A = the set of students who pass in English,

B = the set of students who pass in Mathematics.

We are given that

$n(X)=105$, $n(A)=80$, $n(B)=75$, $n(A \cap B)=60$.

Since $n(A \cup B)=n(A)+n(B)-n(A \cap B)$, therefore,

$$n(A \cup B)=80+75-60=95.$$

The required number $=n(X)-n(A \cup B)$

$$=105-95=10.$$

Thus 10 students fail in both subjects.

Example 2. *Every student of class XI of a certain school takes at least one of the three subjects—Mathematics, Computer Science and Biology. If 61 students take Mathematics, 46 take Biology, 34 take Computer Science, 25 take Mathematics as well as Computer Science, 15 take Biology as well as Computer Science, 21 take Mathematics as well as Biology, and 10 take all the three subjects, find—*

(*a*) *the number of students who take either Mathematics or Computer Science ;*

(*b*) *the number of students in class XI*;

(*c*) *the number of students who take Mathematics and Biology but not Computer Science*;

(*d*) *the number of students who take only Computer Science.*

Solution. Let X = the set of students in class XI,
M = the set of students who take Mathematics,
B = the set of students who take Biology,
C = the set of students who take Computer Science.

We are given that

$n(M)=61$, $n(B)=46$, $n(C)=34$,

$n(M\cap B)=21$, $n(M\cap C)=25$, $n(B\cap C)=15$,

$n(M\cap B\cap C)=10$.

We shall first of all find the number of elements in each of the eight sets into which X can be partitioned by means of the sets M, B and C.

(*i*) Observe that he sets $M\cap B\cap C$ and $M\cap B\cap C'$ are disjoint, and their union is $M\cap B$. Therefore,

$$n(M\cap B)=n(M\cap B\cap C)+n(M\cap B\cap C'),$$

so that $\quad 21=10+n(M\cap B\cap C')$,

i.e., $\quad n(M\cap B\cap C')=11$.

(*ii*) Similarly,

$$n(M\cap C)=n(M\cap B\cap C)+n(M\cap B'\cap C),$$

so that $\quad 25=10+n(M\cap B'\cap C)$,

i.e., $n(M\cap B'\cap C)=15$.

(*iii*) From $n(B\cap C)=n(M\cap B\cap C)+n(M'\cap B\cap C)$,

we have $\quad 15=10+n(M'\cap B\cap C)$,

i.e., $\quad n(M'\cap B\cap C)=5$.

(*iv*) From $n(M)=n(M\cap B\cap C)+n(M\cap B\cap C')$
$\qquad +n(M\cap B'\cap C)+n(M\cap B'\cap C')$,

we have

$$61=10+11+15+n(M\cap B'\cap C'),$$

i.e., $\quad n(M\cap B'\cap C')=25$.

(*v*) From $n(B)=n(M\cap B\cap C)+n(M'\cap B\cap C)+n(M\cap B\cap C')$
$\qquad +n(M'\cap B\cap C')$,

we have $\quad 46=10+5+11+n(M'\cap B\cap C')$,

i.e., $\quad n(M'\cap B\cap C')=20$.

(*vi*) From $n(C)=n(M\cap B\cap C)+n(M'\cap B\cap C)+n(M\cap B'\cap C)$
$+n(M'\cap B'\cap C)$,

we have $$34=10+5+15+n(M'\cap B'\cap C),$$

i.e., $$n(M'\cap B'\cap C)=4.$$

(*vii*) Since every student must take at least one of the three subjects, therefore,

$$n(M'\cap B'\cap C')=0.$$

(*viii*) We are given that

$$n(M\cap B\cap C)=10.$$

We can now draw the Venn diagram for X and label it as shown in Fig. 1.7.

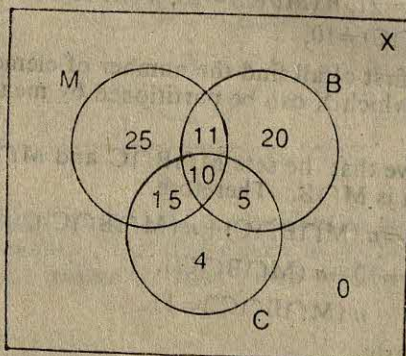


Fig. 1·7.

From the above diagram we immediately have

(*a*) $n(M\cup C)=25+11+15+10+5+4=70$,

(*b*) $n(M\cup B\cup C)=25+11+20+15+10+5+4=90$,

(*c*) $n(M\cap B\cap C')=11$,

(*d*) $n(M'\cap B'\cap C)=4$.

Remarks. 1. After some practice, the number of elements in the various regions can be written down by inspection, proceeding step by step (as in Example 3), without writing down all the relations formally as we have done in the above example.

2. Most problems (in fact all !) concerning the number of elements in a set can be solved by the systematic procedure illustrated above.

3. It is not always necessary to write down all the relations for solving a given problem.

4. If the number of elements in some region turns out to be negative, the conclusion is that the data are inconsistent.

Example 3. *The girls in a certain hostel were enquired about the magazines they purchased. The details are given below :*

20 purchased Femina,

17 purchased Sarita,

13 purchased Dharma Yug,

7 purchased Femina and Sarita both,

5 purchased Femina and Dharma Yug both,

4 purchased Sarita and Dharma Yug both,

1 purchased all the three above magazines.

If each girl purchased at least one of the magazines, how many were there in all ?

Solution. Let F, S, D denote the sets of girls purchasing Femina, Sarita and Dharma Yug respectively. Draw Venn diagram and work backwards.

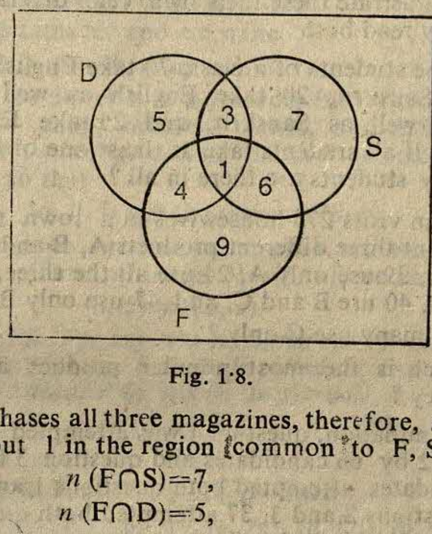


Fig. 1·8.

1 girl purchases all three magazines, therefore, $n(F \cap D \cap S)=1$. We, therefore, put 1 in the region common to F, S and D. Also, given,

$$n(F \cap S)=7,$$
$$n(F \cap D)=5,$$
$$n(S \cap D)=4.$$

Thus there are seven girls common to the circles F and S. We have already put 1 in this region and so we now put 6 more in the region common to F and S only. Similarly put 4 and 3. Now $n(F)=20$, $n(S)=17$ and $n(D)=13$. Of the twenty girls purchasing Femina, $1+6+4=11$ have already been listed in the sets purchasing 3 or 2 magazines including Femina. Thus there are only nine who purchase exclusively Femina. Similarly we put 7 and 5 in the regions exclusive to S and D respectively. Now adding all, we have

$$9+7+5+6+4+3+1=35 \text{ girls in all.}$$

EXERCISE 1 (*d*)

1. Let A be a set containing 4 elements, B be a set containing 3 elements.

   (*a*) What is the maximum number of elements in $A \cup B$, and what is the minimum ?

   (*b*) What is the maximum number of elements in $A \cap B$, and what is the minimum ?

2. Let A be a set consisting of 10 elements, and B be a set containing 15 elements. If $A \cap B$ has 4 elements, then how many elements does $A \cup B$ have ?

3. Let a set A contain 9 elements and let B contain 13 elements. If $A \cup B$ has 15 elements, then how many elements are common to A and B ?

4. Out of 450 students in a school, 193 students read Science Today and 200 students read Junior Statesman, 80 students read neither. Illustrate these facts by a Venn diagram and find out how many read both.

5. Among the students of a class, 70 take English, 40 take Hindi, 40 take Sanskrit, 20 take English as well as Hindi, 15 take Hindi as well as Sanskrit, and 25 take English as well as Sanskrit. If all students take at least one of the three subjects, how many students are there in all ?

6. A salesman visits 275 housewives in a town to find out their views about three different products A, B and C. He finds that 156 use A, 99 use only A, 24 use all the three, 15 use A and C but not B, 40 use B and C, and 47 use only B.

   (*i*) How many use C only ?

   (*ii*) Which is the most popular product according to this survey ?

7. In an examination, question 1 was attempted by 87 candidates, question 2 by 66 candidates and question 3 by 60 candidates. 40 candidates attempted both questions 1 and 2, 17 attempted both questions 2 and 3, 37 attempted both questions 1 and 3, and 5 attempted all the three questions

   (*i*) How many attempted question 1 but not 2 and 3 ?

   (*ii*) How many attempted question 2 but not 1 and 3 ?

8. A TV survey gives the following data for TV viewing :

   35% see programmes A, 55% see programme B, 60% see programme C, 15% see programmes A and B, 25% see programmes B and C, 30% see programmes A and C, and 10% do not see any programme. Draw a Venn diagram and find :

   (*i*) What percent see all the three programmes A, B and C.

(*ii*) What percent see only programme A.

(*iii*) What percent see exactly two programmes.

9. Show that the following report is inconsistent :

In a survey of 100 students concerning ability to read English, Hindi and Urdu, 46 read English, 25 read Hindi, 27 read Urdu, 19 read English and Hindi, 8 read English and Urdu, 10 read Hindi and Urdu, and 3 read all the three languages.

10. In a very hotly fought battle, at least 70% of the soldiers lost an eye, at least 75% lost an ear, at least 80% lost an arm, at least 85% lost a leg. How many lost all four limbs ? (Give the best answer possible in the form : "At least......percent lost all four".)

1·4. CARTESIAN PRODUCT OF TWO SETS

The set $\{a, b\}$ is the same as $\{b, a\}$, because the two sets consist of exactly the same elements. Similarly, all of the expressions

$$\{a, b, c\},\ \{b, c, a\},\ \{b, a, c\}$$

are names for the same set and we write

$$\{a, b, c\}=\{b, c, a\}=\{b, a, c\}.$$

Sometimes we do wish to take order into account and we then speak of ordered pairs, ordered triples, and so on. Thus in the ordered pair (a, b), a is regarded as 'the first' element and b as 'the second' element, so that

$$(a, b) \text{ is not the same as } (b, a).$$

To distinguish an ordered pair from a set, we use parentheses instead of braces. Thus, while $\{a, b\}=\{b, a\}$, we have

$$(a, b) \neq (b, a).$$

Definition 1·5. *For any sets A, B, the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$, is called the* **cartesian product** *of A and B, and is denoted by $A \times B$. In symbols,*

$$A \times B=\{x : x=(a, b), a \in A, b \in B\}.$$

Illustrations :

(*a*) Let $A=\{a, b, c\}$, $B=\{\triangle, \square\}$. Then

$$A \times B=\{(a, \triangle), (b, \triangle), (c, \triangle), (a, \square), (b, \square), (c, \square)\}.$$

(*b*) Let $A=\{1, 2, 3, \ldots\ldots\}$, $B=\{1, 2, 3, \ldots\ldots\}$.

Then
$$\begin{aligned} A \times B=\{&(1, 1), (1, 2), (1, 3), \ldots\ldots\\ &(2, 1), (2, 2), (2, 3), \ldots\ldots\\ &(3, 1), (3, 2), (3, 3), \ldots\ldots\\ &\ldots\ldots\ldots\ldots\ldots\ldots\}\end{aligned}$$

EXERCISE 1 (*e*)

1. Let $A=\{a, b, c\}$, $B=\{*\}$. Compute $A \times B$.
2. Let $A=\{0, 1\}$, $B=\{5, 6, 7\}$. Compute $A \times B$.
3. In problem 2, compute $B \times A$.

4. Let $A=\{\triangle, \odot, \square\}$, $B=\{l, m\}$. Show by a direct comparison that $A\times B\neq B\times A$.

5. If A, B be two non-empty sets, under what condition will
$$A\times B=B\times A\,?$$

6. Let $A=\{x, y, z\}$, $B=\{m, n\}$, $C=\{*\}$.
Compute $A\times(B\times C)$ and $(A\times B)\times C$. Are they same ?

7. If A, B be any two sets, show that if $A\cap B=\phi$, then $(A\times B)\cap(B\times A)=\phi$.

8. If A, B, C are any three sets, prove that :
(*a*) $A\times(B\cup C)=(A\times B)\cup(A\times C)$;
(*b*) $(A\cup B)\times C=(A\times C)\cup(B\times C)$;
(*c*) $A\times(B\cap C)=(A\times B)\cap(A\times C)$;
(*d*) $(A\cap B)\times C=(A\times C)\cap(B\times C)$;
(*e*) $A\times(B\sim C)=(A\times B)\sim(A\times C)$;
(*f*) $A\subset B\Rightarrow(A\times C)\subset(B\times C)$.

9. If A, B, C, D be four sets, show that
(*a*) $(A\times B)\cap(C\times D)=(A\cap C)\times(B\cap D)$;
(*b*) $(A\times B)\cup(C\times D)\subset(A\cup C)\times(B\cup D)$;
(*c*) $(A\subset B)\wedge(C\subset D)\Rightarrow A\times C\subseteq B\times D$.

10. Give an example of sets A, B, C, D for which $(A\times B)\cup(C\times D)$ is a proper subset of $(A\cup C)\times(B\cup D)$.

11. If A and B are any sets, show that
$$(A\sim B)\times(B\sim A)=[(A\cup B)\times(B\sim A)]\cap[(A\sim B)\times(A\cup B)].$$
[**Hint**. Use problem 9 (*a*).]

1·5. RELATIONS

We often use, in our daily lives, such partial sentences as the following :

is the father of,	is less than,
is a friend of,	is a divisor of,
is a member of,	is congruent to.

Each such partial sentence describes a relationship between two objects, one which precedes it and the other which follows it. Thus

'Ram is the father of Sohan'

describes the relationship of Ram to Sohan. A complete sentence like this is either true or false. Thus

'5 is less than 8' is true while '5 is less than 2', is false.

Let us think of this concept of the relation, 'is the father of' in the language of sets.

Let F be the set of male human beings living or dead, and P, the set of all living human beings at a particular moment. Then the relation 'father of' applies to some ordered pairs of the type (a, b) where $a \in F$, $b \in P$ and a is (or was) the father of b. Thus the relation 'father of' brings to our mind a subset of $F \times P$. There may be other subsets of $F \times P$, which may indicate a different relationship such as 'is a brother of', 'is a cousin of', etc. Some of these subsets will have no familiar names, but these may also be admitted as relations.

Definition 1·6. *A* **relation** *R from a set A to a set B, is a subset of $A \times B$. The sets A and B are respectively called the* **set of departure** *and the* **set of destination** *of R.*

If R is a relation from A to B and $a \in A$, $b \in B$, such that $(a, b) \in R$, then a is said to be R-related to b and we denote this by

$$a \, R \, b.$$

If a is not R-related to b, then we write it as $a \not R \, b$.

A relation from a set A to itself is called a relation in (on) the set A. Every subset of $A \times A$ is a relation in A.

Since a relation R in a set A is a subset of $A \times A$, and therefore is a set of ordered pairs, we may have a graph of R.

Illustration :

Let $U = \{1, 2, 3, 4, 5\}$. Then the set $U \times U$ is a set of 25 elements each of which is an ordered pair. Let R be the relation 'less than' in U. Then

$$R = \{(x, y) : x \in U, y \in U, x < y\}.$$

Fig. 1·9 is a graph of the relation R. It is apparent that

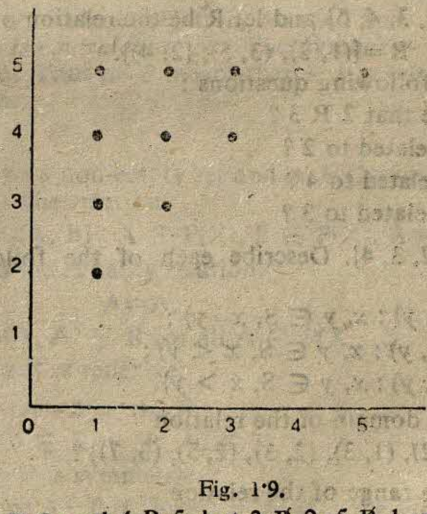


Fig. 1·9.

2 R 3, 1 R 5, and 4 R 5, but $3 \not R \, 2$, $5 \not R \, 1$, etc.

In like manner, we can graph the relation U expressed by $x=y$ or by $x > y$.

1·5·1. Domain and Range of a Relation

If R is a relation in U, then the subset

$$D(R)=\{x : (x, y) \in R\}$$

of U is called the **domain** of R and is denoted by D(R). Similarly the subset

$$R(R)=\{y : (x, y) \in R\}$$

is called the **range** of R and is denoted by R (R).

It is obvious that if R is a relation in A, then $D(R) \subset A$ and, $R \subset (R) \subset A$.

We may note three particular cases :

(*i*) $A \times A$ is a relation in A. It is called the **universal** relation in A, since for each ordered pair (x, y) with $x, y \in A$, we have $x \text{ R } y$.

(*ii*) $\phi \subset A \times A$ is a relation in A. It is called the **void** relation in A since for any ordered pair (x, y) with $x \in A$ and $y \in A$, we have $(x, y) \notin R$ or $x \not R\, y$.

(*iii*) $I_a \subset A \times A$, where $I_a=\{(x, x) : x \in A\}$ is called the **identity** relation in A. For each $x \in A$, $x \, I_a \, x$ is true.

EXERCISE 1 (*f*)

1. Describe all possible relations from $\{1, 2\}$ to $\{5, 6\}$.
2. Describe all possible relations on $\{a, b\}$.
3. Let $S=\{1, 2, 3, 4, 5\}$ and let R be the relation on S defined by
$$R=\{(1, 2), (3, 5), \{2, 4)\}.$$
Answer the following questions :
(*a*) Is it true that 2 R 3 ?
(*b*) Is 4 R-related to 2 ?
(*c*) Is 2 R-related to 4 ?
(*d*) Is 1 R-related to 3 ?
4. Let $S=\{1, 2, 3, 4\}$. Describe each of the following relations in S :
(*a*) $R_1=\{(x, y) : x, y \in S, x=y\}$;
(*b*) $R_2=\{(x, y) : x, y \in S, x < y)$;
(*c*) $R_3=\{(x, y) : x, y \in S, x > y\}$.
5. What is the domain of the relation
$$\{(1, 2), (1, 3), (2, 3), (2, 5), (3, 7)\} ?$$
6. Describe the range of the relation
$$\{(3, 5), (2, 7), (1, 6), (2, 8), (5, 9)\}.$$

7. Let R be a relation defined in the set **Z** of integers as follows :

$$x \text{ R } y \Leftrightarrow x-y \text{ is divisible by } 4.$$

(a) Which of the following statements are true ?
6 R 21, 8 R 12, (−5) R 7, (−2) R (−9), 2 R 18, 20 R 4, 5 R 11, 13 R 2, 7 R 7, 3 R 11.

(b) Insert R or $\not R$ whichever is correct :
12...14, 13...5, (−9)...7, 4...7, (−6)...2, 3...6, 15...4, 6..9.

(c) Which of the following statements are false ?
6 R 11, 12 R 20, (−4) R 7, (−3) R (−4), 6 R 18, 2 R (−4), 4 R 8, 1 R 1, 2 R 7, 5 R 15.

1.6. EQUIVALENCE RELATIONS

Because of their frequent occurrence in Mathematics, the following types of relations are important :

Definition 1·7. *Let R be a relation in the set A. Then*

(i) *R is said to be* **reflexive** *if* $x\,R\,x$ *for each* $x \in A$;

(ii) *R is said to be* **symmetric** *if* $x\,R\,y \Rightarrow y\,R\,x$, *for all* $x, y \in A$;

(iii) *R is said to be* **transitive** *if* $(x\,R\,y \wedge y\,R\,z) \Rightarrow x\,R\,z$, *for all* $x, y, z \in A$.

Definition 1·8. (*Equivalence Relation*) *A relation that is reflexive, symmetric and transitive is called an* **equivalence relation** (*or an RST relation*).

If R is an equivalence relation, we shall find it convenient to read *a* R *b* as '*a is equivalent to b*'. The symbol $\sim$ is also used to denote a relation. Thus $a \sim b$ means that $\sim$ is a relation and that *a* is $\sim$ related to *b*.

Illustrations :

(a) Let X be a non-empty set and let P (X) be the collection of all subsets of X. The relation

$$R = \{(A, B) : A \in P(X), B \in P(X), A = B\}$$

on P(X) is called the equality relation.

Since $\quad A = A,$
therefore, $\quad (A, A) \in R$ for all $A \in P(X)$,
and consequently R is reflexive.

Since $\quad A = B \Rightarrow B = A,$
therefore, $\quad (A, B) \in R \Rightarrow (B, A) \in R,$
and consequently R is symmetric.

Finally, since $A = B \wedge B = C \Rightarrow A = C$, therefore, the relation R is transitive.

The relation in question is, therefore, an equivalence relation.

.8322

(*b*) Let X be a non-empty set and let P(X) be the collection of all subsets of X. Let R be the relation 'is contained in' in P(X), so that

$$(A, B) \in R \Leftrightarrow A \subset B.$$

In other words,

$$R=\{(A, B) : A \in P(X), B \in P(X), A \subset B\}.$$

This relation is reflexive [because $A \subset A$ for all $A \in P(X)$], transitive (because $A \subset B \wedge B \subset C \Rightarrow A \subset C$) but not symmetric (because if A is a proper subset of B, then $A \subset B$ but $B \not\subset A$, *i.e.*, $(A, B) \in R$ but $(B, A) \notin R$).

(*c*) Let $\sim$ be a relation in the set **N** of all natural numbers defined as follows :

For all $m, n \in \mathbf{N}$, $m \sim n$ iff m, n leave the same remainder when divided by 7.

Evidently (*i*) for every $n \in \mathbf{N}$, $n \sim n$;

(*ii*) for all $m, n \in \mathbf{N}$, $(m \sim n \Rightarrow n \sim m$;

(*iii*) for all $m, n, p \in \mathbf{N}$, $(m \sim n) \wedge (n \sim p) \Rightarrow m \sim p$.

The relation $\sim$ is, therefore, an equivalence relation in **N**.

(*d*) Let T be the set of all triangles in a plane. The relation 'is congruent to' in T is an equivalence relation.

(*e*) Let Z be the set of all integers. The relation 'is greater than' is transitive but is neither reflexive nor symmetric.

(*f*) Let L be the set of all straight lines in a plane. Then the relation 'is perpendicular to' in L is symmetric (for if $l_1, l_2 \in L$, then $l_1 \perp l_2 \Rightarrow L_2 \perp l_1$) but is neither reflexive nor transitive.

(*g*) Let $X=\{1, 2, 3, 4\}$. Consider the following eight relations in X :

$R_1=\{1, 1) (2, 2), (3, 3) (4, 4)\}$;

$R_2=\{(2, 2), (3, 3), (4, 4)\}$;

$R_3=\{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2)\}$;

$R_4 \ \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 3), (3, 1), (1, 4), (4, 1)\}$;

$R_5=\{(2, 2), (3, 3), (4, 4), (1, 2)\}$;

$R_6=\{(1, 2), (2, 1)\}$;

$R_7=\{(1, 1), (2, 2), (3, 3), (4, 4), (1, 3), (4, 1)\}$;

$R_8=\{(1, 2), (2, 3)\}$.

It can be easily seen that :

R_1 is reflexive, symmetric and transitive ;

R_2 is symmetric and transitive but not reflexive ;

R_3 is reflexive and transitive but not symmetric ;

R_4 is reflexive and symmetric but not transitive ;
R_5 is transitive but is neither reflexive nor symmetric ;
R_6 is symmetric but is neither reflexive nor transitive ;
R_7 is reflexive but is neither symmetric nor transitive ;
R_8 is neither reflexive, nor symmetric nor transitive.

1·6·1. Equivalence Classes

Definition 1·9. *If* $\sim$ *is an equivalence relation in a set* S *and* $a \in S$*, then the subset*

$$[a]=\{x \in S : x \sim a\}$$

of S is called the **equivalence class** *corresponding to a.*

Illustration : In the set N of natural numbers, let $\sim$ be defined as follows :

$m, n \in \mathbf{N}, m \sim n$ iff m, n leave the same remainder when divided by 5.

Evidently (*i*) for every $n \in \mathbf{N}$, $n \sim n$,

(*ii*) $m \sim n \Rightarrow n \sim m$,

(*iii*) $(m \sim n) \wedge (n \sim p) \Rightarrow m \sim p$.

The relation $\sim$ is, therefore an equivalence relation.

Consider now the set

$$[2]=\{2, 7, 12, 17, 22, \ldots\ldots\}$$

The subset so defined is an equivalence class determined by the element 2 of $\mathbf{N}$ and the relation $\sim$ on $\mathbf{N}$. It consists of all those natural numbers that are equivalent to 2 with respect to the relation in question.

Theorem 1·5. *An equivalence relation* $\sim$ *defined in a set S partitions it into equivalence classes.*

Proof. Let $\sim$ be an equivalence relation defined in a set S. We observe the following :

(*i*) Every element a of S, belongs to some equivalence class. In fact, $\sim$ being reflexive, $a \sim a$, so that $a \in [a]$.

(*ii*) If $a \in [b]$, then $b \in [a]$. For, $\sim$ being symmetric,

$$a \sim b \Rightarrow b \sim a.$$

Thus, $\quad a \in [b] \Rightarrow b \in [a]$.

(*iii*) If $a \in [b]$, then

$$[a]=[b].$$

We prove it as follows :

Let $x \in [a]$, so that $x \sim a$. [Definition 1·9]

Since $a \in [b]$, therefore, $a \sim b$. [Hypothesis]

Now $x \sim a \wedge a \sim b \Rightarrow x \sim b$, [$\sim$ is transitive]

i.e., $x \in [b]$. [Definition 1 9]

Thus $x \in [a] \Rightarrow x \in [b]$.

Hence $[a] \subset [b]$. ...(*A*)

Since by (*ii*), $a \in [b] \Rightarrow b \in [a]$, therefore, as proved above, $[b] \subset [a]$. ...(*B*)

From (*A*) and (*B*), $[a]=[b]$.

(*iv*) If $[a] \cap [b] \neq \phi$, then $[a]=[b]$.

We prove it as follows :

Let $c \in [a] \cap [b]$.

Then $c \in [a] \wedge c \in [b]$.

Now $c \in [a] \Rightarrow [c]=[a]$. [by (*iii*)]

Also, $c \in [b] \Rightarrow [c]=[b]$.

But $[c]=[a] \wedge [c]=[b] \Rightarrow [a]=[b]$.

We may re-state the result obtained in (*iv*) as follows :

(*v*) If $[a]$ and $[b]$ are two equivalence classes in S, then $[a]$ and $[b]$ are either disjoint or identical.

Combining (*i*) and (*v*) we conclude that the equivalence relation $\sim$ in S partitions it into equivalence classes.

EXERCISE 1 (*g*)

1. Let $S=\{1, 2, 3\}$. Which of the following relations on S are equivalence relations ?

(*a*) $\{(1, 1), (2, 2), (3, 3)\}$.

(*b*) $\{(1, 1), (2, 2), (1, 2), (2, 1)\}$.

(*c*) $\{(1, 1), (2, 2), (1, 3), (3, 3)\}$.

(*d*) $\{(1, 1), (2, 2), (1, 3), (3, 1), (3, 3)\}$.

(*e*) $\{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3), (3, 1)\}$.

(*f*) $\{(1, 1), (2, 2), (1, 3), (2, 3)\}$.

2. Consider the following relations on the set of all people living in the world at a particular moment :

(*a*) $x\, R_1\, y \Leftrightarrow x$ is the father of y ;

(*b*) $x\, R_2\, y \Leftrightarrow x$ is the mother of y ;

(*c*) $x\, R_3\, y \Leftrightarrow x$ is a brother of y ;

(*d*) $x\, R_4\, y \Leftrightarrow x$ is a sister of y ;

(*e*) $x\, R_5\, y \Leftrightarrow x$ is elder to y ;

(*f*) $x\, R_6\, y \Leftrightarrow x$ is younger than y ;

(*g*) $x\, R_7\, y \Leftrightarrow x$ has the same colour of hair as y ;

(*h*) $x\, R_8\, y \Leftrightarrow x$ lives in the same city as y ;

(*i*) $x \, R_9 \, y \Leftrightarrow x$ was born on the same day as y;

(*j*) $x \, R_{10} \, y \Leftrightarrow x$ lives next door to y:

Which of the above relations are (*i*) reflexive, (*ii*) symmetric, (*iii*) transitive?

3. Let $S=\{1, 2, 3, 4, 5\}$. Define a relation on S which is:

(*a*) reflexive, symmetric and transitive;

(*b*) reflexive and symmetric but not transitive;

(*c*) reflexive and transitive but not symmetric;

(*d*) symmetric and transitive but not reflexive;

(*e*) reflexive but neither symmetric nor transitive;

(*f*) symmetric but neither reflexive nor transitive;

(*g*) transitive but neither reflexive nor symmetric.

4. Let **R** be the set of all real numbers. Consider the following relations in **R** and point out which ones are (*i*) reflexive, (*ii*) transitive:

(*a*) $x \, R \, y \Leftrightarrow x^2+y^2=1$;

(*b*) $x \, R \, y \Leftrightarrow xy=1$;

(*c*) $x \, R \, y \Leftrightarrow x < y^2$;

(*d*) $x \, R \, y \Leftrightarrow x-y$ is an integer;

(*e*) $x \, R \, y \Leftrightarrow x-y$ is positive;

(*f*) $x \, R \, y \Leftrightarrow x+y$ is negative;

(*g*) $x \, R \, y \Leftrightarrow x^2+y^2=16$;

(*h*) $x \, R \, y \Leftrightarrow x-1 \geqslant y$;

(*i*) $x \, R \, y \Leftrightarrow x+2y+3=0$;

(*j*) $x \, R \, y \Leftrightarrow |x|+|y|=1$;

(*k*) $x \, R \, y \Leftrightarrow x+y=0$;

(*l*) $x \, R \, y \Leftrightarrow x-y$ is an even integer;

(*m*) $x \, R \, y \Leftrightarrow x-y$ is an odd integer;

(*n*) $x \, R \, y \Leftrightarrow xy$ is an even integer;

(*o*) $x \, R \, y \Leftrightarrow x+y$ is a rational number;

(*p*) $x \, R \, y \Leftrightarrow x-y$ is a rational number;

(*q*) $x \, R \, y \Leftrightarrow x/y$ is a rational number;

(*r*) $x \, R \, y \Leftrightarrow x/y$ is a non-zero rational number;

(*s*) $x \, R \, y \Leftrightarrow [x]=[y]$, where $[x]$ denotes the largest integer not greater than x;

(*t*) $x \, R \, y \Leftrightarrow |x-y| \neq 1$;

(*u*) $x \, R \, y \Leftrightarrow 0 < |x-y| < 1$.

5. For the set S and the relation $\sim$ given below, determine whether it is an equivalence relation:

(*a*) S is the set of all real numbers, $a \sim b$ if $a=\pm b$;

(b) S is the set of all integers, $a \sim b$ if both $a>b$ and $b>a$;

(c) S is the set of all integers, $a \sim b$ if $a-b$ is an even integer ;

(d) S is the set of all integers, $n>1$ is a fixed integer, $a \sim b$ if $a-b$ is a multiple of n ;

(e) S is the set of all integers, $a \sim b$ if $a+b$ is an even integer ;

(f) S is the set of integers, $a \sim b$ if $a+b$ is an integral multiple of 3 ;

(g) S is the set of all integers, $a \sim b$ if either a is a factor of b, or b is a factor of a ;

(h) Let $S=\mathbf{Z} \cup \{\sqrt{2}\}$, where $\mathbf{Z}$ denotes the set of all integers. Define $a \sim b$ to mean $a+b \in \mathbf{Z}$;

(i) S is any non-empty set and $a \sim b$ if $a=b$;

(j) $S=\{(a, b) : a, b$ are positive integers$\}$, and $(a, b) \sim (c, d)$ if $ad=bc$;

(k) $S=\{(a, b) : a, b$ are integers$\}$, $(a, b) \sim (c, d)$ if $a+d=b+c$;

(l) $S=\{(a, b) : a, b$ are positive rational numbers$\}$, $(a, b) \sim (c, d)$ if $ad=bc$;

(m) $S=\{(a, b) : a, b$ are rational numbers$\}$, $(a, b) \sim (c, d)$ if $a+d=b+c$;

(n) $S=\{(a, b) : a, b$ are positive real numbers$\}$, $(a, b) \sim (c, d)$ if $ad=bc$;

(o) $S=\{(a, b) : a, b$ are real numbers$\}$, $(a, b) \sim (c, d)$ if $a+d=b+c$;

(p) S is the set of all integers, $a \sim b$ if $a<b$;

(q) $S=\{a, b) : a, b$ are real numbers$\}$, $(a, b) \sim (c, d)$ if $b-d=m(a-c)$, m a fixed real number ;

(r) $S=\{(a, b) ; a, b$ are real numbers$\}$, $(a, b) \sim (c, d)$ if $a-c$ is an integer and $b=d$;

(s) S is the set of all integers, $a \sim b$ if $a-b$ is an odd integer ;

(t) S is the set of all integers, $a \sim b$ if $ab \geqslant 0$;

(u) S is the set of all integers, $a \sim b$ if $a^2=b^2$;

(v) S is the set of all integers, $a \sim b$ if $|a-b|<1$;

(w) S is the set of all people in the world at a particular moment, $a \sim b$ if a and b have an ancestor in common ;

(x) S is the set of all people in the world at a particular moment, $a \sim b$ if a and b have the same father ;

(y) S is the set of all people in the world at a particular moment, $a \sim b$ if a is within 500 km of b ;

(z) S is the set of all points on the surface of the earth, $a \sim b$ if a and b have the same latitude.

6. The following "proof" purports to show that if a relation is symmetric and transitive, then it must be reflexive :

Let S be a non-empty set and let $\sim$ be a relation in S which is symmetric and transitive.
Since $\sim$ is symmetric, therefore, $a \sim b \Rightarrow b \sim a$.
Again, since $\sim$ is transitive, therefore $(a \sim b) \wedge (b \sim a) \Rightarrow a \sim a$.
Hence $\sim$ is reflexive.

Can you point out the flaw in the above "proof" ?

7. Let S be any non-empty set. In P(S), define a relation R as follows :

$$ARB \Leftrightarrow A \cup B \neq \phi.$$

Show that R is symmetric but is neither reflexive nor transitive.

8. Let X be a non-empty set and M a non-empty subset of X. Prove :
   (*a*) the relation R on P(X) defined by setting $ARB \Leftrightarrow M \subset A \cap B$ is symmetric and transitive but not reflexive.
   (*b*) the relation R on P(X) defined by setting $ARB \Leftrightarrow A \triangle B \subset M$ is an equivalence relation.

9. Let X be a non-empty set, Which of the following relations in P(X) have the property of being reflexive, symmetric, transitive ?
   (*a*) $ARB \Leftrightarrow A \cap B = \phi$.
   (*b*) $ARB \Leftrightarrow A \cup B = X$.
   (*c*) $ARB \Leftrightarrow A \triangle B = X$.

10. Which of the following relations in the set **R** of all real numbers are equivalence relations ?
   (*a*) $x \mathrm{R} y \Leftrightarrow |x| = |y|$.
   (*b*) $x \mathrm{R} y \Leftrightarrow |x| \geqslant |y|$.
   (*c*) $x \mathrm{R} y \Leftrightarrow x - y \geqslant 0$.

1·7. FUNCTIONS

We shall now introduce the concept of a function. The concept of a function is perhaps the most important one in Mathematics. A function is an important type of relation.

Definition 1·10. *A relation f from a set A to a set B is said to be a* **function** *(or a* **mapping**) *from (on) A to (into) B, written as $f : A \to B$, if the following conditions are satisfied :*

(*i*) For each $x \in$ A, there exists a $y \in$ B such that $(x, y) \in f$;

(*ii*) Each $x \in$ A is the first coordinate of not more than one element of f.

Condition (*i*) in the above definition says that for a relation from A to B to be a function from A to B, it is necessary that the

domain of the relation f is A. (Note that this is not necessary for an arbitrary relation from A to B).

Condition (*ii*) in the above definition says that if f be a function, then $(x, y) \in f$ and $(x, z) \in f$ together imply that $y=z$.

The above conditions (*i*) and (*ii*) together say that if f be a function from A to B, then each element of A must be the first coordinate of exactly one element of f.

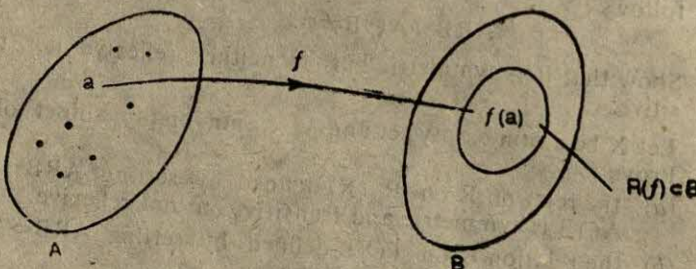


Fig. 1·10.

If f be a function from A to B, then by definition, the domain of the function f is the same as the domain of the relation f, and the range of the function f is the same as the range of the relation f. As a consequence of the definition, it follows that the domain of the function f must be A. The range of f need not, however, be B. We can only say that the range of f must be a subset of B. We say that B is the **co-domain** of f. For an important class of functions, the range happens to be identical with co-domain. (See the definition of surjective function on page 31). As in the case of relations, we shall denote the domain and the range of a function f by $D(f)$ and $R(f)$ respectively.

If f be a function from A to B and $x \in A$, then the element $y \in B$ such that $(x, y) \in f$, is called the **image** of x under f (or the value of f at x) and is denoted by $f(x)$. Also, x is called a **pre-image** of y. In view of this notation, we can write

$$f=\{(x, f(x)) : x \in A\}.$$

If f is a function on A into B, then we may look upon f as a rule that lets correspond to each element $a \in A$, a unique element $f(a) \in B$.

While to each $a \in A$, there corresponds, $f(a) \in B$, there may be some $b \in B$ for which there exists no $a \in A$ such that $b=f(a)$, *i.e.*, while each element of A has an image under f, some elements of B may not have a pre-image under f. While it is possible that two or more elements of A may give rise to the same image, it is impossible for an element of A to have more than one image.

Illustrations :

(*a*) Let $A=\{0, 1, 2, 3, 4\}$, $B=\{x, y, z\}$. Then the relation
$$f=\{(0, x), (1, x), (2, x), (3, y), (4, y)\}$$
is a function from A to B.

Here $f(0)=x$, $f(1)=x$, $f(2)=x$, $f(3)=y$, $f(4)=y$.

To each member of A there corresponds a unique member of B.

Also, $D(f)=\{0, 1, 2, 3, 4\}=A$,
$R(f)=\{x, y\} \subset B$,
so that the domain of f is A, the co-domain of f is B and the range of f is a proper subset of B.

(*b*) In a certain basketball match, the scores of one of the teams were as follows :

Satish 16, Girish 18, Rajesh 10, Vir Singh 0, Mohan 6.

Thus we have a function

f: {Satish, Girish, Rajesh, Vir Singh, Mohan}$\rightarrow \mathbf{Z}$,
where $\mathbf{Z}$ is the set of all integers, given by

f={(Satish, 16), (Girish, 18), (Rajesh, 10), (Vir Singh, 0), (Mohan, 6)}.

Domain of f={Satish, Girish, Rajesh, Vir Singh, Mohan}.

Range of $f=\{16, 18, 10, 0, 6\}$.

(*c*) Let $\mathbf{R}$ be the set of all real numbers and let $f: \mathbf{R}\rightarrow\mathbf{R}$ be the function given by $\{(x, x) : x \in \mathbf{R}\}$.

Here $f(x)=x$, for all $x \in \mathbf{R}$. This function is called the **identity function** on $\mathbf{R}$. Its domain and range are both $\mathbf{R}$.

(*d*) Let k be an arbitrary but fixed real number. Then
$$f=\{(x, k) : x \in \mathbf{R}\}$$
is a function on $\mathbf{R}$ to itself, called a **constant function**. This function may also be described by saying that 'f is a function from $\mathbf{R}$ to itself such that $f(x)=k$ for all $x \in \mathbf{R}$'.

(*e*) Let $$f(x)=\begin{cases} x \text{ if } x>0, \\ -x \text{ if } x\leqslant 0. \end{cases}$$

f is a function on $\mathbf{R}$ to itself, called the **absolute value function.**

$$f=\{(x, x) : x \in \mathbf{R} \wedge x > 0\} \cup \{(x, -x) : x \in \mathbf{R} \wedge x \leqslant 0\}.$$

The domain of f is $\mathbf{R}$ and the range of f is the set of non-negative real numbers.

(*f*) Let $\mathbf{N}$ be the set of all natural numbers and let
$$f=\{(a, b) : a, b \in \mathbf{N}, b=4a\}.$$

Then f is a function on **N** to **N**

(*g*) Let T be the set of all triangles in a plane, and **R** the set of all real numbers. If for each $t \in T$,

$$f(t) = \text{area of the triangle } t,$$

then f is a function on **T** to **R**.

Remark. From the above illustrations it is clear that a function f may be described in either of the following ways :

(*i*) By writing down the set of ordered pairs which are members of the function.

(*ii*) By writing down the domain D of the function and the value $f(x)$ of the function for each $x \in D$.

It may be noted that the above methods of describing a function are equivalent.

EXERCISE 1 (*h*)

1. Let H denote the sets of all human beings living in India at a particular moment. Which of the following relations from H to itself are functions ?

(*a*) $x \text{ R } y \Leftrightarrow y$ is a son of x ;

(*b*) $x \text{ R } y \Leftrightarrow y$ is the father of x ;

(*c*) $x \text{ R } y \Leftrightarrow y$ is a daughter of x ;

(*d*) $x \text{ R } y \Leftrightarrow y$ is the mother of x ;

(*e*) $x \text{ R } y \Leftrightarrow y$ is a brother of x ;

(*f*) $x \text{ R } y \Leftrightarrow y$ is a sister of x ;

(*g*) $x \text{ R } y \Leftrightarrow y$ is an uncle of x ;

(*h*) $x \text{ R } y \Leftrightarrow y$ is the husband of x ;

(*i*) $x \text{ R } y \Leftrightarrow y$ is the wife of x ;

(*j*) $x \text{ R } y \Leftrightarrow y$ is the paternal grandfather of x ;

(*k*) $x \text{ R } y \Leftrightarrow y$ is the maternal grandfather of x ;

(*l*) $x \text{ R } y \Leftrightarrow y$ is a grandfather of x.

2. Let $A = \{1, 2, 3, 4\}$, $B = \{1, 2, 3\}$. State whether the following relations from A to B are functions on A into B :

(*a*) $f = \{(1, 2), (2, 2), (3, 2), (4, 2)\}$;

(*b*) $g = \{(1, 1), (2, 2), (3, 2), (4, 2)\}$;

(*c*) $h = \{(1, 3), (2, 3), (3, 2), (4, 1)\}$.

3. Let $A = \{1, 2, 3, 4, 5\}$, $\mathbf{N} = \{x : x \text{ is a natural number}\}$. Is the following relation a function on A into **N** ?

$$f = \{(1, 1), (2, 4), (3, 9), (4, 16), (5, 25)\}.$$

1·8. SOME SPECIAL TYPES OF FUNCTIONS

By putting some restrictions on a function f, we shall define special types of functions, which would be found useful later on.

Definition 1·11. *A function $f: A \to B$ is said to be a* **univalent function** (*or an* **injective function** *or an* **injection**) *if for all x, $x' \in A$, $f(x)=f(x') \Rightarrow x=x'$.*

Thus, under a univalent function, an element of B can have at the most one element of A as its pre-image. If $x \in A$, then by the definition of a function there exists a unique element $f(x) \in B$. If the function is univalent, we are fully assured that for no other element $x' \in A$, can we have $f(x')=f(x)$. In other words, a univalent function $f: A \to B$ sets up a one-to-one correspondence between the elements of the domain and the range of f.

Definition 1·12. *A function $f: A \to B$ is said to be from A* **onto** *B* (*or a* **surjective function** *or a* **surjection**) *if the range of f is B, i.e., for every $y \in B$, there exists some $x \in A$ such that $f(x)=y$.*

The fact that f is a function from A *onto* B is sometimes also expressed by saying that f maps A *onto* B, and is often expressed as

$$f: A \xrightarrow{\text{onto}} B.$$

Definition 1·13. *A function $f: A \to B$ is called a* **one-to-one function** (*or a* **bijective function** *or a* **bijection**) *if it is univalent and onto.*

Thus if f is a one-to-one function from A onto B, then

(*i*) f is defined for each $x \in A$;

(*ii*) $x, x' \in A \wedge f(x)=f(x') \Rightarrow x=x'$;

(*iii*) $R(f)=B$, *i.e.*, to each $y \in B$ there exists $x \in A$ such that $f(x)=y$.

Illustrations. (*a*) Let $A=\{x, y, z\}$, $B=\{0, 1, 2\}$ and let a function $f: A \to B$ be defined by

$$f(x)=0, \quad f(y)=1, \quad f(z)=2.$$

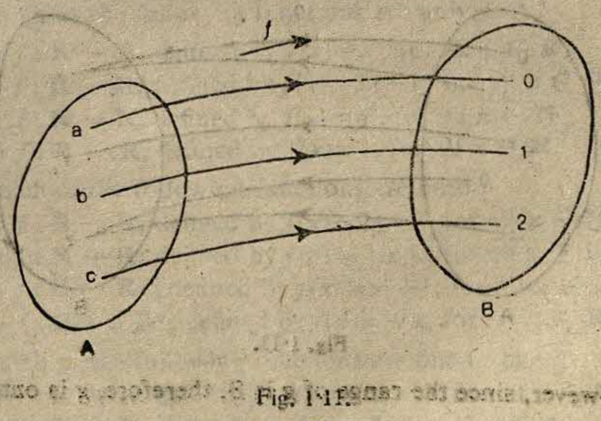


Fig. 1·11.

Here the range of $f=\text{B}$, and therefore, f is onto. Also, since the images of distinct elements of A are distinct elements of B, therefore, f is univalent. Again, since f is both univalent and onto, therefore, it is a one-to-one function from A to B.

(*b*) Let $\text{A}=\{a, b, c\}$, $\text{B}=\{0, 1, 2, 3\}$, and let f be defined by

$$f(a)=0, \qquad f(b)=1, \qquad f(c)=2.$$

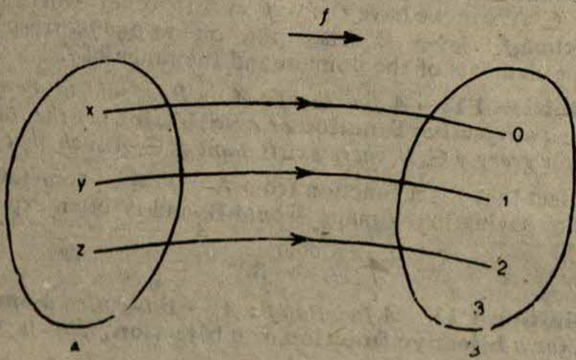


Fig. 1·12.

Since the images of distinct elements of A under f are distinct elements of B, therefore, f is univalent. However, since the range of f is not the whole of B, therefore, f is not onto.

(*c*) Let $\text{A}=\{0, 1, 2, 3, 4\}$, $\text{B}=\{x, y, z\}$, and let g be a function defined by

$$g(0)=g(1)=x, \; g(2)=y, \; g(3)=g(4)=z.$$

The function g is not univalent, since distinct elements 0 and 1 of the domain have the same image.

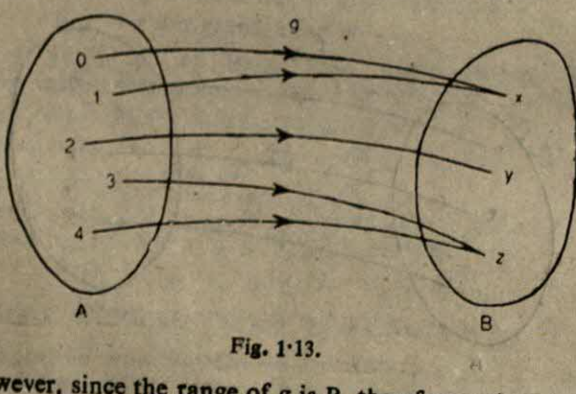


Fig. 1·13.

However, since the range of g is B, therefore, g is onto.

(*d*) Let $A=\{a, b, c\}$, $B=\{0, 1\}$, and let f be a function from A to B defined by $f(a)=f(b)=f(c)=1$.

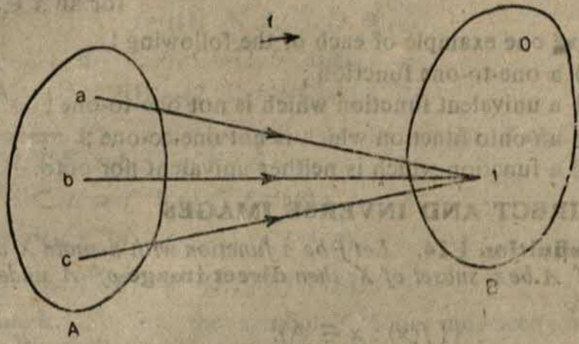


Fig. 1·14.

Since the two distinct elements a and b of A have the same image, namely 1, under f, therefore f is not univalent. Also since the range of f is a proper subset of B, therefore, f is not onto. f is thus, neither univalent nor onto.

The above illustrations show that a function may be univalent as well as onto, univalent but not onto, onto but not univalent, or neither univalent nor onto.

EXERCISE 1 (*i*)

[**R** stands for the set of all real numbers and $\mathbf{R}^+$ stands for the set of all positive real numbers.]

1. Let **N** be the set of natural numbers and $E \subset \mathbf{N}$ be the set of even numbers. For each $n \in \mathbf{N}$, let $f(n)=2n$. Is the function $f: \mathbf{N} \to E$ univalent ?
2. Which of the following functions are univalent ?
   (*a*) $f: \mathbf{R} \to \mathbf{R}$, defined by $f(x)=x^2$, for all $x \in \mathbf{R}$;
   (*b*) $f: \mathbf{R} \to \mathbf{R}$, defined by $f(x)=3x-1$, for all $x \in \mathbf{R}$;
   (*c*) $f: \mathbf{N} \to \mathbf{N}$, defined by $f(n)=n^2$, for all $n \in \mathbf{N}$;
   (*d*) $f: \mathbf{R} \to \mathbf{R}$, defined by $f(x)=1$, for all $x \in \mathbf{R}$.
3. Which of the following functions are onto ?
   (*a*) $f: \mathbf{R} \to \mathbf{R}$, defined by $f(x)=2x+7$, for all $x \in \mathbf{R}$;
   (*b*) $f: \mathbf{R} \to \mathbf{R}$, defined by $f(x)=|x|$, for all $x \in \mathbf{R}$;
   (*c*) $f: \mathbf{R} \to \mathbf{R}^+$, defined by $f(x)=x^2+1$, for all $x \in \mathbf{R}$;
   (*d*) $f: \mathbf{R}+ \to \mathbf{R}^+$, defined by $f(x)=\sqrt{x}$, for all $x \in \mathbf{R}^+$.
4. Which of the following functions are one-to-one ?
   (*a*) $f: \mathbf{R} \to \mathbf{R}$, defined by $f(x)=5x-13$, for all $x \in \mathbf{R}$;
   (*b*) $f: \mathbf{R} \to \mathbf{R}$, defined by $f(x)=x^2$, for all $x \in \mathbf{R}$;

(c) $f : \mathbf{R}^+ \to \mathbf{R}$, defined by $f(x)=2\sqrt{x}+1$, for all $x \in \mathbf{R}^+$;

(d) $f : [-1, 1] \to [-1, 1]$, defined by $f(x)=x^3$,

for all $x \in [-1, 1]$.

5. Give one example of each of the following :

(a) a one-to-one function ;

(b) a univalent function which is not one-to-one ;

(c) an onto function which is not one-to-one ;

(d) a function which is neither univalent nor onto.

1·9. DIRECT AND INVERSE IMAGES

Definition 1·14. *Let f be a function with domain X and range in Y. If A be a subset of X, then* **direct image** *of A under f is the set*

$$\{f(x) : x \in A\}.$$

It is usual to denote the direct image of a set A by $f(A)$.

Illustration. Let $f : \mathbf{R} \to \mathbf{R}$ be a function defined by

$$f(x)=x^2 \text{ for all } x \in \mathbf{R}.$$

If $A=[0, \infty[$

and $B=[-1, 1]$,

then $f(A)=[0, \infty[$,

$f(B)=[0, 1]$.

The following theorem gives some valuable information about direct images :

Theorem 1·6. *Let f be a function with domain X and range in Y, and let A, B be subsets of X. Then*

(a) $A \subset B \Rightarrow f(A) \subset f(B)$;

(b) $f(A \cup B)=f(A) \cup f(B)$;

(c) $f(A \cap B) \subset f(A) \cap f(B)$.

Remark. It can be shown that the inclusion relation in (c) cannot, in general, be replaced by equality. For example, consider the function f defined by

$$f=\{(-1, 1), (0, 0), (1, 1)\}.$$

Let $A=\{-1, 0\}, B=\{0, 1\}$.

Then $A \cap B=\{0\}, f(A \cap B)=\{f(0)\}=\{0\}$,

$f(A)=\{1, 0\}, f(B)=\{0, 1\}, f(A) \cap f(B)=\{0, 1\}$.

Thus $f(A \cap B)$ is a proper subset of $f(A) \cap f(B)$.

Definition 1·15. *Let f be a function with domain X and range in Y, and let B be a subset of Y. The* **inverse image** *of B under f is the set*

$$\{x : f(x) \in B\}.$$

It is usual to denote the inverse image of B under f by $f^{-1}(B)$.

Illustration. Let f be a function defined by

$$f=\{(1, 1), (2, 4), (3, 9)\}.$$

If $A=\{1\}$, $B=\{4, 9\}$, $C=\{1, 9\}$, then

$f^{-1}(A)=\{1\}, f^{-1}(B)=\{2, 3\}, f^{-1}(C)=\{1, 3\}$.

Theorem 1·7. *Let f be a function with domain X and range in Y, and let A, B be subsets of Y. Then*

(*a*) $A \subset B \Rightarrow f^{-1}(A) \subset f^{-1}(B)$;

(*b*) $f^{-1}(A \cup B)=f^{-1}(A) \cup f^{-1}(B)$;

(*c*) $f^{-1}(A \cap B)=f^{-1}(A) \cap f^{-1}(B)$.

Remark. So far, the symbol f^{-1} has not been given any meaning. We shall give it a meaning in section 1·11.

EXERCISE 1 (*j*)

1. Let $f: \mathbf{R} \to \mathbf{R}$ be a function defined by $f(x)=x^3$ for all $x \in \mathbf{R}$ and let $A=]0, 1[$, $B=[0, 1]$, $C=]0, 1]$, $D=[0, \infty[$. Find $f(A)$, $f(B)$, $f(C)$ and $f(D)$.
2. Let $f: \mathbf{R} \to \mathbf{R}$ be a function defined by $f(x)=x^2+1$, for all $x \in \mathbf{R}$ and let $A=]-1, 1[$, $B=]-1, \infty[$, $C=]-3, 3]$, $D=]\infty, 0[$. Find $f(A)$, $f(B)$, $f(C)$ and $f(D)$.
3. Let $f: \mathbf{R} \to \mathbf{R}$ be a function defined by $f(x)=2x+1$ for all $x \in \mathbf{R}$, and let $A=]-\infty, \infty[$, $B=]-3, 3]$, $C=]2, 4[$, $D=]-\infty, 1[$. Find $f^{-1}(A)$, $f^{-1}(B)$, $f^{-1}(C)$ and $f^{-1}(D)$.
4. Let $f: \mathbf{R} \to \mathbf{R}$ be a function defined by $f(x)=|x|$ for all $x \in \mathbf{R}$, and let $A=]-\infty, 0[$, $B=]0, \infty[$, $C=]-1, 1[$, $D=]0, 1[$. Find $f^{-1}(A)$, $f^{-1}(B)$, $f^{-1}(C)$ and $f^{-1}(D)$.
5. Show by means of an example that for a function f, the sets $f(A \cap B)$ and $f(A) \cap f(B)$ may be distinct.

1·10. COMPOSITE OF FUNCTIONS

Definition 1·16. *Let f be a function with domain X and range in Y, and let g be a function with domain Y and range in Z. The function with domain X and range in Z which maps an element $x \in X$, into $g[f(x)]$, is called the* **composite** *of the functions f and g and is written as $g \circ f$.*

Illustration: Let $f=\{(1, 3), (3, 9), (5, 15)\}$,

$g=\{(3, 2), (9, 8), (15, 14), (21, 20)\}$.

Then $g \circ f$ is the function $\{(1, 2), (3, 8), (5, 14)\}$

It is important to note that for $g \circ f$ to be defined, the range of must be contained in the domain of g.

It may happen that for a certain pair of functions f and g, one of the composites $f \circ g$ and $g \circ f$ be defined while the other may not be defined.

For example, let $\mathbf{R}^+$ denote the set of all positive real numbers and let $f : \mathbf{R}^+ \to \mathbf{R}$ be defined by $f(x)=2x-1$ for all $x \in \mathbf{R}^+$ and let $g : \mathbf{R}^+ \to \mathbf{R}^+$ be defined by $g(x)=x^2$ for all $x \in \mathbf{R}^+$. $f \circ g$ is defined but $g \circ f$ is not defined.

It can be easily seen that if f be a function with domain X and range in Y, then $g \circ f$ and $f \circ g$ will be both defined if and only if the domain of g contains the range of f and the range of g is contained in X.

It may well happen that for a given pair of functions f and g, the composites $g \circ f$ and $f \circ g$ are both defined but are not equal. For example, let f be a function from $\mathbf{R}$ to $\mathbf{R}$ defined by $f(x)=x^2$, and let g be a function from $\mathbf{R}$ to $\mathbf{R}$ defined by $g(x)=x+1$; $f \circ g$ and $g \circ f$ are both defined here. In fact,

$$(f \circ g)(x)=(x+1)^2 \text{ for all } x \in \mathbf{R},$$

$$(g \circ f)(x)=x^2+1 \text{ for all } x \in \mathbf{R}.$$

That the two functions are not the same may be seen by considering $x=1$.

Theorem 1·8. *Let $f : X \to Y$, $g : Y \to Z$, $h : Z \to W$ be three functions. Then $(h \circ g) \circ f = h \circ (g \circ f)$.*

Proof. $(h \circ g) \circ f$ and $h \circ (g \circ f)$ are both functions with domain X and range in W. Also, if $x \in X$, then

$$\begin{aligned}((h\circ g) \circ f)(x) &= (h\circ g)(f(x)),\\ &= h(g(f(x))),\\ &= h(g\circ f(x)),\\ &= (h\circ(g\circ f))(x).\end{aligned}$$

Hence $\quad (h\circ g) \circ f = h\circ(g\circ f)$.

EXERCISE 1 (*k*)

1. Let $f : \mathbf{R} \to \mathbf{R}$, $g : \mathbf{R} \to \mathbf{R}$ be defined by
   (*a*) $f(x)=2x+1$, $g(x)=x^3+3$ for all $x \in \mathbf{R}$;
   (*b*) $f(x)=x^2$, $g(x)=2x+5$ for all $x \in \mathbf{R}$;
   (*c*) $f(x)=x^3-2$, $g(x)=x+7$ for all $x \in \mathbf{R}$.

   Compute $f\circ g$ and $g\circ f$ for each of the above pairs of functions.
2. Let $f : \mathbf{R} \to \mathbf{R}$, $g : \mathbf{R}^+ \to \mathbf{R}^+$ be defined by
   (*a*) $f(x)=x^3-1$, for all $x \in \mathbf{R}$, $g(x)=\sqrt{x}$ for all $x \in \mathbf{R}^+$;
   (*b*) $f(x)=x-\frac{1}{2}$ for all $x \in \mathbf{R}$, $g(x)=x^2+\frac{1}{2}$ for all $x \in \mathbf{R}^+$.

Show that in each of the above cases $f o g$ is defined but $g o f$ is not defined.

3. Show by means of examples that for a pair of functions f and g

(*a*) none of $f o g$ and $g o f$ may be defined ;

(*b*) only one of $g o f$ and $f o g$ may be defined ;

(*c*) $f o g$ and $g o f$ may both be defined but not be equal ;

(*d*) $f o g$ may be equal to $g o f$.

1·11. INVERTIBLE FUNCTIONS

Definition 1·17. *Let f be a function with domain X and range Y. If there exists a function g with domain Y and range X such that $g o f$ is the identity function on X, then g is said to be an* **inverse** *of f. A function which possesses an inverse is said to be* **inversible.**

Illustrations. (*a*) Let f and g be defined by

$$f=\{(1, 3), (2, 6), (3, 9)\},$$

$$g=\{(3, 1), (6, 2), (9, 3)\}.$$

Here $g o f$ is the identity function on the domain of f and $f o g$ is the identity function on the domain of g. The functions f and g are inverses of each other.

(*b*) Let $f : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ and $g : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ be defined by

$$f(x)=x^2$$

and
$$g(x)=\sqrt{x}$$

for all $x \in \mathbf{R}^+$. Here $g o f$ is the identity function from $\mathbf{R}^+$ onto $\mathbf{R}^+$ and $f o g$ is the identity function from $\mathbf{R}^+$ onto $\mathbf{R}^+$. The functions f and g are inverses of each other.

(*c*) Let $f : \mathbf{R} \rightarrow \mathbf{R}^+ \cup \{0\}$ be the function defined by

$$f(x)=x^2$$

for all $x \in \mathbf{R}$. It can be easily seen that there is no function

$$g : \mathbf{R}^+ \cup \{0\} \rightarrow \mathbf{R}$$

for which $g o f$ is the identity function on $\mathbf{R}$. That is, f is not invertible.

We state below (without proof) the main results on invertible functions :

Theorem 1·9. *If a function g be an inverse of a function f, then f is an inverse of g.*

Remark. In view of the above theorem, it follows that the relation 'is an inverse of' among functions is symmetric and consequently it is often more appropriate to talk of a pair of functions as inverses of each other, rather than talking of one of them as the inverse of the other.

The next theorem shows that a function cannot possess more than one inverse.

Theorem 1·10. *If g and h be inverses of a function f, then $g=h$.*

The above theorem shows that if a function possesses an inverse it must be unique. In view of this, we may talk of the inverse of a function rather than an inverse. It is usual to denote the inverse of a function f by the symbol f^{-1}.

Theorem 1·11. *Every one-to-one function is invertible.*

The above theorem shows that every one-to-one function is invertible. The following theorem shows that one-to-one functions are precisely the invertible functions.

Theorem 1·12. *If a function is invertible, then it must be one-to-one.*

Theorem 1·13. *Let f be a one-to-one function with domain X and range Y, and let g be a one-to-one function with domain Y and range Z Then*

$$(g \circ f)^{-1}=f^{-1} \circ g^{-1}$$

EXERCISE 1 (*l*)

1. Verify that for each of the following pairs of functions f and g, the function $g \circ f$ is the identity function on the domain of f:

   (*a*) $f(x)=2x+1$ for all $x \in \mathbf{R}$, $g(x)=\frac{1}{2}(x-1)$ for all $x \in \mathbf{R}$;
   (*b*) $f(x)=x^2$ for all $x \in \mathbf{R}^+$, $g(x)=\sqrt{x}$ for all $x \in \mathbf{R}^+$;
   (*c*) $f(x)=1/x$ for all $x \in \mathbf{R}^+$, $g(x)=1/x$ for all $x \in \mathbf{R}^+$.

2. Which of the following functions are invertible ?

   (*a*) $f : \mathbf{R} \to \mathbf{R}$, where $f(x=3x+2$ for all $x \in \mathbf{R}$;
   (*b*) $f : \mathbf{R} \to \mathbf{R}^+ \cup (0)$, where $f(x)=x^2$ for all $x \in \mathbf{R}$;
   (*c*) $f : \mathbf{R}^+ \to \mathbf{R}^+$, where $f(x)=1/x^2$ for all $x \in \mathbf{R}^+$.

3. Find the inverse of each of the following functions :

   (*a*) $f : \mathbf{R} \to \mathbf{R}$ where $f(x)=3x+7$ for all $x \in \mathbf{R}$;
   (*b*) $f : \mathbf{R}^+ \to \mathbf{R}^+$, where $f(x)=2x^2$ for all $x \in \mathbf{R}^+$;
   (*c*) $f : [0, 1] \to [0, 1]$, where $f(x)=x^3$ for all $x \in [0, 1]$.

4. Construct an invertible function whose domain is the set of all natural numbers.

1·12. BINARY OPERATIONS

Consider the operation of addition in integers. We may look upon '+' as a machine such that when we feed integers m and n into this machine, the output is the integer $m+n$. The operation '+' thus assigns to each element (m, n) of $\mathbf{Z} \times \mathbf{Z}$ (where $\mathbf{Z}$ stands for

the set of all integers) a uniquc integer $m+n$. Addition of integers

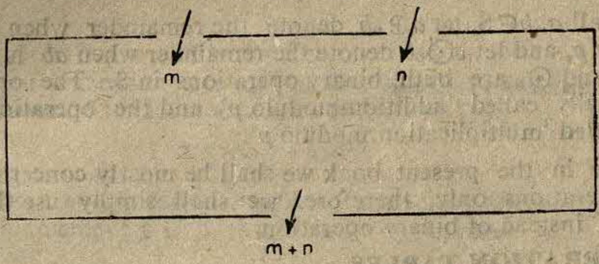


Fig. 1·15.

is an example of a binary operation (or a binary composition). More generally, we have the following :

Definition 1·18. *Let S be a non-empty set. Any function from* $S \times S$ *to S is called a* **binary composition** (*or a* **binary operation**) *in S.*

If $f : S \times S \to S$ be a binary composition in S, and $x, y \in S$, then $f(x, y)$ is called the composite of x and y under the composition f. It is usual to denote a binary composition by any of the following symbols :

$*, \top, \perp, \oplus, \odot, \ominus, +, .,$ juxtaposition.

If we denote a binary operation in a set S by $*$ (resp. $\top$, $\perp$, $\oplus$, $\odot$, θ, $+$, $.$, juxtaposition) and $x, y \in S$, then the composite of x and y under this operation is denoted by $x*y$ (resp. by $x \top y$, $x \perp y$, $x \oplus y$, $x \odot y$, $x \,\theta\, y$, $x+y$, $x.y$, xy).

Illustrations. (*a*) Addition is a binary operation in

(α) the set **N** of all natural numbers ;

(β) the set **Z** of all integers ;

(γ) the set **Q** of all rational numbers ;

(δ) the set **R** of all real numbers ;

(ε) the set **C** of all complex numbers.

(*b*) Multiplication is a binary operation in each of the sets (α)–(ε) described in illustration (*a*) above.

(*c*) Subtraction is not a binary operation in **N**.

(*d*) For all $a, b \in \mathbf{Z}$, let

$$a \oplus b = a + b - ab.$$

$\oplus$ is a binary operation in **Z**.

(*e*) For all $a, b \in \mathbf{N}$, let

$$a * b = a^b.$$

$*$ is a binary operation in **N**.

(*f*) Let p be a fixed positive integer and let

$$S=\{0, 1, 2, \ldots, p-1\}.$$

For all $a, b \in S$, let $a \oplus_p b$ denote the remainder when $a+b$ is divided by p, and let $a \odot_p b$ denote the remainder when ab is divided by p. $\oplus_p$ and $\odot_p$ are both binary operations in S. The operation $\oplus_p$ is usually called 'addition modulo p' and the operation $\odot_p$ is usually called 'multiplication modulo p'.

Since in the present book we shall be mostly concerned with binary operations only, therefore, we shall simply use the word 'operation' instead of binary operation.

1·13. OPERATION TABLES

If S be a finite set, consisting of n elements say, then an operation '*' in S can be described by means of a table consisting of n rows and n columns in which the entry at the intersection of row headed by an element $a \in S$ and the column headed by an element $b \in S$ is $a * b$. Such tables are called operation tables.

Illustration. Let $S=\{a, b, c\}$. The table in Fig. 1·16 gives an operation θ in S. From this table we find that

$$\begin{array}{lll} a\,\theta\, a=a, & a\,\theta\, b=b, & a\,\theta\, c=a, \\ b\,\theta\, a=b, & b\,\theta\, b=b, & b\,\theta\, c=c, \\ c\,\theta\, a=a, & c\,\theta\, b=c, & c\,\theta\, c=c. \end{array}$$

θ	a	b	c
a	a	b	a
b	b	b	c
c	a	c	c

Fig. 1·16.

Definition 1·19. *Let S be a non-empty set and let * be an operation in S.*

(*i*) The operation * is said to be **associative** if for all $x, y, z \in S$,

$$x * (y * z)=(x * y) * z.$$

(*ii*) The operation $*$ is said to be **commutative** if for all $x, y \in S$,

$$x * y = y * x.$$

(*iii*) An element $n \in S$ is said to be a **neutral element** (or an **identity element**) for the operation $*$ if for all $x \in S$.

$$x * n = n * x = x$$

(*iv*) Let the operation $*$ possess a neutral element n, and let $x \in S$. An element $y \in S$ is said to be an **inverse** of x for the operation $*$ if

$$x * y = y * x = n.$$

(*v*) Let $*_1$ and $*_2$ be two operations in S. The $*_2$ is said to be **distributive** over $*_1$ if

$$x *_2 (y *_1 z) = (x *_2 y) *_1 (x *_2 z)$$

for all $x, y, z \in S$.

Illustrations. (*a*) Let X be a non-empty set, and let P (X) be the set of all subsets of X. $\cap$ and $\cup$ are operations in P (X).

Since $A \cap B = B \cap A$,

and $A \cup B = B \cup A$,

for all $A, B \in P(X)$, therefore, both these operations are commutative.

Again, since

$$A \cap (B \cap C) = (A \cap B) \cap C,$$

and

$$A \cup (B \cup C) = (A \cup B) \cup C,$$

for all $A, B, C \in P(X)$, therefore, both these operations are associative.

Since $A \cap X = X \cap A = A$.

for all $A \in P(X)$, therefore, X is a neutral element for the operation $\cap$.

Again, since

$$A \cup \phi = \phi \cup A = A,$$

for all $A \in P(X)$, therefore, ϕ is a neutral element for the operation $\cup$.

Finally, since

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C),$$

and

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C),$$

for all A, B, C $\in$ P (X), therefore, $\cap$ distributes itself over $\cup$, and $\cup$ distributes itself over $\cap$.

(*b*) Addition in **N** is commutative and associative. There is no neutral element for addition in **N**.

(*c*) Subtraction in **Z** is neither commutative nor associative.

(*d*) Multiplication in **N** is commutative and associative. The natural number 1 is a neutral element for multiplication. The only natural number which has inverse with respect to multiplication is 1.

(*e*) Multiplication in the set **Q** of rational numbers is commutative and associative. 1 is a neutral element for multiplication. If x be any rational number other than zero, then $1/x$ is an inverse of x with respect to multiplication.

(*f*) The multiplication composition in the set **Q** of all rational numbers distributes itself over addition. For,

$$a\,(b+c)=ab+ac,$$

for all $a, b, c \in \mathbf{Q}$.

Addition in **Q** does not distribute itself over multiplication. For, the statement $a+bc=(a+b)(a+c)$ is not true for all possible triples a, b, c of rational numbers.

EXERCISE 1 (*m*)

1. For each operation defined below, determine whether $*$ is associative :

   (*a*) On **Z**, define $*$ by $a * b=b-a$.

   (*b*) On **Q**, define $*$ by $a * b=ab+2$.

   (*c*) On **Q**, define $*$ by $a * b=ab/3$.

   (*d*) On **Q**, define $*$ by $a * b=a$.

   (*e*) On **R**, define $*$ by $a * b=a+b+ab$.

   (*f*) On **N**, define $*$ by $a * b=a^2+b^2$.

2. Let $S=\{a, b, c, d\}$. Complete the missing entries in the table in Fig. 1.17 and Fig. 1.18 so that the operation $*$ may be associative.

*	a	b	c	d
a	a		c	d
b	b	a	c	d
c	c	d	c	
d				

Fig. 1·17.

*	a	b	c	d
a	a	b	c	
b	b	d		c
c	c	a	d	b
d	d			a

Fig. 1·18.

3. Let S={*a*, *b*, *c*, *d*}. Complete the missing entries in the table in Fig. 1.18 so that the operation * may be commutative.

4. Let S be a set consisting of 2 elements. How many different operations can be defined on X ?

5. How many different operations can be defined on a set S consisting of 3 elements ?

6. How many different commutative operations can be defined on a set consisting of 10 elements ?

7. How many different commutative operations can be defined on a set consisting of n elements ?

8. Which of the following statements are true and which are false ?

   (*a*) An operation in a set S assigns at least one element of S to each ordered pair of elements of S.

   (*b*) An operation in a set S assigns not more than one element of S to each ordered pair of elements of S.

   (*c*) An operation in a set S assigns exactly one element of S to each ordered pair of elements of S.

   (*d*) If $*$ be a commutative operation in a set S, then
   $$a*(b*c)=(c*b)*a$$
   for all $a, b, c \in S$.

   (*e*) Every operation in a set consisting of 3 elements is commutative.

   (*f*) Every operation in a set consisting of 6 elements is commutative.

   (*g*) If $*$ be an associative operation in a set S, then
   $$a*(b*c)=(c*b)*a$$
   for all $a, b, c \in S$.

9. Let S be a set with an associative operation $*$. Show that
   $$(a*b)*(c*d)=((a*b)*c)*d$$
   for all $a, b, c, d \in S$.

10. Let **Z** be the set of integers. We define two operations $\oplus$, $\odot$, in $\mathbf{Z} \times \mathbf{Z}$ as follows :
   $$(a, b) \oplus (c, d)=(a+c, b+d),$$
   $$(a, b) \odot (c, d)=(ac-bd, ad+bc).$$
   Show that

   (*a*) $\oplus$ is commutative as well as associative.

   (*b*) $(0, 0)$ is a neutral element for the operation $\oplus$.

TEST YOUR UNDERSTANDING I

In each of the problems overleaf, four alternatives are given. Put a tick-mark (✓) on the correct alternative :

1. Let A={2, 3, 4, 5}, B={3, 4, 6}, C={1, 2, 3}. (A ∪ B) ∩ C equals

   (*a*) {2, 3} (*b*) {1, 2, 3, 4}

   (*c*) {3} (*d*) {2}.

2. Let A={2, 3, 4}, B={3, 4, 5}. Then A ∼ B equals

   (*a*) {3, 4} (*b*) {2}

   (*c*) {5} (*d*) {2, 3}.

3. In the set **Z** of integers, let $*$ denote the operation of multiplication, that is, let $a * b = ab$ for all $a, b \in \mathbf{Z}$. The neutral element for $*$ is

   (*a*) 1 (*b*) 0

   (*c*) —1 (*d*) 2.

4. In the set **N** of natural numbers, let $*$ be defined by $a * b =$ lcm a, b for all $a, b, \in \mathbf{N}$. The neutral element for $*$ is

   (*a*) 1 (*b*) 2

   (*c*) 5 (*d*) 10.

5. In the set **Z** of integers, let $*$ be defined by $a * b = a - b$ for all $a, b \in \mathbf{Z}$. Then

   (*a*) 0 is a neutral element for $*$

   (*b*) 1 is a neutral element for $*$

   (*c*) there is no neutral element for $*$

   (*d*) -1 is a neutral element for $*$

6. In the set **Z** of integers, let $*$ be the binary operation defined by $a * b = a - b$ for all $a, b \in \mathbf{Z}$. The operation $*$ is

   (*a*) commutative as well as associative

   (*b*) associative but not commutative

   (*c*) commutative but not associative

   (*d*) neither associative nor commutative.

7. In the set **N** of natural numbers, let $*$ be the binary operation of addition. Then

   (*a*) 1 is the neutral element for $*$

   (*b*) there is no neutral element for $*$

   (*c*) 2 is a neutral element for $*$

   (*d*) 10 is a neutral element for $*$.

8. The inverse of 2 for the binary operation 'addition' in the set **R** of real numbers is

(a) $\frac{1}{2}$ (b) 1
(c) -2 (d) 0.

9. The inverse of -1 for the binary operation 'multiplication' in the set **Z** of integers is

(a) 2 (b) 0
(c) 1 (d) -1.

10. Let $A=\{1, 2, 3, 4\}$, $B=\{a, b, c, d\}$. The number of elements in $A \times B$ is

(a) 1 (b) 4
(c) 8 (d) 16.

REVIEW EXERCISE I

1. Show that the number of subsets of a set consisting of elements is 2^5.

2. Give an example of a binary operation on **N** which is not commutative.

3. Show, by means of an example, that a binary operation on a set may fail to be associative.

4. Let f be the function on the set **R** of real numbers defined by $f(x)=x^2$, for all $x \in \mathbf{R}$. Is f surjective ? Is it invertible ?

5. Let $f(x)=3x-5$ for all $x \in \mathbf{R}$. Find the inverse of f.

6. Let f and g be functions from **R** to **R** defined by $f(x)=x^2$, $g(x)=2x-5$ for all $x \in \mathbf{R}$. Find $f \circ g$ and $g \circ f$.

7. In the set **Z** of integers let $*$ be the operation 'multiplication of integers'. Show that the only elements of **Z** having inverses for $*$ are -1 and 1.

8. In the set **N** of natural numbers, the binary operation $*$ is defined by

$$p * q = gcd\,(p, q), \text{ for all } p, q \in \mathbf{N}.$$

Is the operation $*$ commutative ? Is it associative ?

9. In the set **N** of natural numbers, the binary operation $*$ is defined by

$$p * q = lcm\,(p, q), \text{ for all } p, q \in \mathbf{N}$$

Is the operation $*$ commutative ? Is it associative ?

10. In the set **N** of natural numbers, the binary operation $*$ is defined by

$$p * q = p^q.$$

Isthe operation $*$ commutative ?

11. The report of one survey of 100 students stated that the numbers studying the various languages were: Sanskrit, Hindi and Tamil, 5; Hindi and Sanskrit, 10; Tamil and Sanskrit, 8; Hindi and Tamil, 20; Sanskrit, 30; Hindi, 23; Tamil, 50. The surveyor who prepared this report was fired. Why?

(*Roorkee Entrance 1983*)

12. In a pollution study of 1500 Indian rivers the following data were reported, 520 were polluted by sulphur compounds, 335 were polluted by phosphates, 425 were polluted by crude oil, 100 were polluted by both crude oil and sulphur compounds, 180 were polluted by both sulphur compounds and phosphates, 150 were polluted by both phosphates and crude oil and 28 were polluted by sulphur compounds, phosphates and crude oil. How many of the rivers were polluted by at least one of the three impurities? How many of the rivers were polluted by exactly one of the three impurities?

(*Roorkee Entrance 1987*)

SUMMARY

1. If A, B, C be any subsets of a set X, then

(*a*) $A \cup A = A$, $A \cap A = A$.

(*b*) $A \cup B = B \cup A$, $A \cap B = B \cap A$.

(*c*) $(A \cup B) \cup C = A \cup (B \cup C)$, $(A \cap B) \cap C = A \cap (B \cap C)$.

(*d*) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(*e*) $X \sim (A \cup B) = (X \sim A) \cap (X \sim B)$,

$X \sim (A \cap B) = (X \sim A) \cup (X \sim B)$.

2. If A and B are sets, then

$A \times B = \{(x, y) : x \in A, y \in B\}$.

3. Let S be a non-empty set. A function from $S \times S$ to S is called a binary operation on S.

4. Let S be a non-empty set and let * be a binary operation on S.

(*i*) * is said to be associative if $a * (b * c) = (a * b) * c$ for all $a, b, c \in S$.

(*ii*) * is said to be commutative if $a * b = b * a$ for all $a, b \in S$.

(*iii*) An element $e \in S$ is said to be a neutral element for '*' if $a * e = e * a = a$, for all $a \in S$.

An element $b \in S$ is said to be an inverse of an element $a \in S$ it $a * b = b * a = e$.

HISTORICAL NOTE

Set Theory was created by the German mathematician *Georg Cantor* (1845—1918). The rule 'complement of the union of a family of sets is the intersection of their complements, and complement of the intersection of a family of sets is the union of their complements' is due to the British mathematician *Augustus De Morgan* (1806—1871). Incidentaly, De Morgan was born in India. Venn diagrams were introduced by the English logician *John Venn* (1834—19.3). The name of the famous Swiss Mathematician Leonhard Euler (1707—1787) is also associated with these diagrams and therefore, they are sometimes also called *Euler-Venn diagrams*.

□ □

GIUSEPPE PEANO (1858—1932)

Giuseppe Peano was born in Italy in the year 1858. He undertook the task of deducing the properties of natural numbers from a set of explicitly stated assumptions, now known as *Peano's Axioms*. This gave a new impetus to the abstract axiomatic method. With the help and collaboration of several other mathematicians of his times, he recast a considerable part of mathematics in accordance with his new method. The principle of mathematical induction is a re-statement of one of the Peano's Axioms.

CHAPTER 2

Mathematical Induction

2·1. INTRODUCTION

Consider the set N of natural numbers. **N** has two characteristic properties :

(*i*) **N** contains the natural number 1.

(*ii*) **N** is closed with respect to addition of 1 to each of its numbers.

Therefore, to determine whether a set K consisting of natural numbers is the set of all natural numbers, we have to verify the following two conditions on K :

(*i*) Does $1 \in K$?

(*ii*) For each natural number $k \in K$, is it true that $k+1 \in K$?

When the answer to both the questions is 'yes', then (and only then) K is N. It gives several important principles for establishing the truth of certain classes of statements. In the present chapter we shall describe the simplest of all such principles.

2·2. PRINCIPLE OF FINITE INDUCTION (PFI)

The following principle, commonly known as the principle of finite induction (abbreviated as PFI) is the simplest of all the principles of mathematical induction. It may be stated as follows :

Let $\{T(n) : n \in \mathbf{N}\}$ be a set of statements, one for each natural number n. If $T(1)$ is true and the truth of $T(k)$ implies that of $T(k+1)$, then $T(n)$ is true for all n.

The method of induction is a powerful tool (perhaps the most powerful single tool !) for proving theorems. A proof by induction may be likened to climbing an infinite ladder (at least mathematicians can imagine such ladders !). Just as to climb such a ladder one must climb the first rung of the ladder, and having climbed any such rung, the next rung should be climbed, similarly to work out a proof by PFI one must show that the statement under consideration holds for $n=1$, and that whenever it holds for a natural number k, it also holds for $k+1$. For the sake of convenience, we shall refer to these two requirements as step 1 and step 2 respectively. Both the steps happen to be equally important. Failure of either of these may lead to dire consequences.

Throughout our study of mathematics we shall have numerous occasions to use PFI. For the sake of illustration, let us consider the following :

Example 1. *Show that the number of subsets of a set consisting of n elements is 2^n.*

Solution. Here T(n) is the statement that 'if S is a set having n elements, then P(S) has 2^n elements'.

Step 1. T (1) is true.

In fact, if $S=\{a\}$, then $P(S)=\{\phi, \{a\}\}$, so that if S is a set having one element, then P(S) has 2^1 elements.

Step 2. Assume that T (k) is true, *i.e.*, assume that every set consisting of k members has 2^k subsets.

Consider now a set S consisting of $k+1$ elements. Let

$$S=\{a_1, a_2, \cdots, a_{k+1}\}.$$

For each subset F of S, exactly one of the following is true :

Either $a_{k+1} \in F$ or $a_{k+1} \notin F$.

Now the collection of all those subsets of S that do not contain a_{k+1} is precisely $P(S^*)$, where

$$S^*=\{a_1, a_2, \ldots\ldots, a_k).$$

Since S^* contains k elements, therefore, by our hypothesis $P(S')$ consists of 2^k elements.

In other words, there are exactly 2^k subsets of S none of which contains the element a_{k+1}.

Again, each subset G of S containing a_{k+1} can be obtained from one subset F of S^* by adding a_{k+1} to it (and all sets G thus obtained obviously satisfy the property that $G \in P(S)$ and $a_{k+1} \in G$), and therefore, there are exactly 2^k subsets of S each of which contains a_{k+1}.

The total number of subsets of S is, therefore, 2^k+2^k, *i.e.*, 2^{k+1}, so that T($k+1$) is true.

The proof is now complete by PFI.

Example 2. *Let T(k) be the statement*

$$1+3+5+\ldots+(2k-1)=k^2+10.$$

Show that T(k) is true ⇒ T(k+1) is true. Can we conclude by applying PFI that T(n) is true for all n ?

Solution. Assume that T(k) is true.

Then
$$\begin{aligned} 1+3+\ldots+(2k-1)+(2k+1), \\ &=(k^2+10)+(2k+1), \\ &=(k+1)^2+10,\end{aligned}$$

so that T($k+1$) is true.

We have proved that for the given statement, the truth of $T(k)$ implies that of $T(k+1)$. But what about $T(1)$? Obviously, $T(1)$ is not true and this leads to the breakdown of a possible proof by PFI.

Conclusion : $T(n)$ is not true for all n.

Example 3. *Find the fallacy in the following 'proof' by PFI of the statement 'all numbers in a set of n natural numbers are equal'.*

Proof. Denote the statement in question by $T(n)$,

Step 1. $T(1)$ is clearly true.

Step 2. Let us assume that $T(k)$ is true and let

$$\{a_1, a_2, \cdots, a_{k+1}\}$$

be an arbitrary set consisting of $k+1$ natural nnmbers.

By our hypothesis, all members of the set $a_1, a_2, \cdots, a_k$ consisting of k elements are equal,

i.e., $$a_1=a_2=\cdots=a_k,$$

and similarly all elements of the set $\{a_2, a_3 \cdots a_{k+1}\}$ consisting of k elements are equal,

i.e., $$a_2=a_3=\ldots=a_{k+1}.$$

By the transitivity property of equality it follows that

$$a_1=a_2=\ldots=a_k=a_{k+1},$$

and consequently $T(k+1)$ is true.

By PFI, $T(n)$ is true for all n.

Argument. Proof of step 2 is fallacious in as much as it is not valid for $k=1, 2$, as in these cases transitivity of equality is not applicable.

Example 4. *Use mathematical induction to prove that $2 \cdot 7^n + 3.5^n - 5$ is divisible by 24 for all $n>0$.* *(I.I.T., J.E.E. 1985)*

Solution. Let $T(n)$ be the statement :

$$f(n)=2.7^n+3 \cdot 5^n-5 \text{ is divisible by 24 for } n>0.$$

Step 1. $f(1)=2.7+3.5-5=24$,

which is divisible by 24.

Therefore, $T(1)$ is true.

Step 2. Let us assume that $T(k)$ is true.

$$f(k+1)=2.7^{k+1}+3.5^{k+1}-5,$$
$$f(k)=2.7^k+3.5^k-5,$$
$$f(k+1)-f(k)=2(7^{k+1}-7^k)+3(5^{k+1}-5^k),$$
$$=12.7^k+12.5^k,$$
$$=12(7^k+5^k), \text{ which is divisible by 24 since } 7^k+5^k \text{ is always even.}$$

Now $f(k+1)-f(k)$ is a multiple of 24, and $f(k)$ is also a multiple of 24. It follows that $f(k+1)$ is a multiple of 24, and consequently $T(k+1)$ is true.

By PFI, $T(n)$ is true for all n.

EXERCISE 2(*a*)

Prove that the following statements are true for all $n \in N$:

1. $1+2+3+\dots+n=\frac{1}{2}n(n+1)$.
2. $1^2+2^2+3^2+\dots+n^2=\frac{1}{6}n(n+1)(2n+1)$.
3. $1^3+2^3+3^3+\dots+n^3=\frac{1}{4}n^2(n+1)^2$.
4. $\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\cdots\left(1+\frac{2n+1}{n^2}\right)=(n+1)^2$.
5. $1+2.2+3.2^2+\dots+n.2^{n-1}=1+(n-1)2^n$.
6. $3^{3n}<2^{5n}$.
7. n^5-n is divisible by 30.
8. $2^{6n}+3^{2n-2}$ is divisible by 5.
9. $2^{3n}-1$ is divisible by 7.
10. $n(n^2+5)$ is divisible by 6.
11. $5^{2n+2}-24n-25$ is divisible by 576.
12. $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\dots+\frac{1}{n(n+1)}<1$.
13. If $x>-1$, then $(1+x)^n \geqslant 1+nx$.
14. If $a_1, a_2, \cdots, a_n$ are any real numbers, then
$|a_1+a_2+\dots+a_n| \leqslant |a_1|+|a_2|+\dots+|a_n|$.
15. A length $\sqrt{n}$ units can be constructed by means of a ruler and a compass.
16. $3^{2n}-1$ is divisible by 8.
17. $1+\frac{1}{2^2}+\frac{1}{3^2}+\dots+\frac{1}{n^2} \leqslant 2-\frac{1}{n}$.
18. Use PFI to prove that if the sum of the digits of a natural number is divisible by 9, so is the number.
19. If $1+2+\dots+n=\frac{1}{8}(2n+1)^2$ is true for some natural number $n=k$, then show that it is also true for $n=k+1$. Can we conclude that the statement is true for all n ?

20. Determine whether the following statements are true for all natural numbers n:
(a) n^2-n+41 is a prime.
(b) n^2+n+17 is a prime.
(c) $2n^2+29$ is a prime.

2·3. A REMARK ABOUT THE PRINCIPLE OF FINITE INDUCTION

The principle of finite induction can also be stated in the following equivalent form:

Let $\{T(n) : n \in \mathbf{N}\}$ be a set of statements, one for each natural number n. If (i) $T(l)$ is true, and (ii) if for each natural number k, the truth of $T(m)$ for all $m<k$ implies the truth of $T(k)$, then $T(n)$ is true for all n.

Two important consequences of this form are the following theorems which we state without proof.

Theorem 1. (Generalised Associative Law for Addition). *Any sum $a_1+a_2+...+a_n$ of natural numbers is independent of the position of parentheses.*

Theorem 2. (Generalised Associative Law for Multiplication). *Any product of natural numbers $a_1, a_2, ..., a_n$ is independent of the position of parentheses.*

Example 5. *The number $(2+\sqrt{3})^n+(2-\sqrt{3})^n$ is an integer for each natural number n.*

Solution. Let T(n) stand for the proposition that '$(2+\sqrt{3})^n+(2-\sqrt{3})^n$ is an integer'.

Step 1. T(1) is true.

Step 2. Let T(m) be true for all $m<k$.

Then $a^k+b^k=(a^{k-1}+b^{k-1})(a+b)-ab(a^{k-2}+b^{k-2})$,
where $a=2+\sqrt{3},\ b=2-\sqrt{3}$.

Now $a+b$, ab are integers. By our hypothesis, $a^{k-1}+b^{k-1}$ and $a^{k-2}+b^{k-2}$ are also integers. This implies that a^k+b^k is an integer. Thus by PFI, $(2+\sqrt{3})^n+(2-\sqrt{3})^n$ is an integer for all $n \in \mathbf{N}$.

Example 6. *Let $a_n=a_{n-1}+a_{n-2}$ for all $n>2$, and $a_1=1$, $a_2=1$. Then $a_n<\left(\frac{7}{4}\right)^n$ for all $n \in \mathbf{N}$.*

Solution. Let T (n) be the statement that $a_n<\left(\frac{7}{4}\right)^n$.

Step 1. Since $a_k=1<\left(\frac{7}{4}\right)^1$, therefore, T(1) is true.

Step 2. Let T(n) be true for all $n<k$.

Now,
$$a_1=a_{k-1}+a_{k-2},$$
$$<\left(\frac{7}{4}\right)^{k-1}+\left(\frac{7}{4}\right)^{k-2},$$
$$=\left(\frac{7}{4}\right)^{k-2}\left(\frac{7}{4}+1\right).$$

$$=\left(\frac{7}{4}\right)^{k-2}\frac{11}{4},$$
$$<\left(\frac{7}{4}\right)^{k-2}\left(\frac{49}{16}\right),$$
$$=\left(\frac{7}{4}\right)^{k}.$$

By PFI, $a_n<\left(\frac{7}{4}\right)^n$ for all $n\in\mathbf{N}$.

Example 7. *Let $R=(5\sqrt{5}+11)^{2n+1}$ and $f=R-[R]$, where [ ] denotes the greatest integer function. Prove that $Rf=4^{2n+1}$.*

(*I.I.T., J.E.E. 1988*)

Solution. Here [R] stands for the integral part of R, and f is the fractional part of R.

We shall show that the fractional part of R is $(5\sqrt{5}-11)^{2n+1}$. In order to do so, we must prove that

(*i*) $(5\sqrt{5}+11)^{2n+1}-(5\sqrt{5}-11)^{2n+1}$ is an integer for all n.

(*ii*) $(5\sqrt{5}-11)^{2n+1}$ lies between 0 and 1 for all n.

Let us denote both the above statements by T(n).

Step 1. Since $(5\sqrt{5}+11)-(5\sqrt{5}-11)=22$, which is a positive integer,

and $$0<(5\sqrt{5}-11)<1,$$

therefore, T(0) is true.

Step 2. Assuming that T(n) is true for all $n<k$, we shall show that T(k) is true.

Consider the identity

$$a^{2k+1}-b^{2k+1}=(a^{2k-1}-b^{2k-1})(a^2+b^2)-a^2b^2(a^{2k-3}-b^{2k-3}).$$

If we take $a=5\sqrt{5}+11$, $b=5\sqrt{5}-11$, we find that by the induction hypothesis $a^{2k-1}-b^{2k-1}$ and $a^{2k-3}-b^{2k-3}$ are integers. Also, $a^2+b^2=492$, $a^2b^2=16$ are integers. Consequently $a^{2k+1}-b^{2k+1}$ is an integer. Since $a^{2k+1}>1$, and $b^{2k+1}<1$ for all k, therefore, $a^{2k+1}-b^{2k+1}$ is a positive integer.

Hence T(k) is true. Consequently f, the fractional part of R, is $(5\sqrt{5}-11)^{2n+1}$.

$$Rf=(\sqrt{55}+11)^{2n+1}(5\sqrt{5}-11)^{2n+1},$$
$$=\{(5\sqrt{5}+11)(5\sqrt{5}-11)\}^{2n+1},$$
$$=4^{2n+1}.$$

Remark. Note that in the above example induction starts with $n=0$.

EXERCISE 2(*b*)

1. Prove that for all natural numbers n, $(3+\sqrt{5})^n+(3-\sqrt{5})^n$ is divisible by 2^n.
2. Prove that the integer next greater than $(\sqrt{7}+\sqrt{3})^{2n}$ is divisible by 2^{2n}.

3. If $a_1, a_2, a_3,$............are natural numbers such that $a_1=6$, $a_2=9$, and

$$a_n=3a_{n-1}+18a_{n-2} \text{ for } n>2$$

show that for all natural numbers n, a_n is divisible by 3^n.

2·4. ANOTHER REMARK ABOUT THE PRINCIPLE OF FINITE INDUCTION

The principle of finite induction can also be stated as follows :

Let $\{T(n) : n \in \mathbf{N}\}$ be a set of statements, one for each natural number n. If (i) $T(a)$ is true for some natural number a, and (ii) $T(k)$ is true implies $T(k+1)$ is true for all $k \geqslant a$, then $T(n)$ is true for all $n \geqslant a$.

We now give an example to illustrate its application.

Example 8. *Show that for each natural number $n>2$, $3^n > 3n+10$.*

Solution. For $n \in \mathbf{N}$, let $T(n)$ be the statement that '$3^n > 3n+10$'.

Step 1. Since $3^3=27=3.3+10+8,$
$$> 3.3+10,$$
therefore, T(3) is true.

Step 2. Let $T(k)$ be true and $k>2$.

Then $$3^{k+1}=3\,(3^k) > 3\,(3k+10),$$
$$=9k+30,$$
$$=3(k+1)+10+6k+17,$$
$$>3\,(k+1)+10,$$

Therefore, $T(k+1)$ is true.

By PFI, the result is true for $n \geqslant 3 > 2$.

EXERCISE 2(c)

1. Show that for all natural numbers $n \geqslant 3$, $2^n > 2n+1$.
2. Show that for all natural numbers $n>1$,
$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+...+\frac{1}{\sqrt{n}} > \sqrt{n}.$$
3. Show that for all natural numbers $n \geqslant 5$, $2^n > n^2$.
4. Show that for all natural numbers $n \geqslant 4$, $2^n < 1.2.3...n$.
5. If $n \geqslant 2$, prove that
$$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)...\left(1-\frac{1}{n}\right)=\frac{(n+1)}{2^n}.$$
6. A diary costs Rs 3 and a pen costs Rs 7. Show by using the principle of induction that any amount (in exact rupees) exceeding Rs 11 can be spent in buying diaries and pens.

2·5. RECURSIVE DEFINITIONS

An important application of the axiom of induction is in recursive definitions. Suppose a is a given natural number. What do we mean by the symbol a^n, n being a natural number ? The answer is 'a multiplied by itself n times'. But there is a difficulty.

Multiplication is a binary operation. How to overcome this difficulty ? This is done by setting $a^1=a$, $a^{n+1}=a^n\,a$. By PFI, a^n is thus defined for each $n \in \mathbf{N}$. Such definitions as this one are called recursive (or inductive).

We shall come across many examples of recursive definitions throughout the book.

REVIEW EXERCISE II

1. Use the principle of mathematical induction to prove that $1+4+7+\ldots+(3n-2)=\frac{1}{2}n\,(3n-1)$. *(A.I.S.S.E., 1987)*
2. Apply the principle of mathematical induction to prove that for all natural numbers n,
$$1.3+3.5+5.7+\cdots+(2n-1)(2n+1)=\frac{1}{3}n(4n^2+6n-1).$$
(A.I.S.S.E., 1985)
3. Apply the principle of mathematical induction to prove that for all natural numbers n, $10^{2n-1}+1$ is divisible by 11. *(A.I.S.S.E., 1986)*
4. Prove that $x(x^{n-1}-na^{n-1})+a^n(n-1)$ is divisible by $(x-a)^2$ for all positive integers $n > 1$. *(I.I.T., J.E.E., 1977)*
5. Use mathematical induction to prove that if n is any odd positive integer, then $n\,(n^2-1)$ is divisible by 24.
6. Let $u_1=1$, $u_2=1$, and $u_{n+2}=u_{n+1}+u_n$ for $n > 1$. Use mathematical induction to show that
$$u_n=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right], \text{ for all } n \geqslant 1.$$
(I.I.T., J.E.E., 1981)
7. Use mathematical induction to prove that for all natural numbers n, $11^{n+2}+12^{2n+1}$ is divisible by 133. *(Roorkee Entrance, 1982)*
8. If p be a natural number, then prove that $p^{n+1}+(p+1)^{2n-1}$ is divisible by p^2+p+1 for every positive integer n. *(I.I.T., J.E.E., 1984)*
9. Apply mathematical induction to prove that for all natural numbers n, $7^{2n}+2^{3n-3}, 3^{n-1}$ is divisible by 25. *(I.I.T., J.E.E., 1982)*

Prove each of the following by applying the principle of finite induction :

10. $1.2+2.3+3.4+\cdots\cdots+n(n+1)=\frac{1}{3}n(n+1)(n+2)$.
11. $1.2.3+2.3.4.+3.4.5+\ldots+n(n+1)(n+2)=\frac{1}{4}n(n+1)(n+2)(n+3)$.
12. $\dfrac{1}{1.3}+\dfrac{1}{3.5}+\cdots\cdots+\dfrac{1}{(2n-1)(2n+1)}=\dfrac{n}{2n+1}$.
13. $\dfrac{1^2}{1.3}+\dfrac{2^2}{3.5}+\dfrac{3^2}{5.7}+\cdots\cdots+\dfrac{n^2}{(2n-1)(2n+1)}=\dfrac{n(n+1)}{2(2n+1)}$.

14. $\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots\ldots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{3n+1}$.

15. $\frac{1}{1.5}+\frac{1}{5.9}+\frac{1}{9.13}+\ldots\ldots+\frac{1}{(4n-3)(4n+1)}=\frac{n}{4n+1}$.

16. n distinct straight lines in a plane, all passing through one point, divide the plane into $2n$ parts.

17. $\frac{1}{n+1}+\frac{1}{n+2}+\ldots\ldots+\frac{1}{2n}>\frac{13}{24}$, for all natural numbers $n>1$.

*18. $\frac{4^n}{n+1}<\frac{(2n)!}{(n!)^2}$.

**19. $\sin x+2\sin 2x+3\sin 3x+\ldots\ldots+n\sin nx$.

$$=\frac{(n+1)\sin nx-n\sin(n+1)x}{4\sin^2(x/2)}.$$

**20. $\frac{1}{2}+\cos x+\cos 2x+\ldots\ldots+\cos nx=\frac{1}{2}\sin\left(n+\frac{1}{2}x\right)$

$\cos c\left(\frac{1}{2}x\right)$.

SUMMARY

1. Let $\{T(n) : n \in N\}$ be a set of statements, one for each natural number n. If (*i*) T(1) is true and (*ii*) the truth of T(k) implies that of T($k+1$), then T (n) is true for all n.
2. Let $\{T(n) : n \in N\}$ be a set of statements, one for each natural number n. If (*i*) T(1) is true and (*ii*) if for each natural number k, the truth of T (m) for all $m < k$ implies the truth of T (k), then T(n) is true for all n.
3. Let $\{T(n) : n \in N\}$ be a set of statements, one for each natural number n. If (i) T(a) is true for some natural number a, and (*ii*) T (k) is true implies T ($k+1$) is true for all $k \geqslant a$, then T(n) is true for all n $\geqslant a$.

HISTORICAL NOTE

The discovery of the principle of mathematical induction is generally attributed to the French mathematician *Blaise Pascal* (1623-1662). However, the principle had been used earlier by the Italian mathematician *Francesco Maurolycus* (1494-1575) in his writings. The writings of *Bhaskaracharya* (1150 A.D.) also lead us to believe that he knew of this principle.

The first one to use the name 'induction' was the English mathematician *John Wallis* (1616-1703). Later on the Swiss mathematician *James Bernoulli* (1655-1705) used the principle to provide a proof of the Binomial Theorem about wh ch we shall learn later in this book. He did not, however, use the name induction.

The name 'mathematical induction' in the modern sense was used by the Engllsh mathemat cian, *Augustus De Morgan* (1806-1871) in his article '*Induction*' (Mathematics) in Penny Cyclopedia, London, 1838. The name gained immediate acceptance by the mathematical community, and during the next fifty years or so it was universally accepted.

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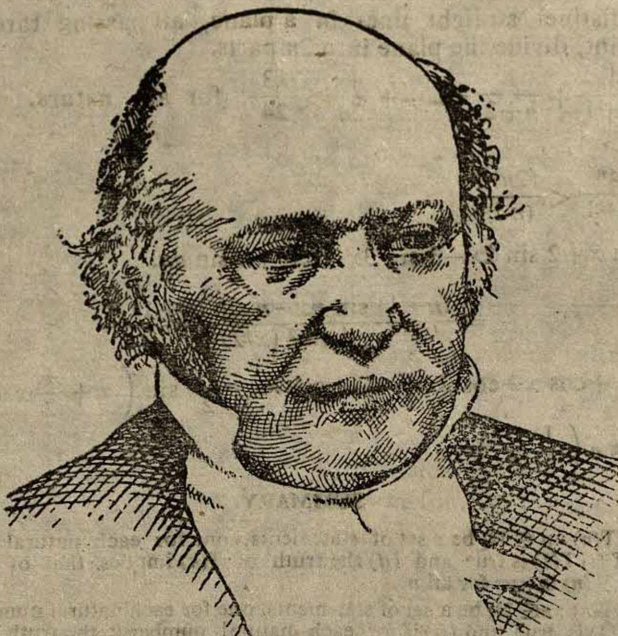
*To be attempted after studying Chapter 6.
**To be attempted after studying Chapter 13.



SIR WILLIAM ROWAN HAMILTON (1805-1865)

Sir William Rowan Hamilton was born at midnight between the 3rd and 4th of August, 1805. He showed promise of extraordinary brilliance even as a child. At the age of four he could read Greek, Hebrew and Latin. By the age of ten, he had learnt French, Italian, Persian and several other languages. At the age of 22 he became professor at Trinity College, Dublin. During the years 1824-1832 he made important contributions to the theory of optics. In 1833 he developed the notion of a complex number as an ordinary pair of real numbers. In 1835 he was elected President of the Royal Irish Academy. October 16, 1843 was the greatest day of his life, for it was on this day that he discovered the Quaternions. The next twenty years of his life were devoted to the study of Quaternions on which he wrote two monumental books.

In 1865 he became a victim of a severe attack of gout which lasted for some months until his death on the 2nd September, 1865.

CHAPTER 3

Complex Numbers

3·1. INTRODUCTION

Historically, imaginary and complex numbers arose as a result of the efforts to solve algebraic equations like

$$x^2+1=0,$$

which have no real roots.

Leonhard Euler was the first to use the symbol i, for $\sqrt{-1}$, having the property

$$i^2=-1.$$

He was thus able to discover the hitherto unknown roots of the equation

$$x^2+1=0.$$

This discovery of Euler is an important landmark in the history of mathematical progress, for it enabled the number system to be extended. Euler called the symbol i imaginary, and the numbers developed before the creation of the symbol i, came to be known as real numbers.

As a sequel to the introduction of the symbol i, the symbol $a+bi$ (where a and b are any real numbers) also came into being. This symbol was called a *complex number*. A systematic use of the symbols i and $a+bi$ led to quite a number of interesting results. It was found that many results which can be stated and demonstrated by the sole use of real numbers could be established much more easily by using complex numbers, and this provided a justification for their use.

For a long time, real and complex numbers were used without providing a logically sound basis. The theory of complex numbers was put on a solid foundation by Sir William Rowan Hamilton (1805-65) and by K.F. Gauss only after the superstructure had been raised.

3·2. COMPLEX NUMBERS IN THE FORM a+bi

Since the square of every real number is non-negative, therefore, there does not exist any real number whose square is -1. More

generally, there is no real number whose square is $-m^2$ (where m is a real number other than zero). In other words, the equation

$$x^2=-m^2, \text{ (m a non-zero real number)}$$

has no solution in **R**. In order to be able to work with such equation we need another set of numbers, namely the set of numbers whose squares are negative real numbers. Elements of this set are called *imaginary* numbers. For example, $\sqrt{-1}$, $\sqrt{-5}$, $\sqrt{-17}$ are imaginary numbers. A typical element of this set is $\sqrt{-m^2}$, where m is real.

Since $\sqrt{-m^2}=\sqrt{m^2(-1)}=\sqrt{m^2}\sqrt{-1}=mi$, where $i=\sqrt{-1}$, therefore, every imaginary number can be written in the form mi, where m is a real number and $i=\sqrt{-1}$.

If mi and ni are two imaginary numbers, then $mi+ni$, $mi-ni$ are either imaginary numbers or zero. Also, $(mi)(ni)=mni^2=-mn$, so that the product of two imaginary numbers is real, Similarly the quotient of two imaginary numbers is a real number.

Illustrations :

$$7i+5i \quad =12i,$$
$$7i+(-7)i=0,$$
$$8i-3i \quad =5i,$$
$$8i-(8i) \quad =0,$$
$$(7i).(5i) \quad =35i^2=-35,$$
$$(7i)/(5i) \quad =7/5.$$

Powers of i can also be simplified by using the relation $i^2=-1$. For example, $i^3=i^2 \,.\, i=-i$, $i^4=(i^2)^2=(-1)^2=1$, $i^5=i^4 \,.\, i=i$, $i^6=i^4 \,.\, i^2=i^2=-1$, and so on.

When we add a real number and an imaginary number, the expression so formed as called a *complex number*, *e.g.*, $3+2i$, $7-5i$ are both complex numbers. The general form of a complex number is $a+bi$ or $(a+ib)$, where a and b can have any real value (including zero). If $a=0$, we get numbers of the form bi, *i.e.*, imaginary numbers. If $b=0$, we get numbers of the form a, *i.e.*, real numbers. Therefore, the set of complex numbers includes the set of real numbers as well as the set of imaginary numbers. We shall denote the set of complex numbers by **C**.

3·3. REPRESENTATION OF COMPLEX NUMBERS BY POINTS IN A PLANE

In 1797 a Danish surveyor, Casper Wessel, published an article describing how to interpret complex numbers geometrically, but his work remained unnoticed for quite some time. In 1806 a French accountant J.R. Argand re-discovered this geometric interpretation. A fuller treatment was given by Karl Friedrich Gauss in 1831.

The idea of representing a complex number by a point in the plane is a very simple one. Draw, in a plane Σ, a pair of rectangular axes OX, OY and choose a scale of measurement. With a complex number $z=x+iy$, associate a point Z whose co-ordinates relative to OX, OY and the chosen scale of measurement are (x, y). The point Z, associated with the complex number z is called the *representative* or *image point* or simply the *image* of z.

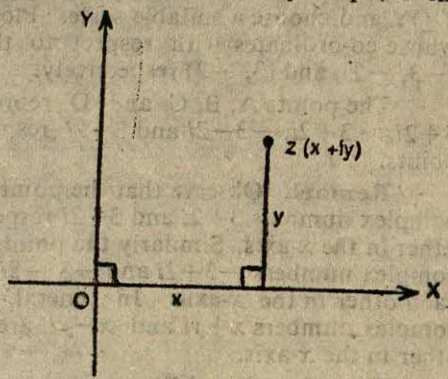


Fig. 3·1.

Every complex number z determines just one point Z in the manner stated above. Conversely each point W whose co-ordinates with respect to OX, OY are (u, v) in the complex plane is the image of a unique complex number $u+iv$. The complex number $u+iv$ is called the *affix* or *complex co-ordinate* of the point W.

There is, thus, a one-to-one correspondence between the set of all complex numbers and the set of all points in a plane. The representative plane is called the **Gauss plane** or the **Cauchy plane** or the **Argand plane** or the **z-plane**. To every set S of complex numbers there corresponds a set of image points. The graph of these points is called the *Argand diagram* for the set S.

Example 1. *Represent the complex numbers $3+2i, -3+2i, -3-2i$ and $3-2i$ on the Argand plane.*

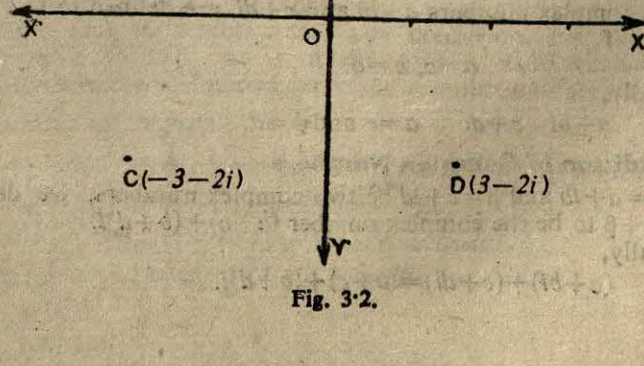


Fig. 3·2.

Solution. Draw a pair of perpendicular lines X'OX and Y'OY and choose a suitable scale. Plot the points A, B, C and D whose co-ordinates with respect to the axes are $(3, 2)$, $(-3, 2)$, $(-3, -2)$, and $(3, -2)$ respectively.

The points A, B, C and D represent the complex numbers $3+2i$, $-3+2i$, $-3-2i$ and $3-2i$ respectively, and are the desired points.

Remark. Observe that the point A and D representing the complex numbers $3+2i$ and $3-2i$ respectively are reflections of each other in the x-axis. Similarly the points B and C representing the complex numbers $-3+2i$ and $-3-2i$ respectively are reflections of each other in the x-axis. In general, the points representing the complex numbers $x+yi$ and $x-yi$ are always reflections of each other in the x-axis.

The points A and B representing the complex numbers $3+2i$ and $-3+2i$ are reflections of each other in the y-axis. Also, the points C and D representing the complex numbers $-3-2i$ and $3-2i$ are the reflections of each other in the y-axis. In general, the points representing the complex numbers $x+yi$ and $-x+yi$ are reflections of each other in the y-axis.

The points A and C representing the complex numbers $3+2i$ and $-3-2i$ are reflections of each other in the origin. Similarly B and D are reflections of each other in the origin. In general, the points representing the complex numbers $x+yi$ and $-x-yi$ are reflections of each other in the origin.

EXERCISE 3(*a*)

1. Represent the complex numbers $3+4i$, $3-4i$, $4+3i$ and $4-3i$ on the Argand plane.
2. The co-ordinates of a point are $(7, 5)$. What complex number does it represent ?
3. Given that $p=5$, $q=3$, represent the complex numbers $p+qi$, $q-pi$, $-p+qi$, $-q-pi$ on the Argand plane.

3·4. ALGEBRA OF COMPLEX NUMBERS

3·4·1. Equality of Complex Numbers

Two complex numbers $a+bi$ and $c+di$ are defined to be *equal* if and only if

$$a=c,\ b=d.$$

Symbolically,

$$a+bi=c+di \Leftrightarrow a=c \text{ and } b=d.$$

3·4·2. Addition of Complex Numbers

If $\alpha=a+ib$ and $\beta=c+id$ be two complex numbers, we define the sum $\alpha+\beta$ to be the complex number $(a+c)+(b+d)i$.
Symbolically,

$$(a+bi)+(c+di)=(a+c)+(b+d)i.$$

For example,

$$(3+4i)+(1-2i)=4+2i,$$
$$(2p+qi)+(p-3qi)=3p-2qi.$$

We shall now state and prove the four fundamental properties of addition on complex numbers.

A1. Addition of complex numbers is associative.

For, if $\alpha=a+bi$, $\beta=c+di$, $\gamma=e+fi$ be any three complex numbers, then

$$\begin{aligned}(\alpha+\beta)+\gamma &=[(a+bi)+(c+di)]+(e+fi),\\ &=[(a+c)+(b+d)\,i]+(e+fi),\\ &=(\overline{a+c}+e)+(\overline{b+d}+f)i,\\ &=(a+\overline{c+e})+(b+\overline{d+f})i,\\ &=(a+bi)+[(c+e)+(d+f)i],\\ &=(a+bi)+[(c+di)+(e+fi)],\\ &=\alpha+(\beta+\gamma).\end{aligned}$$

A2. Addition of complex numbers is commutative.

For, if $\alpha=a+bi$, $\beta=c+di$ be any two complex numbers, then

$$\begin{aligned}\alpha+\beta &=(a+bi)+(c+di),\\ &=(a+c)+(b+d)i,\\ &=(c+a)+(d+b)i,\\ &=(c+di)+(a+bi),\\ &=\beta+\alpha.\end{aligned}$$

A3. Existence of zero

The complex number $0+0i$ *is the zero (or identity) for addition*

For, if $\alpha=a+bi$ be any complex number, then

$$\begin{aligned}\alpha+(0+0i) &=(a+bi)+(0+0i),\\ &=(a+0)+(b+0)i,\\ &=a+bi,\\ &=\alpha.\end{aligned}$$

Notation. In future we shall denote the complex number $0+0i$ by 0. This need not create any confusion, for we shall always be able to find out from the context, as to whether the symbol 0 denotes the real number 0 or the complex number $0+0i$.

A4. Existence of Negatives

The complex number $(-a)+(-b)i$ *is the negative of* $a+bi$.

For, $\quad (a+bi)+((-a)+(-b)i)=0+0i.$

Notation. The negative of $a+bi$ is also written as

$$-(a+bi) \qquad \text{or} \qquad -a-bi.$$

3·4·3. Subtraction of Complex Numbers

If α, β be two complex numbers, then we have by definition,

$$\alpha-\beta=\alpha+(-\beta).$$

If $\alpha=a+bi,\ \beta=c+di,$

then
$$\begin{aligned}\alpha-\beta&=\alpha+(-\beta),\\&=(a+bi)+[(-c)+(-d)i],\\&=[a+(-c)]+[b+(-d)]i,\\&=(a-c)+(b-d)i.\end{aligned}$$

For example,

$$\begin{aligned}(3+4i)-(1-2i)&=(3-1)+(4+2)i,\\&=2+6i.\end{aligned}$$

3·4·4. Multiplication of Complex Numbers

If $\alpha=a+bi$ and $\beta=c+di$ be two complex numbers, we define the product $\alpha\beta$ to be the complex number $(ac-bd)+(ad+bc)\,i$. In symbols, we write

$$(a+bi)(c+di)=(ac-bd)+(ad+bc)i.$$

For example,

$$\begin{aligned}(3+2i)(-1+5i)&=(3(-1)-2.5)+(3.5+2(-1))i,\\&=-13+13i.\end{aligned}$$

We shall now prove the four fundamental properties of multiplication of complex numbers.

M1. Multiplication of complex numbers is associative.

For, if $\alpha=a+bi$, $\beta=c+di$, $\gamma=e+fi$ be any three complex numbers, then

$$\begin{aligned}(\alpha\beta)\,\gamma&=[(a+bi)(c+di)](e+fi),\\&=[(ac-bd)+(ad+bc)i]\,(e+fi),\\&=[(ac-bd)\,e-(ad+bc)\,f]+[(ac-bd)f+(ad+bc)\,e]i,\\&=[a(ce-df)-b(cf+de)]+[a(cf+de)+b(ce-df)]i,\\&=(a+bi)[(ce-df)+(cf+de)i],\\&=(a+bi)[(c+di)(e+fi)],\\&=\alpha(\beta\gamma).\end{aligned}$$

M2. Multiplication of complex numbers is commutative.

For, if $\alpha=a+bi$, $\beta=c+di$ be any two complex numbers, then

$$\begin{aligned}\alpha\beta&=(a+bi)(c+di),\\&=(ac-bd)+(ad+bc)i,\\&=(ca-db)+(da+cb)i,\\&=(c+di)(a+bi),\\&=\beta\alpha.\end{aligned}$$

M3. Existence of Unity

The complex number $1+0i$ is an identity for multiplication.

For, if $\alpha=a+bi$ be any complex number, then

$$\begin{aligned}\alpha(1+0i)&=(a+bi)(1+0i),\\&=(a.1-b.0)(a.0+b.1)i,\\&=a+bi=\alpha.\end{aligned}$$

The complex number $1+0i$ behaves in the same way for multiplication of complex numbers as the real number 1 does for multiplication of real numbers. In future we shall denote the complex number $1+0i$ by the symbol $\bar{1}$. As we shall see, this will not create any confusion.

M4. Existence of Inverses (or Reciprocals)

Every complex number other than $0+0i$ possesses an inverse with respect to multiplication.

We shall show that corresponding to any complex number $a+bi$ other than $0+0i$ (so that a and b are not both zero), there always exists a complex number $x+yi$ such that

$$(a+bi)(x+yi)=1+0i.$$

For, $(a+bi)(x+yi)=1+0i,$

$$\Rightarrow \quad (ax-by)+(ay+bx)i=1+0i,$$

$$\Rightarrow \left[\begin{array}{ll} ax-by=1 & \quad ...(1)\\ ay+bx=0. & \quad ...(2)\end{array}\right.$$

Solving (1) and (2) for x and y, we have

$$x=a/(a^2+b^2),\ y=-b/(a^2+b^2).$$

Thus every non-zero complex number possesses a multiplicative inverse. Denoting by α^{-1} the inverse of α, we have

$$(a+bi)^{-1}=\frac{a}{a^2+b^2}+\frac{-b}{a^2+b^2}\ i.$$

The multiplicative inverse of a complex number is also called its reciprocal.

So far we have established properties which related either to addition or to multiplication. We shall now establish a property which connects addition and multiplication.

3·4·4. Distributivity of Multiplication with Respcct to Addition

If α, β, γ be any three complex numbers, then

$$\alpha(\beta+\gamma)=\alpha\beta+\alpha\gamma.$$

Writing

$$\alpha=(a+bi),$$
$$\beta=(c+di),$$
$$\gamma=(e+fi),$$

we have

$$\begin{aligned}\alpha(\beta+\gamma)&=(a+bi)[(c+di)+(e+fi)],\\&=(a+bi)[(c+e)+(d+f)i],\\&=[a(c+e)-b(d+f)]+[a(d+f)+b(c+e)]i,\\&=[(ac-bd)+(ad+bc)i]+[(ae-bf)+(af+be)i],\\&=(a+bi)(c+di)+(a+bi)(e+fi),\\&=\alpha\beta+\alpha\gamma.\end{aligned}$$

3·4·5. Division of Complex Numbers

If α, β be any two complex numbers ($\beta\neq 0$), then by definition

$$\alpha\div\beta=\alpha\beta^{-1}.$$

The symbols $\dfrac{\alpha}{\beta}$ or α/β are also used for $\alpha\div\beta$.

3·4·6. Powers of Complex Numbers

Positive integral powers :

Let α be a complex number. We define α^n, where n is a positive integer, by the following rule :

$$\alpha^1=\alpha,$$
$$\alpha^n=\alpha.\alpha^{n-1}.$$

It can be easily proved that

$$\alpha^p\alpha^q=\alpha^{p+q},$$
$$(\alpha^p)^q=\alpha^{pq},$$

p, q being arbitrary positive integers.

Negative integral powers of a non-zero complex number :

If α be a non-zero complex number, and k be any positive integer, then we define α^{-k} as $(\alpha^k)^{-1}$. Also we agree to write $\alpha^0=1$. It can be shown that if α be any complex number other than zero, and p, q be any integers, then

$$\boxed{\alpha^{-p}=(\alpha^{-1})^p,\ \alpha^p.\alpha^q=\alpha^{p+q},\ (\alpha^p)^q=\alpha^{pq}.}$$

Example 2. *Given that $z_1=2+3i$, $z_2=4-5i$, express each of the numbers z_1+z_2, z_1-z_2, z_1z_2, z_1^2 in the form $a+bi$.*

Solution. Since $z_1=2+3i$, $z_2=4-5i$, therefore,

$$\begin{aligned}z_1+z_2&=(2+3i)+(4-5i),\\&=(2+4)+(3+(-5))i,\\&=6-2i.\end{aligned}$$

$$
\begin{aligned}
z_1 - z_2 &= (2+3i)-(4-5i),\\
&= (2-4)+(3+5)i,\\
&= -2+8i.\\
z_1 z_2 &= (2+3i)\,(4-5i),\\
&= (2.4-3(-5))+(2(-5)+3.4)i,\\
&= 23+2i.\\
z_1^2 &= (2+3i)^2,\\
&= 4+12i+9i^2,\\
&= 4+12i-9,\\
&= -5+12i.
\end{aligned}
$$

Remark. To find z_1z_2 we could as well multiply $2+3i$ and $4-5i$ as if they are binomials in i, and then put $i^2=-1$. You will find it more convenient than to directly apply the definition of the product of complex numbers. If we do so, we shall have

$$
\begin{aligned}
z_1 z_2 &= (2+3i)\,(4-5i),\\
&= 8+12i-10i-15i^2,\\
&= 8+2i+15, \text{ since } i^2=-1,\\
&= 23+2i.
\end{aligned}
$$

Example 3. *Find the inverse of $3+2i$.*

Solution. Let the inverse of $3+2i$ be $x+yi$.

Then $(3+2i)\,(x-yi)=1,$

i.e., $(3x-2y)+(2x+3y)i=1=1+0i.$ ...(1)

From (1), we have

$$
\begin{aligned}
3x-2y&=1\\
2x+3y&=0
\end{aligned} \qquad (...2)
$$

Solving the equations (2) for x and y, we have

$$x=\frac{3}{13},\ y=-\frac{2}{13}.$$

Therefore, the inverse of $3+2i$ is

$$\frac{3}{13}-\frac{2}{13}i.$$

EXERCISE 3 (*b*)

1. Add the following pairs of complex numbers :

   (*a*) $2-5i$ and $3+4i$ (*b*) $4+2i$ and $3-5i$

   (*c*) $2-7i$ and $4+9i$ (*d*) $a+bi$ and $2a-3bi$.

2. Subtract the second number from the first in each of the following pairs of complex numbers :

   (*a*) $3-4i$ and $2-i$ (*b*) $6-8i$ and $3+2i$

   (*c*) $1+3i$ and $2+5i$ (*d*) $1+4i$ and $3-6i$.

3. Simplify :

(*a*) $(2-3i)(3+4i)$ (*b*) $(5-4i)(7+i)$

(*c*) $(3-4i)(3+4i)$ (*d*) $(2+i)^2$

(*e*) $(2+i)^3$ (*f*) $(1+i)^3-(1-i)^3$.

4. Find the inverse of :

(*a*) $2-i$ (*b*) $1-3i$

(*c*) $3+4i$ (*d*) $5+12i$.

5. Given that $z_1=3-2i$ and $z_2=4+3i$, find z_1^{-1}, z_2^{-1}, z_1/z_2 and z_2/z_1.

6. Given that $z=1+i$, find z^{-1}, z^2, z^{-3}.

3·5. REAL AND IMAGINARY PARTS OF A COMPLEX NUMBER

If $z=x+iy$ be any complex number, then x, y respectively are called the *real* and *imaginary* parts of z. We write

$$x=\text{Re}(z), \quad y=\text{Im}(z).$$

Thus, for example, if $z=3-4i$, $\text{Re}(z)=3$, $\text{Im}(z)=-4$. A complex number z is said to be purely real if $\text{Im}(z)=0$, (*i.e.*, if its imaginary part is zero) and purely imaginary if $\text{Re}(z)=0$ (*i e.*, if its real part is zero). The purely real number $x+0i$ is written simply as x. In particular, the number $1+0i$ is denoted by 1 and the number $0+0i$ is written simply as 0. The purely imaginary number $0+yi$ is written simply as yi.

The complex number $x-iy$ is called the *complex-conjugate* (or simply the *conjugate*) of the complex number z. We use the symbol $\bar{z}$ to denote the complex conjugate of the complex number z. Thus, if $z=x+iy$, then $\bar{z}=x-iy$.

It is obvious that if $\bar{z}$ be the complex-conjugate of z, then z is the complex-conjugate of $\bar{z}$. The relation of conjugacy between complex numbers is a symmetric one. We may, therefore, say that z, $\bar{z}$ are conjugate complex numbers rather than saying that $\bar{z}$ is the complex-conjugate of z.

The following facts are worth noting :

I The sum of two conjugate complex numbers is a real number.

For, if $z=x+iy$, then $\bar{z}=x-iy$, so that

$$z+\bar{z}=2x=2\,\text{Re}(z).$$

II The difference of two conjugate complex numbers is, in general, purely imaginary.

For, if $z=x+yi$, then $\bar{z}=x-yi$,

$$z-\bar{z}=2yi=(2\,\text{Im}(z))i,$$

which is purely imaginary except when $\text{Im}(z)=0$.

III The product of two conjugate complex numbers is a non-negative real number.

For, if $z=x+yi$, then $\bar{z}=x-yi$, and

$$z\bar{z}=(x+yi)(x-yi)=x^2+y^2,$$

which is a non-negative real number.

The above property III is often used to express the quotient of two complex numbers as a complex number in the form $a+ib$, where a and b are real numbers. This is illustrated in the following example.

Example 4. *Express* $\dfrac{2-5i}{1+3i}$ *in the form* $p+qi$ *where* p, q *are real numbers.*

Solution. Multiplying the numerator and denominator of the given expression by the complex-conjugate of the denominator, *i.e.*, by $1-3i$, we have

$$\frac{2-5i}{1+3i}=\frac{(2-5i)(1-3i)}{(1+3i)(1-3i)},$$

$$=\frac{[2.1-(-5)(-3)]+[2(-3)+(-5).1]i}{1^2+3^2},$$

$$=\frac{-13+(-11)i}{10},$$

$$=-\frac{13}{10}+\left(-\frac{11}{10}\right)i.$$

Remark. The above process is often referred to as *realizing the denominator.*

EXERCISE 3(c)

1. Find the real and imaginary parts of :
   (a) $(3-i)(2+i)$ (b) $(3-2i)^2$.
2. Solve the following equations for x and y :
   (a) $x+yi=(2-i)(3+2i)$ (b) $x+yi=(1+i)(4-3i)$.
3. Realize the denominator of each of the following and hence express each in the form $p+qi$:
   (a) $\dfrac{3}{1+i}$ (b) $\dfrac{3-i}{4+3i}$
   (c) $\dfrac{3i}{4-i}$ (d) $\dfrac{1+i}{1-i}$.
4. Solve the following equations for x and y :
   (a) $3-4i=(x+yi)(1-i)$ (b) $\dfrac{2-5i}{1+i}=x+yi$.

5. Given that $z=1+3i$, express $z+\dfrac{2}{z}$ in the form $a+bi$, where a and b are real.
6. Given that $(1-5i)\,p-2q=3-7i$, find p and q where p, q are conjugate complex numbers.
7. If z_1, z_2 be any two complex numbers, prove that
   (a) $\text{Re}(z_1+z_2)=\text{Re}(z_1)+\text{Re}(z_2)$
   (b) $\text{Re}(z_1-z_2)=\text{Re}(z_1)-\text{Re}(z_2)$
   (c) $\text{Im}(z_1+z_2)=\text{Im}(z_1)+\text{Im}(z_2)$
   (d) $\text{Im}(z_1-z_2)=\text{Im}(z_1)-\text{Im}(z_2)$.
8. If z_1, z_2 be two complex numbers, show by suitable examples that
   (a) $\text{Re}(z_1\,z_2)$ and $\text{Re}(z_1)\,\text{Re}(z_2)$ need not be equal.
   (b) $\text{Im}(z_1\,z_2)$ and $\text{Im}(z_1)\,\text{Im}(z_2)$ need not be equal.

If z_1, z_2 be any two complex numbers, prove that

9. $\overline{z_1+\bar{z}_1}=\bar{z}_1+z_1$
10. $\overline{z_1-z_2}=\bar{z}_1-\bar{z}_2$
11. $\overline{\bar{z}_1\,\bar{z}_2}=z_1\,z_2$
12. $\overline{(z_1\div z_2)}=\bar{z}_1\div\bar{z}_2$, provided $z_2\neq 0$.

3·6. MODULUS AND ARGUMENT OF A COMPLEX NUMBER

Let $\alpha=a+bi$ be any complex number. Let r be a non-negative real number, and let θ be a real number such that

$$a+bi=r(\cos\theta+i\sin\theta),$$

so that
$$a=r\cos\theta, \quad \ldots(1)$$
$$b=r\sin\theta. \quad \ldots(2)$$

From (1) and (2), we have
$$r=\sqrt{a^2+b^2}. \quad \ldots(3)$$

As you will see later, θ can be chosen in infinitely many ways and any two values of the same differ from each other by a multiple of 2π.

The number r, which by (3) is the non-negative square root of a^2+b^2, is called the *modulus* of α and is denoted by $|\alpha|$.

Thus $\quad |\alpha|=|a+ib|=\sqrt{a^2+b^2}$

Also, θ is called an *argument* (or *amplitude*) of α. We write

$$\theta=\text{arg. }\alpha \text{ (or amp. } \alpha).$$

The value of θ, such that $-\pi<\theta\leqslant\pi$ is called the *principal value* of the argument and is generally denoted by

$$pr.\,arg.\ \alpha.$$

Section 3·6 and onwards should be studied after Chapter 13.

If there is no chance of confusion, pr arg. α is also called the argument of 'α'.

Example 5. *Find the modulus and principal argument of $1+i$.*

Solution. Let $1+i=r(\cos\theta+i\sin\theta)$. ...(1)

Comparing real and imaginary parts, we have from (1),

$$1=r\cos\theta, \quad ...(2)$$

$$1=r\sin\theta. \quad ...(3)$$

Squaring and adding (2) and (3), we have

$$2=r^2,$$

Therefore, $r=\sqrt{2}$. ...(4)

From (2), (3) and (4), we have

$$\cos\theta=1/\sqrt{2}, \sin\theta=1/\sqrt{2}. \quad ...(5)$$

The value of θ satisfying (5) and such that

$$-\pi<\theta\leqslant\pi \text{ is } \pi/4.$$

Thus the modulus and principal argument of $1+i$ are $\sqrt{2}$ and $\pi/4$ respectively.

Example 6. *Find the modulus and principal argument of*

$$\frac{12+12i}{7-(2-3i)^2}.$$

Solution. $\dfrac{12+12i}{7-(2-3i)^2}=\dfrac{12+12i}{7-(-5-12i)}$

$$=\frac{12+12i}{12+12i}=1.$$

Let $1=r(\cos\theta+i\sin\theta)$. ...(1)

Comparing real and imaginary parts on both sides, we have

$$1=r\cos\theta, \quad ...(2)$$

$$0=r\sin\theta. \quad ...(3)$$

From (2) and (3) we have $r=1$, $\theta=0$.

Hence the modulus is 1 and the principal argument is 0.

Example 7. *Find the modulus and principal argument of the complex number -6.*

Solution. Since $-6=6(\cos\pi+i\sin\pi)$, therefore, the modulus is 6 and principal argument is π.

Example 8. *Find the modulus and principal argument of the complex number $3i$.*

Solution. Writing $3i=3(\cos\pi/2+i\sin\pi/2)$, we find that the modulus is 3 and principal argument is $\pi/2$.

Example 9. *If z be any complex number, show that*

(*a*) $|z|=0 \Leftrightarrow z=0$ (*b*) $z\bar{z}=|z|^2$

(*c*) $-|z| \leqslant \text{Re}\,(z) < |z|$.

Solution. (*a*) Let $z=x+iy$.

Then $$|z|^2=x^2+y^2.$$

Now $|z|=0 \Leftrightarrow x^2+y^2=0 \Leftrightarrow x=0$ and $y=0 \Leftrightarrow z=0$.

(*b*) If
$$z=x+iy,$$
$$\bar{z}=x-iy,$$
$$z\bar{z}=(x+iy)\,(x-iy)=x^2+y^2=|z|^2.$$

(*c*) If
$$z=x+iy,$$
$$[\text{Re}\,(z)]^2=x^2 \leqslant x^2+y^2=|z|^2.$$

Since $[\text{Re}\,(z)]^2 \leqslant |z|^2$, we have
$$-|z| \leqslant \text{Re}\,(z) \leqslant |z|.$$

3·6·1. Two Important Properties of the Moduli

We shall now state and prove two important properties of the moduli of complex numbers, the first one of which relates to the modulus of the product of two complex numbers, and the second one relates to the modulus of the sum of two complex numbers.

Property I. *If* z_1, z_2 *be any two complex numbers, then*
$$|z_1z_2|^2=|z_1|\,|z_2|.$$

Proof.
$$|z_1z_2|^2=(z_1z_2)\overline{(z_1z_2)},$$
$$=(z_1z_2)\,(\bar{z}_1\bar{z}_2),$$
$$=(z_1\bar{z}_1)\,(z_2\bar{z}_2),$$
$$=|z_1|^2\,|z_2|^2,$$
$$=(|z_1|\,|z_2|)^2. \qquad ...(1)$$

Since the modulus of a complex number is non-negative, by taking the square roots of both sides of (1), it follows that

$$\boxed{|z_1z_2|=|z_1|\,|z_2|.}$$

The above property is often expressed in words by saying that *the modulus of the product of two complex numbers is equal to the product of the moduli of the complex numbers.*

Property I has an important corollary, which we state (and prove) below.

Corollary. *If* z_1, z_2 *be any two complex numbers and* $z_2 \neq 0$, then $|z_1/z_2|=|z_1|/|z_2|$.

Proof. Since $z_2 \neq 0$, we may write $z_1=(z_1/z_2).z_2$.

Since the modulus of the product of two complex numbers equals the product of their moduli, therefore,

$$|z_1| = |(z_1/z_2).z_2| = |z_1/z_2| \; |z_2|.$$

Dividing both sides by the non-zero real number $|z_2|$, we have

$$|z_1| / |z_2| = |z_1/z_2|,$$

i.e.,
$$|z_1/z_2| = |z_1| / |z_2|.$$

Property II. If z_1, z_2 be any two complex numbers, then

$$|z_1+z_2| \leqslant |z_1| + |z_2|.$$

Proof. $|z_1+z_2|^2 = (z_1+z_2)\overline{(z_1+z_2)}$,

$$= (z_1+z_2)(\bar{z}_1+\bar{z}_2),$$
$$= z_1\bar{z}_1+z_2\bar{z}_2+(z_1\bar{z}_2+\bar{z}_1z_2),$$
$$= |z_1|^2+|z_2|^2+(z_1\bar{z}_1+\bar{z}_1z_2). \qquad ...(1)$$

Since $z_1\bar{z}_2$ and $\bar{z}_1z_2$ are conjugate complex numbers, therefore,

$$z_1\bar{z}_2+\bar{z}_1z_2 = 2\,\mathrm{Re}(z_1\bar{z}_2) \leqslant 2\,|z_1\bar{z}_2|. \qquad ...(2)$$

From (1) and (2) we have,

$$|z_1+z_2|^2 \leqslant |z_1|^2+|z_2|^2+2\,|z_1\bar{z}_2|,$$

or
$$|z_1+z_2|^2 \leqslant |z_1|^2+|z_2|^2+2\,|z_1|\,|\bar{z}_2|,$$

or
$$|z_1+z_2|^2 \leqslant |z_1|^2+|z_2|^2+2\,|z_1|\,|z_2|, \qquad ...(3)$$

since $|\bar{z}_2| = |z_2|$.

Taking positive square roots of both sides of (3), we have

$$\boxed{|z_1+z_2| \leqslant |z_1| + |z_2|.}$$

The above property is often expressed in words by saying that the *modulus of the sum of two complex numbers is always less than or equal to the sum of theese moduli of these complex numbers.*

It is usually referred to as *the triangle inequality.* Property II has an important corollary which we state (and prove) below :

Corollary. *If* z_1, z_2 *be any complex numbers, then*

$$|z_1-z_2| \geqslant \big|\,|z_1|-|z_2|\,\big|.$$

Proof. $|z_1| = |(z_1-z_2)+z_2| \leqslant |z_1-z_2| + |z_2|$,

so that $|z_1| - |z_2| \leqslant |z_1-z_2|$. ...(1)

Interchanging z_1 and z_2 in (1), we have

$$|z_2| - |z_1| \leqslant |z_2-z_1|,$$

or
$$-(|z_1| - |z_2|) \leqslant |z_1-z_2|, \qquad ...(2)$$

since $|z_2-z_1| = |z_1-z_2|$.

From (1) and (2), we find that

$$|z_1-z_2| \geqslant |z_1|-|z_2|,$$

and

$$|z_1-z_2| \geqslant -(|z_1|-|z_2|).$$

Since the non-negative real number $|z_1-z_2|$ is either greater than or equal to $|z_1|-|z_2|$ and its negative $-(|z_1|-|z_2|)$, therefore,

$$|z_1-z_2| \geqslant \big|\, |z_1|-|z_2| \,\big|.$$

In the following examples we shall see that the above two properties I and II can be easily generalized to the case of n complex numbers.

Example 10. *Let $z_1, z_2, \ldots, z_n$ be any complex numbers. Show that*

(*a*) $|z_1+z_2+\cdots+z_n| \leqslant |z_1|+|z_2|+\cdots+|z_n|$

(*b*) $|z_1z_2\cdots z_n| = |z_1|\,|z_2|\cdots|z_n|$.

Solution. (*a*) We shall prove the result by the principle of finite induction.

Step 1. The result holds for $n=2$.

Step 2. Assume that the result holds for any k complex numbers $z_1, z_2, \ldots, z_k$, *i.e.*, assume that

$$|z_1+z_2+\cdots+z_k| \leqslant |z_1|+|z_2|+\cdots+|z_k|,$$

for any k complex numbers $z_1, z_2, \ldots, z_k$.

If $z_1, z_2, \ldots, z_{k+1}$ be any $k+1$ complex numbers, then

$$\begin{aligned} |z_1+z_2+\cdots+z_{k+1}| &= |(z_1+z_2+\cdots+z_k)+z_{k+1}| \\ &\leqslant |z_1+z_2+\cdots+z_k| + |z_{k+1}|, \\ &\leqslant (|z_1|+|z_2|+\cdots+|z_k|)+|z_{k+1}| \\ &= |z_1|+|z_2|+\cdots+|z_{k+1}|. \end{aligned}$$

Thus the result holds for $n=k+1$.

By FPI the proof is complete.

(*b*) We shall prove the result by the finite principle of induction.

Step 1. The result holds for $n=2$.

Step 2. Assume that the result holds for k complex numbers $z_1, z_2, \ldots, z_k$, *i.e.*, assume that

$$|z_1z_2\cdots z_k| = |z_1|\,|z_2|\cdots|z_k|$$

for any k complex numbers $z_1, z_2, \ldots, z_k$.

If $z_1, z_2, \ldots z_{k+1}$ be any $k+1$ complex numbers, then

$$\begin{aligned} |z_1z_2\cdots z_{k+1}| &= |z_1z_2\cdots z_k\cdot z_{k+1}|, \\ &= |z_1z_2\cdots z_k|\,|z_{k+1}|, \\ &= (|z_1|\,|z_2|\cdots|z_k|)\,|z_{k+1}|, \\ &= |z_1|\,|z_2|\cdots|z_{k+1}|. \end{aligned}$$

Thus the result holds for $n=k+1$. By PFI the proof is complete.

EXERCISE 3(*d*)

1. Find the modulus and the principle argument of each of the following complex numbers :
(*a*) $1-i$ (*b*) $-1+i$ (*c*) $-1-i$
(*d*) -3 (*e*) $2i$ (*f*) $-4i$.

2. Express in the form $r(\cos\theta+i\sin\theta)$:
(*a*) $\sqrt{3}+i$ (*b*) $1+\sqrt{3}i$ (*c*) $-1+\sqrt{3}\,i$
(*d*) $i(1+i)$ (*e*) $i^2(1-i)$ (*f*) $\dfrac{1-i}{1+i}$.

3. If $z_1=7-i$, $z_2=1+i$, find the modulus of
(*a*) z_1-z_2 (*b*) z_1z_2 (*c*) $\dfrac{z_1-z_2}{z_1z_2}$.

4. If $u=3+i$, $v=1-2i$, find the modulus of
(*a*) $2u+3v$ (*b*) $\dfrac{u}{2v}$.

5. Verify that
$$2 \text{ pr. arg. } (-1) \neq \text{pr. arg. } (-1)^2.$$

6. Verify that
pr. arg. $(1+i)$+pr. arg. $(1-i)$=pr. arg. $\{(1+i)(1-i)\}$.

3·7. SQUARE ROOTS OF A COMPLEX NUMBER

Can you name the real numbers whose squares are equal to 1 ? The answer is 1 and -1. The numbers 1 and -1 are both square roots of 1. Similarly, $\sqrt{2}$ and $-\sqrt{2}$ are both square roots of 2. Every real number has two square roots.

Just as every real number has two square roots, similarly every complex number also has two square roots.

The following example will illustrate the method of finding the square roots of a complex number.

Example 11. *Find the square roots of* $3-4\sqrt{7}i$.

Solution. Let $\sqrt{\{3-4\sqrt{7}i\}}=x+iy$, ...(1)

where x and y are real numbers.

Squaring both sides of (1), we have

$$3-4\sqrt{7}i=(x+iy)^2=x^2-y^2+2xyi. \qquad ...(2)$$

Equating real and imaginary parts on both sides of (2), we have

$$x^2-y^2=3, \qquad ...(3)$$

$$2xy=-4\sqrt{7}. \qquad ...(4)$$

From (3) and (4), we have

$$(x^2+y^2)^2=(x^2-y^2)^2+4x^2y^2,$$

$$=3^2+(-4\sqrt{7})^2=9+112=121, \quad \ldots(5)$$

or $$x^2+y^2=11. \quad \ldots(6)$$

(Observe that since x and y are real numbers, therefore, x^2+y^2 must be positive, and consequently we have taken only the positive sign while taking the square roots of both sides of (5)).

From (3) and (6), we have

$$x^2=7, \text{ so that } x=\pm\sqrt{7}. \quad \ldots(7)$$

When $x=\sqrt{7}$, from (4), we have $y=-2$. ...(8)

When $x=-\sqrt{7}$, from (4), we have $y=2$. ...(9)

Hence the required square roots are

$$\sqrt{7}-2i \text{ and } -\sqrt{7}+2i, \text{ i.e., } \pm(\sqrt{7}-2i).$$

EXERCISE 3(*e*)

Find the square roots of

1. $8+6i$
2. $6+8i$
3. $-8i$
4. $1+2\sqrt{6}i$
5. $4ab-2(a^2-b^2)i$
6. $13-20\sqrt{3}i$
7. $3+4\sqrt{7}i$
8. $x+i\sqrt{x^4+x^2+1}$.

3·8. CUBE ROOTS AND FOURTH ROOTS OF UNITY

We have seen that every complex number has two square roots. In particular, ± 1 are the two square roots of unity. These are only special cases of the general result that every complex number has n nth roots, *i.e.*, given a complex number w, there are n complex numbers (some of which may be equal !) the nth power of each of which equals w. When the given number is taken to be 1, we get the nth roots of unity.

Instead of considering the general case, we shall only consider the cases $n=3$ and $n=4$, *i.e.*, we shall restrict our attention to cube roots and fourth roots of unity.

3·8·1. Cube Roots of Unity

Let $z=u+vi$ be a cube root of unity, where u and v are real numbers.

Then $$z^3=(u+vi)^3=1,$$

or $$u^3+3u^2vi+3uv^2i^2+v^3i^3=1,$$

or $$(u^3-3uv^2)+(3u^2v-v^3)i=1.$$

Comparing real and imaginary parts on both sides, we have

$$u^3-3uv^2=1, \quad \ldots(1)$$

$$3u^2v-v^3=0 \quad \ldots(2)$$

From (2), we have three different possibilities :

$$v=0,\ v=\sqrt{3}u,\ v=-\sqrt{3}u.$$

Let us consider them one by one.

(*i*) Let $v=0$. From (1), we have $u^3=1$, *i.e.*, $u=1$.

(*ii*) Let $v=\sqrt{3}u$. From (1), we have $-8u^3=1$, so that

$$u=-\frac{1}{2},\ v=-\frac{\sqrt{3}}{2}.$$

(*iii*) Let $v=-\sqrt{3}u$. From (1), we have $-8u^3=1$, so that

$$u=-\frac{1}{2},\ v=\frac{\sqrt{3}}{2}.$$

From (*i*), (*ii*) and (*iii*) we find that the three values of $u+vi$ are

$$1,\ -\frac{1}{2}-\frac{\sqrt{3}}{2}i, \text{ and } -\frac{1}{2}+\frac{\sqrt{3}}{2}i.$$

Thus **the three cube roots of unity are**

$$1,\ -\frac{1}{2}+\frac{\sqrt{3}}{2}i,\ -\frac{1}{2}-\frac{\sqrt{3}}{3}i.$$

Observe that of the three cube roots of unity, one is purely real, and the remaining two are complex-conjugates of each other.

The two complex cube-roots of unity are related to each other in a very special way :

$$\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^2=\left(\frac{-1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}i\right)^2+2\left(-\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}i\right),$$

$$=\left(\frac{1}{4}-\frac{3}{4}\right)-\frac{\sqrt{3}}{2}i,$$

$$=-\frac{1}{2}-\frac{\sqrt{3}}{2}i, \qquad \ldots(1)$$

and

$$\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^2=\left(\frac{-1}{2}\right)^2+\left(\frac{-\sqrt{3}}{2}i\right)^2+2\left(-\frac{1}{2}\right)\left(\frac{-\sqrt{3}}{2}i\right),$$

$$=\left(\frac{1}{4}-\frac{3}{4}\right)+\frac{\sqrt{3}}{2}i,$$

$$=-\frac{1}{2}+\frac{\sqrt{3}}{2}i. \qquad \ldots(2)$$

From (1) and (2) we find that *each of the two complex cube roots of unity is the square of the other.* If we denote either of them by the greek letter ω, the other would then be denoted by ω^2. Henceforth we shall denote the cube roots of unity by $1, \omega, \omega^2$, where ω is any one of the complex numbers $-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$.

The cube roots of unity have the interesting property that their sum is zero. For, $1+\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)+\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)=0$. Thus we find that if $1, \omega, \omega^2$ be the three cube roots of unity, then

$$\boxed{1+\omega+\omega^2=0.}$$

3·8·2. Fourth Roots of Unity

Let $z=u+iv$ be a fourth root of unity, where u, v are real numbers.

Then $\quad (u+iv)^4=1,$

or $\quad [(u+iv)^2]^2=1,$

or $\quad [(u^2-v^2)+2uvi]^2=1,$

or $\quad [(u^2-v^2)^2-4u^2v^2]+4uv\,(u^2-v^2)i=1. \qquad ...(1)$

Equating real and imaginary parts on both sides of (1), we have

$$(u^2-v^2)^2-4u^2v^2=1, \qquad ...(2)$$

$$4uv(u^2-v^2)=0. \qquad ...(3)$$

From (3) we have $uv(u^2-v^2)=0$. Three different cases arise according as $u=0$, $v=0$, or $u^2-v^2=0$. Let us consider them one by one.

Case (i). $u=0$. From (2), we then have $v^4=1$, which gives $v^2=1$, since v^2 is non-negative. This, in turn gives $v=\pm 1$. Therefore, $u+iv=\pm i$.

Case (ii). $v=0$. From (2), we then have $u^4=1$, and consequently $u=\pm 1$. Therefore, $u+iv=\pm 1$.

Case (iii). $u^2-v^2=0$. From (2), we then have $-4u^2v^2=1$, which is not possible.

From cases (*i*)–(*iii*) above we have the four fourth roots of unity to be $\pm 1, \pm i$.

Remark. Since $i^2=-1$, $i^3=i^2.\, i=-i$, $i^4=(i^2)^2=(-1)^2=1$, therefore, the fourth roots of unity can be written as i, i^2, i^3 and i^4.

Example 12. *If $1, \omega, \omega^2$ be the three cube roots of unity, prove that*

$$(2-\omega)(2-\omega^2)(2-\omega^{10})(2-\omega^{11})=49.$$

Solution. L.H.S. $=(2-\omega)(2-\omega^2)(2-\omega^{10})(2-\omega^{11}),$

$=(2-\omega)(2-\omega^2)(2-\omega^9\,.\,\omega)(2-\omega^9\,.\,\omega^2).$

$=(2-\omega)(2-\omega^2)(2-\omega)(2-\omega^2)$, since $\omega^9=1$,

$=\{(2-\omega)(2-\omega^2)\}^2,$

$$=\{4-2(\omega+\omega^2)+\omega^3\}^2,$$
$$=(4+2+1)^2, \text{ since } \omega+\omega^2=-1,\ \omega^3=1,$$
$$=49.$$

EXERCISE 3(*f*)

If 1, ω, ω^2 *are the three cube roots of unity, prove that*

1. $(1-\omega+\omega^2)(1+\omega-\omega^2)=4$.
2. $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^5)=9$.
3. $(4-3\omega-3\omega^2)^3=343$.
4. $(x+y)(x+\omega y)(x+\omega^2 y)=x^3+y^3$.
5. $(x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)=x^3+y^3+z^3-3xyz$.
6. $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)\ldots$to 10 factors$=1$.
7. Prove that

$$\left[\frac{-1+\sqrt{(-3)}}{2}\right]^{10}+\left[\frac{-1-\sqrt{(-3)}}{2}\right]^{10}=-1.$$

8. Prove that

$$\left[\frac{-1+\sqrt{(-3)}}{2}\right]^{12}+\left[\frac{-1-\sqrt{(--3)}}{2}\right]^{12}=2.$$

9. Find the value of $(1+2\omega+3\omega^2)(3+2\omega+\omega^2)$.

3·9. GEOMETRY OF COMPLEX NUMBERS*

The properties of a set of complex numbers can often be studied easily by considering the corresponding Argand diagram. In the next few pages we propose to develop this technique for studying properties of complex numbers.

3·9·1. DISTANCE BETWEEN TWO POINTS

Let $z_1=x_1+iy_1$, and $z_2=x_2+iy_2$, be two complex numbers and let Z_1, Z_2 be the points on the Argand plane representing z, z_2 respectively.

Then

$$Z_1Z_2=\sqrt{\{(x_2-x_1)^2+(y_2-y_1)^2\}},$$
$$=|(x_2-x_1)+i(y_2-y_1)|,$$
$$=|z_2-z_1|.$$

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*This section may be omitted on first reading and may be studied after studying chapters IX and X.

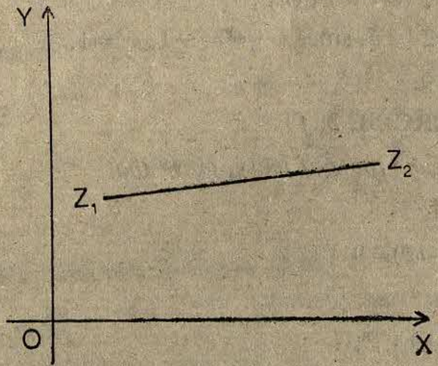

Fig. 3·3.

Thus, we find that if Z_1, Z_2 *represent the complex numbers* z_1, z_2 *respectively on the Argand plane, then* $Z_1Z_2 = |z_2 - z_1|$.

The above fundamental result has many important applications, one of which is contained in the following theorem :

Theorem 3·1. *Let* z_1, z_2, z_3 *be any three complex numbers such that their images A, B, C on the Argand plane are non-collinear, and let P, Q be the images of the complex numbers* $z_2 - z_1$, $z_3 - z_1$ *respectively on the Argand plane. The triangles ABC and OPQ are congruent.*

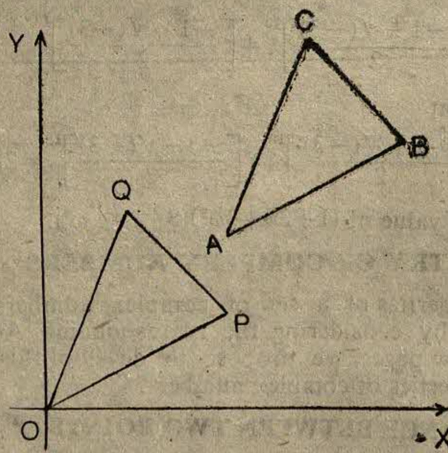

Fig. 3·4.

Proof.

$$\begin{aligned} OP &= |z_2 - z_1|, \\ &= AB, \\ OQ &= |z_3 - z_1|, \\ &= AC, \\ PQ &= |(z_3 - z_1) - (z_2 - z_1)|, \\ &= |z_3 - z_2|, \\ &= BC. \end{aligned}$$

Since OP=AB, PQ=BC, QO=CA. therefore, the triangles OPQ and ABC are congruent.

Example 13. *Prove that the points P, Q, R, S representing the complex numbers -1, $3i$, $3+2i$, and $2-i$ respectively on the Argand plane are the vertices of a square.*

Solution.

$$PQ = | 3i+1 | = \sqrt{10},$$
$$QR = | (3+2i)-3i | = | 3-i | = \sqrt{10},$$
$$RS = | (2-i)-(3+2i) | = | -1-3i | = \sqrt{10},$$
$$SP = | -1-(2-i) | = | -3+i | = \sqrt{10}.$$

Since $PQ=QR=RS=SP$, therefore, PQRS is a rhombus.

Also, $PR = | (3+2i)-(-1) | = | 4+2i | = \sqrt{20}$,

therefore, $PR^2 = PQ^2 + QR^2$. ...(1)

From (1) we find that $\angle PQR$ is a right angle.

Hence PQRS is a square.

EXERCISE 3(*g*)

1. The complex numbers $1+i$, $-4+4i$ and $4+6i$ are represented on the Argand plane by the points A, B and C respectively. Prove that the triangle ABC is isosceles.
2. Prove that the points representing the complex numbers $3+3i$, $-3-3i$, $-3\sqrt{3}+3\sqrt{3}i$ on the Argand plane are the vertices of an equilateral triangle.
3. Prove that the triangle formed by the points representing the complex numbers $10+8i$, $-2+4i$ and $-11+31i$ on the Argand plane is right-angled.
4. Prove that the points representing the complex numbers $3+7i$, $9+9i$, and $11+3i$ are the vertices of a right-angled isosceles triangle.
5. Prove that the points representing the complex numbers -1, $3+i$, $2+2i$, and $-2+i$ on the Argand plane are the vertices of a parallelogram.
6. Prove that the points representing the complex numbers $3+i$, $4+6i$, $-1+5i$ and -2 on the Argand plane are the vertices of a rhombus.
7. Show that the points representing the complex numbers $2i$, $1+i$, $4+4i$ and $3+5i$ on the Argand plane are the vertices of a rectangle.
8. Show that the points representing the complex numbers $5+3i$, $1+2i$, $2-2i$ and $6-i$ on the Argand plane are the vertices of a square.
9. Show that the points representing the complex numbers $3+4i$, $5-2i$ and $-1+16i$ on the Argand plane are collinear.

10. Show that the points representing the complex numbers $1+3i$, $5+i$ and $3+2i$ on the Argand plane are collinear.

3·9·2. POINT DIVIDING A LINE-SEGMENT IN A GIVEN RATIO

Let $$z_1=x_1+iy_1,$$
and $$z_2=x_2+iy_2,$$
be the affixes of the points Z_1, Z_2 respectively, in the Argand plane. If λ be a real number ($\neq -1$), then there is a unique point Z on Z_1Z_2 such that

$$Z_1Z : ZZ_2=\lambda : 1.$$

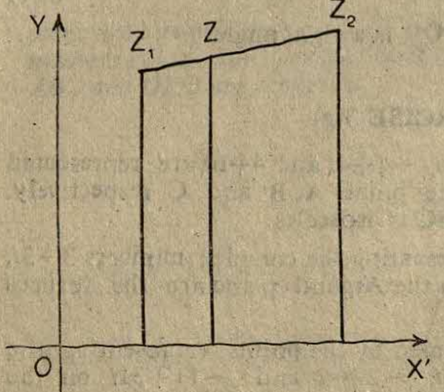

Fig. 3·5.

The co-ordinates of Z are
$$\left(\frac{x_1+\lambda x_2}{1+\lambda},\ \frac{y_1+\lambda y_2}{1+\lambda}\right).$$
The affix of Z is, therefore,
$$\frac{z_1+\lambda z_2}{1+\lambda}.$$
Remarks. 1. The affix of the mid-point of Z_1Z_2 is $\frac{1}{2}(z_1+z_2)$.

2. If z_1, z_2, z_3 be the affixes of the vertices of a triangle in the Argand plane, the centroid of the triangle has affix $\frac{1}{3}(z_1+z_2+z_3)$.

Example 14. *ABCD is a parallelogram on the Argand plane. The affixes of A, B, C are $8+5i$, $-7-5i$ and $-5+5i$ respectively. Find the affix of D.*

Solution. Let the affix of D be z.

The affix of the mid-point of AC is $\frac{1}{2}\{(8+5i)+(-5+5i)\}$, *i.e.*, $\frac{3}{2}+5i$.

The affix of the mid-point of BD is $\frac{1}{2}\{(-7-5i+z)\}$. Since the diagonals of a parallelogram bisect each other, therefore,
$$\tfrac{1}{2}\{(-7-5i)+z)\}=\tfrac{3}{2}+5i,$$
whence $$z=10+15i.$$

Hence the affix of D is $10+15i$.

EXERCISE 3(*h*)

1. The complex numbers $8+10i$ and $7+6i$ are represented by the points A and B respectively on the Argand plane. Find the complex number represented by the point dividing AB internally in the ratio 4 : 3.

2. The points A, B on the Argand plane represent the complex numbers $-7+8i$ and $4-3i$ respectively. If C divides the join of A and B externally in the ratio 5 : 7, what complex number does C represent ?
3. A and B are the points representing the complex numbers $1+2i$ and $-9+7i$. Find the affixes of the points P and Q which divide the line segment AB internally and externally in the ratio 2 : 3.
4. The complex numbers $-4+3i$, $5+12i$ are represented on the Argand plane by the points A and B respectively. If C and D divide AB internally and externally in the ratio 5 : 13, find the complex numbers represented by C and D.
5. The points A, B on the Argand plane represent the complex numbers $-5-5i$ and $25+10i$ respectively. Find the affixes of the points of trisection of the line segment AB.
6. The points A, B, C on the Argand plane represent the complex numbers $1-2i$, $4+7i$ and $-5-20i$ respectively. In what ratio does C divide the join of A and B ?
7. Show that the points representing the complex numbers $6-i$, $7+3i$, $8+2i$ and $7-2i$ on the Argand plane are the vertices of a parallelogram.
8. The complex numbers $8+5i$, $-3+i$, $-2-3i$ are represented on the Argand plane by the points A, B, C respectively. Find the modulus and argument of the complex number represented by the centroid of the triangle ABC.
9. The points A, B, C, represent the complex numbers z_1, z_2, z_3 respectively and G is the centroid of the triangle ABC. If $4z_1+z_2+z_3=0$, show that the origin is the mid-point of AG.
10. If the vertices of a triangle ABC represent z_1, z_2, z_3 respectively, show that the orthocentre represents
$(az_1 \sec A+bz_2 \sec B+cz_3 \sec C)/(a \sec A+b \sec B+c \sec C)$.
Determine also the complex numbers represented by the in-centre and the circumcentre.

3·9·3. GEOMETRICAL REPRESENTATION OF ADDITION IN C*

Let Z_1, Z_2, be the image points in the Argand plane of the complex numbers $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$ respectively.

If Z is the fourth vertex of the parallelogram of which OZ_1, OZ_2 are adjacent sides, then from elementary geometry we know that the co-ordinates of Z are (x_1+x_2, y_1+y_2). Hence the affix of Z is $(x_1+x_2)+i\,(y_1+y_2)$, *i.e.*, z_1+z_2. The above construction for the image point Z of the sum of the affixes of the points Z_1, Z_2 can also be stated in the following form :

*An elementary idea of vectors has been used in sections 3·9·3 and 3·9·4.

If the complex numbers z_1, z_2 have for images the points Z_1, Z_2, their sum

$$z=z_1+z_2$$

has for image the point Z such that the points O, Z_1, Z, Z_2 are the vertices of a parallelogram taken in order.

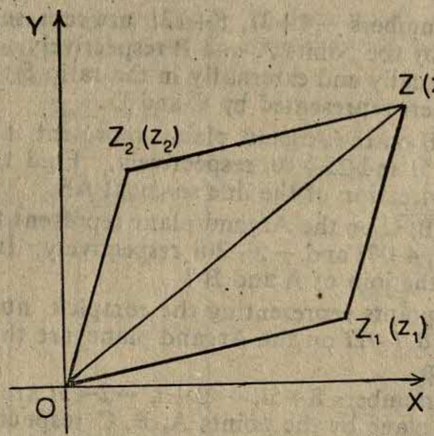

Fig. 3·6.

Remarks 1. Since the moduli of z_1, z_2, z_1+z_2 are given respectively by the lengths OZ_1, OZ_2 (which is equal in length to Z_1Z), OZ and since the length of OZ cannot exceed the sum of the lengths OZ_1, Z_1Z, it follows that

$$|z_1+z_2| \leqslant |z_1| + |z_2|.$$

Hence *the modulus of the sum of two complex numbers never exceeds the sum of their moduli.*

We have already established the above inequality by algebraic methods in section 3·6·1.

2. In the language of vectors, the above result can be stated thus :

If the complex numbers z_1, z_2 are representai by the vectors $\mathbf{OZ_1}$, $\mathbf{OZ_2}$ *respectively, the complex number*

$$z=z_1+z_2$$

is represented by the vector sum

$$\mathbf{OZ}=\mathbf{OZ_1}+\mathbf{OZ_2}$$

of the corresponding vectors.

3·9·4. GEOMETRICAL REPRESENTATION OF SUBTRACTION IN C

Let Z_1, Z_2, be the image points in the Argand plane of the complex numbers $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$ respectively. If Z_2O be produced to a point Z_3 such that O is the mid-point of Z_2Z_3, then Z_3 is symmetric to Z_2 with respect to the origin O. The affix of Z_3 is,

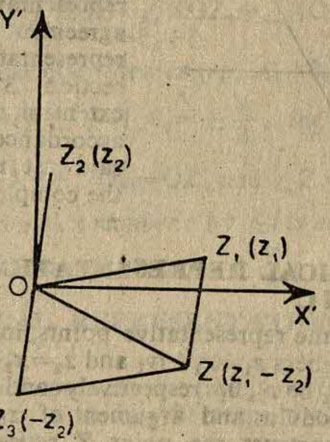

Fig. 3·7.

therefore, $-z_2$. (Alternatively, the co-ordinates of Z_3 are $(-x_2, -y_2)$ so that its affix is $(-x_2)+i(-y_2)$, *i.e.*, $-z_2$. If we complete the parallelogram $OZ_3\,Z\,Z_1$, Z will be the image point of the complex number $z_1+(-z_2)$, *i.e.*, of z_1-z_2.

Remarks 1. In the language of vectors, the above result can be stated thus :

If the complex numbers z_1, z_2 are represented by the vectors $\mathbf{OZ_1}$, $\mathbf{OZ_2}$ *respectively, the complex number*

$$z=z_1-z_2$$

is represented by the vector difference

$$\mathbf{OZ}=\mathbf{OZ_1}-\mathbf{OZ_2}$$

of the corresponding vectors.

2. If Z_1, Z_2, Z be the image points of z_1, z_2, z_1-z_2 respectively, then OZZ_1Z_2 is a parallelogram, so that Z_2Z_1 and OZ are parallel and in the same sense, and have equal lengths.

Therefore,

$$\mathbf{Z_2Z_1}=\mathbf{OZ}.$$

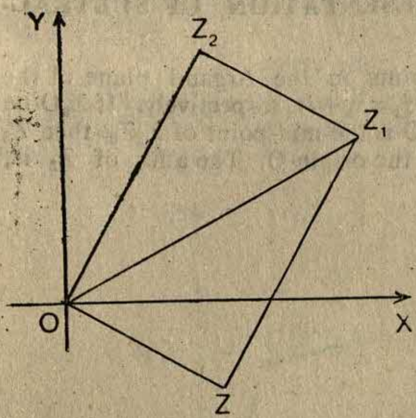

Fig .3·8·

We may, therefore, take the vector $\mathbf{Z_2Z_1}$ as representing the complex number $z_1 - z_2$ where z_1, z_2 are the affixes of the terminal and initial points respectively, of $\mathbf{Z_2Z_1}$.

It may be noted that this representation is in perfect agreement with the geometrical representation discussed in section 3·3 and is only an extension of the same. For, in accordance with what has been said just now, **OZ** represents the complex number $z-0$, *i.e.*, z.

3·9·5. GEOMETRICAL REPRESENTATION OF MULTIPLICATION IN C

Let Z_1, Z_2 be the representative points, in the Argand plane, of the complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. If the modulus and argument of z_1 be r_1, θ_1 respectively and those of z_2 be r_2, θ_2 respectively, the modulus and argument of z_1z_2 are r_1r_2 and $\theta_1+\theta_2$ respectively. To construct the point Z representing the complex number z_1z_2 we have to construct the point with polar co-ordinates

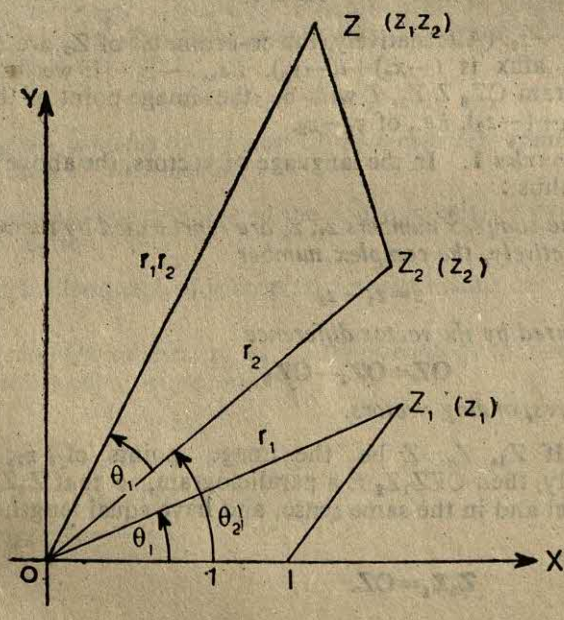

Fig. 3·9.

$(r_1r_2, \theta_1+\theta_2)$. This can be done by taking the point I representing the number 1 (and therefore, having cartesian co-ordinates (1, 0) and drawing the triangle OZ_2Z similar to the triangle OIZ_1. The point Z is the required point. For,

$$\frac{OZ}{OZ_2}=\frac{OZ_1}{OI},$$

so that

$$OZ=OZ_1.OZ_2=r_1r_2,$$

and $\qquad \angle Z_2OZ=\angle XOZ_1=\theta_1,$

so that $\qquad \angle XOZ=\angle XOZ_2+\angle XOZ_1,$

$$=\theta_2+\theta_1.$$

Hence Z has polar co-ordinates $(r_1r_2, \theta_1+\theta_2)$.

Remark. If $z_2=\pm i$, $r_2=1$, $\theta_2=\pm\frac{\pi}{2}$, the polar co-ordinates of Z are $\left(r_1, \theta_1\pm\frac{\pi}{2}\right)$, *i.e.*, $OZ=OZ_1$ and $\angle Z_1OZ$ is a right angle. Hence *multiplying a complex number z by $\pm i$ is equivalent to rotating its representative vector through an angle $\pm\frac{\pi}{2}$*.

3·9·6. GEOMETRICAL REPRESENTATION OF DIVISION IN C

Let Z_1, Z_2 be the representative points, in the Argand plane, of the complex numbers z_1 and $z_2(\neq 0)$ respectively. If the modulus

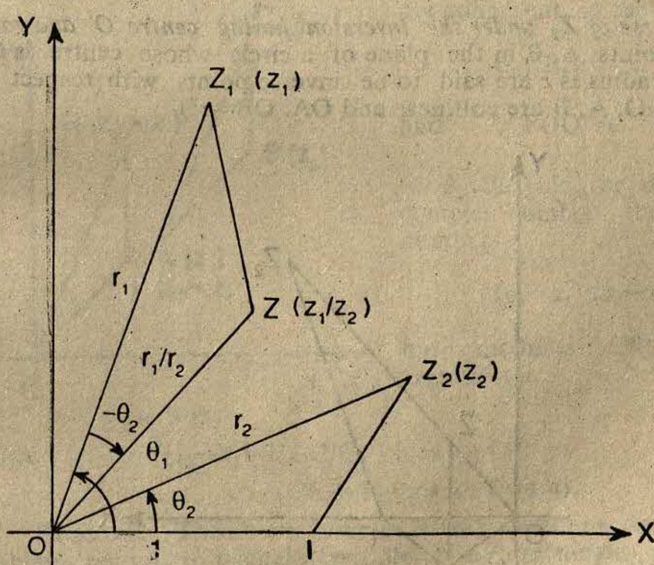

Fig. 3·10.

and argument of z_1 be r_1, θ_1 respectively and those of z_2 be r_2, θ_2 respectively, the modulus and argument of z_1/z_2 are r_1/r_2 and $\theta_1-\theta_2$ respectively.

To construct the point Z representing the complex number z_1/z_2, we have to construct the point with polar co-ordinates $(r_1/r_2, \theta_1-\theta_2)$. This can be done by taking the point I representing the number 1 and drawing the triangle OZ_1Z directly similar to the triangle OZ_2I. The point Z so obtained is the required point. For,

$$\frac{OZ_2}{OI}=\frac{OZ_1}{OZ},$$

so that $\quad OZ=r_1/r_2.$

Also, $\quad \angle XOZ=\angle XOZ_1-\angle ZOZ_1=\angle XOZ_1-\angle XOZ_2=\theta_1-\theta_2.$

Hence Z has polar co-ordinates $(r_1/r_2, \theta_1-\theta_2)$.

Remarks. 1. If $z_1=1$, the point Z_1 coincides with I. The lines OZ_2 and OZ are symmetric with respect to XO and

$$|OZ|=\frac{1}{|OZ_2|},$$

i.e., $\quad |OZ|\,.\,|OZ_2|=1.$

Therefore, if Z be the reflection of $\overline{Z}$ in the line OX, the points $\overline{Z}$, Z_2 are corresponding points under an inversion having centre O and radius I.

Hence *the representative point of* $\dfrac{1}{z_2}$ *is the reflection in OX of the inverse of* Z_2 *under the inversion having centre O and radius 1.* (Two points A, B in the plane of a circle whose centre is O and whose radius is r are said to be inverse points with respect to the circle if O, A, B are collinear and OA. OB$=r^2$).

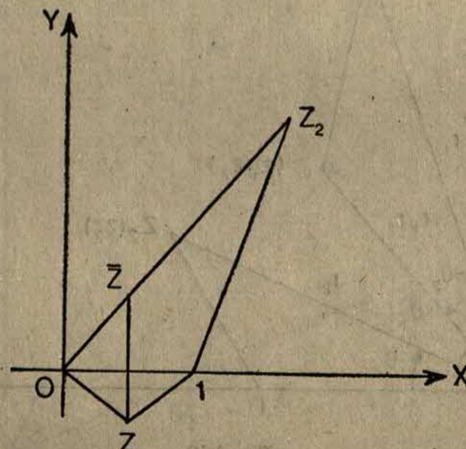

Fig. 3·11.

2. If z_1, z_2, z_3, z_4 be the affixes of four points Z_1, Z_2, Z_3, Z_4 respectively, in the Argand plane, then Z_4Z_2 is inclined to Z_3Z_1 at an angle arg. $\frac{z_1-z_3}{z_2-z_4}$. Therefore, Z_3Z and Z_4Z_2 are at right angles provided

$$\arg.\ \frac{z_1-z_3}{z_2-z_4}=\pm\frac{\pi}{2},$$

i.e., provided $z_1-z_3=\pm ik(z_2-z_4)$, where k is a non-zero red number.

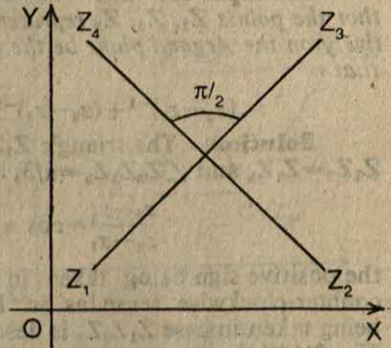

Fig. 3·12.

3. The following result is a straight-forward application of the geometrical representation of division of complex numbers :

An important result. *If* z_1, z_2, z_3 *be the affixes of the vertices of a triangle ABC described in counter-clockwise sense, then*

$$(z_3-z_1)/(z_2-z_1)=(CA/BA)\ (\cos\alpha+i\sin\alpha),$$

where $\angle BAC=\alpha$.

Proof. Let P and Q be the points in the Argand plane having affixes z_2-z_1 and z_3-z_1 respectively. The triangles OPQ and ABC are congruent, so that

$$CA/BA=QO/PO,$$

and $\quad \angle POQ=\alpha.$

By the rule for division of complex numbers, the complex number

$$(z_3-z_1)/(z_2-z_1)$$

has modulus OQ/OP and argument α.

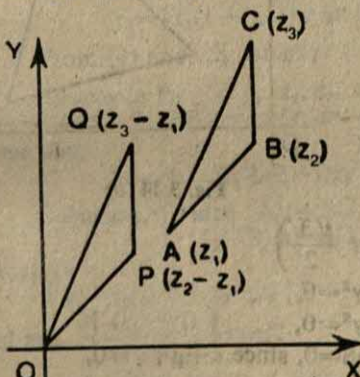

Fig. 3·13.

Hence $\quad (z_3-z_1)/(z_2-z_1)=(OQ/OP)\ (\cos\alpha+i\sin\alpha)$

$$=(CA/BA)\ (\cos\alpha+i\sin\alpha).$$

Remark. The above theorem is very useful in dealing with problems relating to triangles, parallelograms, rectangles, rhombuses and squares. In fact, it is useful whenever the magnitude of any angle is involved.

Example 15. *Show that a necessary and sufficient condition that the points Z_1, Z_2, Z_3 representing the numbers z_1, z_2, z_3 respectively on the Argand plane be the vertices of an equilateral triangle is that*

$$(z_2-z_3)^{-1}+(z_3-z_1)^{-1}+(z_1-z_2)^{-1}=0.$$

Solution. The triangle $Z_1Z_2Z_3$ is equilateral if and only if $Z_1Z_2=Z_1Z_3$ and $\angle Z_3Z_1Z_2=\pi/3$, and this holds if and only if

$$\frac{z_3-z_1}{z_2-z_1}=\cos \pi/3 \pm i \sin \pi/3, \qquad \text{...(1)}$$

the positive sign being taken in case $Z_1Z_2Z_3$ is described in the counter-clockwise sense [as in Fig. 3·14(*a*)], and the negative sign being taken in case $Z_1Z_2Z_3$ is described in the clockwise sence [as in Fig. 3·14(*b*)].

Putting $z_2-z_3=\alpha$, $z_3-z_1=\beta$, $z_1-z_2=\gamma$ so that $\alpha+\beta+\gamma=0$, we find that (1) is equivalent to

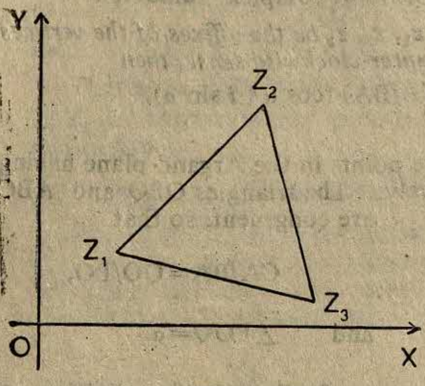

Fig. 3·14 (*a*)

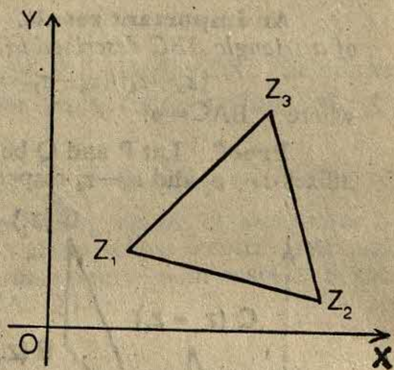

Fig. 3·14 (*b*)

$$\beta=-\gamma\left(\frac{1}{2}\pm i\,\frac{\sqrt{3}}{2}\right),$$

$$\Leftrightarrow \quad (\beta+\tfrac{1}{2}\gamma)^2+\tfrac{3}{4}\gamma^2=0,$$

$$\Leftrightarrow \quad \beta^2+\beta\gamma+\gamma^2=0,$$

$$\Leftrightarrow \quad \beta\gamma+\gamma\alpha+\alpha\beta=0, \text{ since } \alpha+\beta+\gamma=0,$$

$$\Leftrightarrow \quad 1/\alpha+1/\beta+1/\gamma=0,$$

$$\Leftrightarrow \quad (z_2-z_3)^{-1}+(z_3-z_1)^{-1}+(z_1-z_2)^{-1}=0.$$

Example 15. *ABCD is a rhombus in the Argand plane. If the affixes of the vertices be z_1, z_2, z_3, z_4 respectively and angle CBA be $\pi/3$, show that*

(*i*) $2z_2=z_1(1+i\sqrt{3})+z_3(1-i\sqrt{3})$,

(*ii*) $2z_4=z_1(1-i\sqrt{3})+z_3(1+i\sqrt{3})$.

Solution. Since the diagonals of a parallelogram bisect each other, and since every rhombus is a parallelogram, therefore the affix of the point of intersection of the diagonals is

$$\tfrac{1}{2}(z_2+z_4)=\tfrac{1}{2}(z_1+z_3). \quad \ldots(1)$$

Again, since

$$AB=BC,$$

and $\angle CBA=\pi/3$,

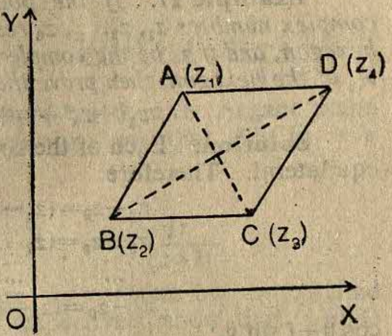

Fig. 3·15.

therefore

$$(z_1-z_2)/(z_3-z_2)=\cos \pi/3+i \sin \pi/3,$$

$$=\frac{1}{2}+i\frac{\sqrt{3}}{2} \quad \ldots(2)$$

Solving (2) for z_2 in terms of z_1 and z_3, we have

$$z_2=\tfrac{1}{2}z_1(1+i\sqrt{3})+\tfrac{1}{2}z_3(1-i\sqrt{3}). \quad \ldots(3)$$

From (1), we have

$$z_2+z_4=z_1+z_3. \quad \ldots(4)$$

Subtracting both sides of (3) from the corresponding side of (4), we have

$$z_4=\tfrac{1}{2}z_1(1-i\sqrt{3})+\tfrac{1}{2}z_3(1+i\sqrt{3}). \quad \ldots(5)$$

From (3) and (5), we have the desired result.

Example 16. *If z_1, z_2, z_3, z_4 be the affixes of the vertices of a square ABCD in the Argand plane taken in the anti-clockwise order, prove that*

$$z_3=-iz_1+(1+i)z_2,\ z_4=(1-i)z_1+iz_2.$$

Solution. Since $AB=BC$,

$$\angle CBA=\pi/2,$$

therefore

$$|z_1-z_2|=|z_3-z_2|,$$

and arg. $\left\{\dfrac{(z_1-z_2)}{(z_3-z_2)}\right\}=\pi/2.$

Thus $\dfrac{z_1-z_2}{z_3-z_2}=i,$

i.e., $\quad z_1-z_2=i(z_3-z_2) \quad \ldots(1)$

Similarly

$$z_4-z_1=i(z_2-z_1), \quad \ldots(2)$$

From (1) and (2), we have

$$z_3=-iz_1+(1+i)z_2$$

$$z_4=(1-i)z_1+iz_2.$$

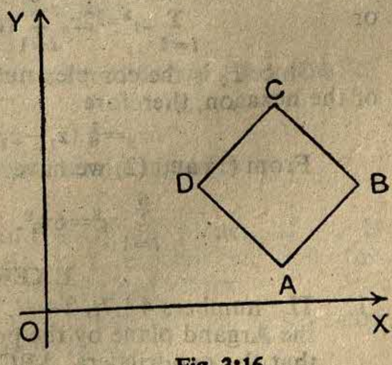

Fig. 3·16.

Example 17. *If the points A_1, A_2, ..., A_6 representing the complex numbers z_1, z_2, ..., z_6 respectively, be the vertices of a regular hexagon, and if z_0 be the complex number corresponding to the centroid A_6 of the hexagon, then prove that*

$$z_1^2+z_2^2+\cdots+z_6^2=6z_0^2.$$

Solution. Each of the six triangles $A_0A_1A_2$, ..., $A_0A_6A_1$ is equilateral. Therefore

$$z_2-z_0=(z_1-z_0) \text{ cis } \pi/3,$$
$$z_3-z_0=(z_2-z_0) \text{ cis } \pi/3,$$
$$\ldots \qquad \ldots \qquad \ldots$$
$$z_6-z_0=(z_5-z_0) \text{ cis } \pi/3,$$ where cis θ stands for $\cos\theta+i\sin\theta$.

From the above relations we have

$$z_2-z_0=(z_1-z_0) \text{ cis } \pi/3$$
$$z_3-z_0=(z_1-z_0) \text{ cis } 2\pi/3$$
$$z_4-z_0=(z_1-z_0) \text{ cis } 3\pi/3$$
$$z_5-z_0=(z_1-z_0) \text{ cis } 4\pi/3$$
$$z_6-z_0=(z_1-z_0) \text{ cis } 5\pi/3$$

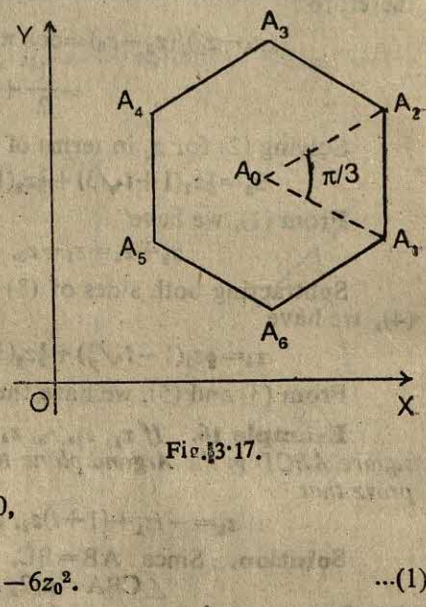

Fig. 3·17.

Squaring and adding, we have

$$\sum_{i=1}^{6} (z_i-z_0)^2$$

$$=(z_1-z_0)^2 \sum_{r=0}^{5} \text{cis } (2r\pi/3)$$

$$=0, \text{ since } \sum_{r=0}^{5} \text{cis } (2r\pi/5)=0,$$

or $$\sum_{i=1}^{6} z_i^2=2z_0\sum_{i=1}^{6} z_i-6z_0^2. \qquad \ldots(1)$$

Since z_0 is the complex number corresponding to the centroid of the hexagon, therefore

$$z_0=\tfrac{1}{6}(z_1+z_2+\ldots+z_6) \qquad \ldots(2)$$

From (1) and (2) we have

$$\sum_{i=1}^{6} z_i^2=6z_0^2.$$

EXERCISE 3 (*i*)

1. The numbers $4+7i$, $2+4i$, $5+2i$ and $7+5i$ are represented on the Argand plane by the points A, B, C, D respectively. Show that the quadrilateral ABCD is a square and find the length of its side.

2. Reduce each of the numbers

$$3-\frac{4}{i},\ \frac{3}{2}(1-i)\ (-1+7i),\ \frac{2(-18+i)}{(1+i)^2},\ \frac{5(-3+i)}{1+i}$$

to the form $x+iy$. If they are represented on the Argand plane by the points A, B, C, D respectively, show that ABCD is a square.

3. Reduce each of the numbers

$$-1-\frac{10}{i},\ \frac{-2-7i}{i},\ \frac{3+5i}{i-1},\ \frac{5(5+i)}{(1-i)(2+i)}$$

to the form $x+iy$. If they are represented on the Argand plane by the points A, B, C, D respectively, show that ABCD is a square.

4. The numbers $2+3i$, $8+11i$, $17i$ are represented on the Argand diagram by the points A, B, C respectively. Show that A, B, C are three vertices of a square and find the complex number represented by the fourth vertex.

5. A, B, C, D are the vertices of a square described in the counter-clockwise sense. If the affixes of A and B are $-1+4i$, and -3 respectively, find the affixes of the other vertices and of the centre of the square.

6. In the Argand plane, ABCD is a square described in the counter-clockwise sense. A represents the complex number $1-2i$ and the centre of the square represents the complex number $6-i$. Find the numbers represented by B, C, and D.

7. The vertices of a square taken in counter-clockwise order represent the numbers z_1, z_2, z_3 and z_4. Prove that

$$z_2=\tfrac{1}{2}(1+i)\,z_1+\tfrac{1}{2}(1-i)z_3,$$
$$z_4=\tfrac{1}{2}(1-i)\,z_1+\tfrac{1}{2}(1+i)z_3.$$

8. OPQ is an equilateral triangle in the Argand plane, named in the counter-clockwise order. If P and Q represent the complex numbers z_1 and z_2 respectively, show that

$$z_1=z_2(\tfrac{1}{2}-i\sqrt{3}/2).$$

9. Show that the origin and the points representing the roots of the equation

$$z^2+pz+q=0$$

on the Argand plane form an equilateral triangle if

$$p^2=3q.$$

10. ABC is an equilateral triangle in the Argand plane. If A, B represent the complex numbers 1, i respectively, find the affixes of two possible positions of the third vertex.

11. Show that a necessary and sufficent condition that z_1, z_2, z_3 be the affixes of an equilateral triangle is that

$$z_1^2+z_2^2+z_3^2=z_2z_3+z_3z_1+z_1z_2.$$

12. Prove that the complex numbers z_1, z_2, z_3 are represented on the Argand plane by the vertices of an equilateral triangle if and only if

$$\begin{vmatrix} z_1 & z_2 & z_3 \\ z_2 & z_3 & z_1 \\ 1 & 1 & 1 \end{vmatrix}=0$$

13. Prove that the complex numbers z_1, z_2, z_3 are represented on the Argand plane by the vertices of an equilateral triangle if and only if

$$(z_2-z_3)^2+(z_3-z_1)^2+(z_1-z_2)^2=0.$$

14. If z_1, z_2, z_3 be the affixes of the vertices of an equilateral triangle, and z_0 be the affix of the centroid of the triangle prove that

$$z_1^2+z_2^2+z_3^2=3z_0^2.$$

15. If $z_1, z_2, \cdots, z_n$ be the affixes of the vertices of a regular polygon of n sides and z_0 be the affix of the centroid of the polygon, prove that

$$z_1^2+z_2^2+\cdots+z_n^2=nz_0^2.$$

16. The complex numbers z_1, z_2, z_3, z_4 are represented on the Argand plane by the points A, B, C, D respectively. If $z_1-z_2=z_4-z_3$, show that ABCD is a parallelogram and that the affix of the centre of the parallelogram is

$$\tfrac{1}{4}(z_1+z_2+z_3+z_4).$$

17. ABCD is a rhombus, described in the clockwise sense, in the Argand plane. If the affixes of the vertices be z_1, z_2, z_3, z_4 respectively and angle CBA be $2\pi/3$, show that

$$2\sqrt{3}z_2=z_1(\sqrt{3}-i)+z_3(\sqrt{3}+i),$$
$$2\sqrt{3}z_4=z_1(\sqrt{3}+i)+z_3(\sqrt{3}-i).$$

18. Show that the points representing the complex numbers

$$a\left\{\cos\left(\alpha+\frac{2r\pi}{n}\right)+i\sin\left(\alpha+\frac{2r\pi}{n}\right)\right\},\ r=0, 1, 2, \dots, n-1$$

form the vertices of a regular n-sided polygon.

19. If $Z_1 Z_2 Z_3 Z_4 Z_5 Z_6$ is a regular hexagon in the Argand plane and the affixes of Z_1, Z_4 are the complex numbers z_1, z_4 respectively, then show that the affixes of the remaining vertices are

$$\tfrac{1}{2}(z_1+z_4)+\tfrac{1}{2}(z_1-z_4)(\cos\theta+i\sin\theta),$$

where θ has the values $\pm\pi/3, \pm 2\pi/3$.

20. Show that the numbers

$$5\left(1-\cos\frac{2r\pi}{5}-i\sin\frac{2r\pi}{5}\right),\ r=0, 1, 2, 3, 4$$

are the affixes of the vertices of a regular pentagon. What is the affix of its centre ?

TEST YOUR UNDERSTANDING III

In each of the following problems only one of the several alternative answers is correct. Write down the letter corresponding to the correct answers.

1. Given that p and q are real numbers such that $p+4i=3+qi$, the value of p^2+q^2 is
   (*a*) 12 (*b*) 7
   (*c*) 25 (*d*) -25.
2. $(3+4i)(4-3i)$ equals
   (*a*) $9i$ (*b*) $24+7i$
   (*c*) $7i$ (*d*) $12-7i$
3. If $\bar{z}=3-4i$, then z equals
   (*a*) $-3-4i$ (*b*) $4-3i$
   (*c*) $4+3i$ (*d*) $3+4i$.
4. If $z^{-1}=-1+i$, then z equals
   (*a*) $-1-i$ (*b*) $2(-1+i)$
   (*c*) $\frac{1}{2}(-1-i)$ (*d*) $\frac{1}{2}(1+i)$.
5. If $z_1z_2=3+i$ and $z_1=1+2i$, then z_2 equals
   (*a*) $1-i$ (*b*) $1+i$
   (*c*) $5+3i$ (*d*) $\frac{1}{5}(1+i)$.
6. The modulus of $5-4i$ is
   (*a* 1 (*b*) 9
   (*c*) $\sqrt{41}$ (*d*) 20.
7. The principal argument of $-1-i$ is
   (*a*) $\frac{\pi}{4}$ (*b*) $\frac{-\pi}{4}$
   (*c*) $\frac{-3\pi}{4}$ (*d*) $\frac{5\pi}{4}$.
8. The modulus of a complex number is 2, and the principal argument is $\frac{\pi}{3}$. The complex number is
   (*a*) $-1-\sqrt{3}i$ (*b*) $-1+\sqrt{3}i$
   (*c*) $1+\sqrt{3}i$ (*d*) $1-\sqrt{3}i$.
9. If ω is a complex cube root of unity, the value of $\omega^{99}+\omega^{100}+\omega^{101}$ is
   (*a*) 1 (*b*) -1
   (*c*) 3 (*d*) 0.
10. $i^{1000}+i^{1001}+i^{1002}+i^{1003}$ equals
    (*a*) 0 (*b*) i
    (*c*) $-i$ (*d*) 1.

REVIEW EXERCISE III

1. Mark in an Argand diagram the points representing the complex numbers $3+4i$, $3-4i$ and $\frac{3+4i}{3-4i}$.

2. Express the complex number $\frac{12+5i}{4+3i}$ in the form $a+bi$.

3. If $z=x+yi$, find the real and imaginary parts of $z+\frac{1}{z}$.

4. If $z=4+3i$, express $z+\frac{25}{z}$ in its simplest form.

5. The complex numbers $z_1=\frac{p}{1-i}$, $z_2=\frac{q}{1-2i}$ where p and q are real, are such that $z_1+z_2=1$. Find p and q.

6. Given that $z=\sqrt{3}+i$, find the modulus and principal argument of $\bar{z}$.

7. If $z=x+iy$ and $z^2=p+qi$, where x, y, p, q are real, prove that
$$2x^2=\sqrt{(p^2+q^2)}+p.$$

8. Given that the complex number z and its conjugate $\bar{z}$ satisfy $z\bar{z}+2i(z+\bar{z})=12+8i$, find the possible values of z.

9. Find the two square roots of $9i$ in the form $a+bi$.

10. If $1, \omega, \omega^2$ are the three cube roots of unity, find the value of $\frac{(1+\omega)^2}{\omega}$.

SUMMARY

1. $a+bi=c+di \Leftrightarrow a=b$ and $b=d$.
2. $(a+bi)+(c+di)=(a+c)+(b+d)i$
3. $(a+bi)-(c+di)=(a-c)+(b-d)i$
4. $(a+bi)(c+di)=(ac-bd)+(ad+bc)i$
5. $\text{Re}(x+yi)=x$, $\text{Im}(x+yi)=y$.
6. If $z=x+yi$, then $\bar{z}=x-yi$.
7. $z+\bar{z}$ is purely real and $z-\bar{z}$ is purely imaginary.
8. $|a+bi|=\sqrt{a_2+b_2}$.
9. If z_1, z_2 be complex numbers, then
$$|z_1+z_2| \leq |z_1|+|z_2|,$$
$$|z_1 z_2| = |z_1|\,|z_2|.$$
10. Every complex number has two square roots.
11. The cube roots of unity $(1, \omega, \omega^2)$ satisfy the relations
$$1+\omega+\omega^2=0,\ \omega^3=1.$$
12. The four fourth roots of unity are
$$\pm 1, \pm i.$$

HISTORICAL NOTE

The Greek mathematicians seem to have realized that the square root of a negative real number does not exist in the set of real numbers. However, it was the Jain mathematician *Mahaviracharya*, who in his work *Ganita Sara Sangraha* written in 850 A.D. first stated this difficulty clearly. He writes "as in the nature of things a negative (quantity) is not a square (quantity), it has, therefore, no square root". *Bhaskaracharya*, another famous Hindu mathematicians writes in his well-known work *Bijaganita* (aronnd 1150 A.D.). "There is no square-root of a negative quantity, for it is not a square".

Leonhard Euler (1707-87), the famous Swiss mathematician was the first to use the symbol i for $\sqrt{-1}$. He called the symbol i, imaginary. The French mathematician *Rene Descartes* (1596-1650) used the words *real* and *imaginary* in his famous work "*La Geometrie*" written in 1637. Another equally great French mathematician *Augustin Louis Cauchy* (1789-1857) is credited with coining the word *modulus* in 1821. The credit for giving the name *complex number* to a number of the form $a+b\sqrt{-1}$ goes to *Karl Friedrich Gauss* (1777-1855), the prince of mathematicians. The theory of complex numbers was put on a firm foundation by the Irish mathematician *Sir Willian R. Hamilton* (1805 1865). The Danish surveyor *Casper Wessel*, the French accountant *James R. Argand* and the German mathematician *Karl Friedrich Gauss* were responsible for the geometrical representation of complex numbers.

□ □

NIELS HENRIK ABEL (1802-1829)

Niels Henrik Abel was born on the 5th August, 1802 in the family of a poor Norwegian country minister. When he was 18, his father expired and thus he inherited a destitute family of six children from his widowed mother. Devoid of any resources he managed to enter the university with financial aid from his friends and teachers. In the university he earned the reputation of being a genius. He published his first paper at the age of 20. At the age of 22, he proved that it was impossible to solve the general equation of the fifth degree by radicals. He died of tuberculosis on 8th April, 1829 at the early age of 27.

Abel was a genius of the highest order. It was unfortunate that his work was not recognised during his life-time.

CHAPTER 4

Quadratic Equations

4·1. IDENTITIES AND EQUATIONS

Consider the following statements :

(1) $x^2-4=(x+2)(x-2)$.

(2) $x^3+y^3=(x+y)(x^2-xy+y^2)$.

(3) $\dfrac{1}{x-1}+\dfrac{1}{x+1}=\dfrac{2}{x^2-1}$.

(4) $2x+4=x-3$.

(5) $\dfrac{1}{x+1}+\dfrac{2}{x+2}=\dfrac{3}{x+3}$.

(1) holds true for all values of x.

(2) holds true for all values of x and y.

(3) holds true for all *permissible* values of x, *i.e.*, for all those values of x for which the two sides have a meaning. (Both sides of (3) have a meaning for all values of x except $x=1$ and $x=-1$.)

(4) does not hold true for all permissible values of x. Here both sides have a meaning for all values of x but the statement holds true only when $x=-7$.

(5) does not hold true for all permissible values of x. Here both sides have a meaning except when $x=-1$, -2, or -3, but the statement holds true only when $x=-3/2$.

A statement that holds true for all permissible values of the variables is called an **identity**. Since (1), (2) and (3) hold true for all permissible values of the variables, they are identities.

A statement that does not hold true for all the permissible values of the variables is called a **conditional equation,** or simply an **equation**. Since (4) and (5) do not hold true for all the permissible values of x, they are not identities ; they are equations.

In the present chapter we shall devote ourselves to the study of quadratic equations.

4·2. SOLUTION OF A QUADRATIC EQUATION

An equation of the form

$$ax^2+bx+c=0 \qquad ...(1$$

where $a \neq 0$, and $a, b, c \in \mathbf{R}$ is called a quadratic equation with real coefficients. If $a, b, c \in \mathbf{C}$, then (1) is said to be a quadratic equation with complex coefficients.

A complex number α is said to be a root of (1) if and only if

$$a\alpha^2+b\alpha+c=0.$$

The process of finding the roots of an equation is often called 'solving the equation'. There are several methods available for solving a quadratic equation. We shall describe some of them one by one. The methods are applicable to equations with complex coefficients but we shall concern ourselves with quadratic equations with real coefficients only, unless stated otherwise.

4·2·1. Solution of a Quadratic Equation by Factorization

If it is possible to express the left hand side of (1) as a product of linear factors, then (1) can be solved easily. For example, consider the equation

$$2x^2-5x+3=0.$$

$$\begin{aligned} & 2x^2-5x+3=0, \\ \Leftrightarrow\quad & 2x^2-3x-2x+3=0, \\ \Leftrightarrow\quad & x(2x-3)-(2x-3)=0, \\ \Leftrightarrow\quad & (x-1)(2x-3)=0, \\ \Leftrightarrow\quad & x-1=0 \text{ or } 2x-3=0, \\ \Leftrightarrow\quad & x=1 \text{ or } 3/2. \end{aligned}$$

Therefore, the roots of the given equation are 1 and 3/2.

Factorization method is useful when the roots are rational, specially when the left hand side can be factorized easily.

4·2·2. Solution of a Quadratic Equation by Completing the Square

Consider the equation

$$ax^2+bx+c=0, \text{ where } a \neq 0,\ a, b, c \in \mathbf{R}, \qquad \ldots(1)$$

$$\begin{aligned} \Leftrightarrow\quad & x^2+\frac{b}{a}x+\frac{c}{a}=0, \text{ (Dividing throughout by } a) \\ \Leftrightarrow\quad & x^2+2\cdot\frac{b}{2a}x=-\frac{c}{a}, \\ \Leftrightarrow\quad & x^2+2\cdot\frac{b}{2a}x+\left(\frac{b}{2a}\right)^2=\left(\frac{b}{2a}\right)^2-\frac{c}{a}, \\ \Leftrightarrow\quad & \left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}, \\ \Leftrightarrow\quad & x+\frac{b}{2a}=\pm\frac{\sqrt{(b^2-4ac)}}{2a}, \\ \Leftrightarrow\quad & x=-\frac{b}{2a}\pm\frac{\sqrt{(b^2-4ac)}}{2a}, \end{aligned}$$

$$\Rightarrow \quad x=\frac{-b\pm\sqrt{(b^2-4ac)}}{2a}.$$

Remarks. 1. The above method is called 'the method of completing the square' because we have added $b^2/4a^2$ to both sides in order to express the left hand side as a perfect square.

2. Since each step in the above working is reversible, therefore, it follows that the equation (1) has two roots.

4·2·3. Solution of a Quadratic Equation by the Formula Method

The above working gives the following formula for writing down the roots of equation $ax^2+bx+c=0$:

$$\boxed{x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}} \quad ...(A)$$

Instead of factorizing or completing the square, we can directly use formula (A) for solving a quadratic equation. When we do so, we say that we have solved the equation by the formula method.

Remark. Formula (A) is valid even when a, b, c are complex numbers. We are, therefore, in a position to solve the equation

$ax^2+bx+c=0$, where $a\neq 0$, $a, b, c \in \mathbf{C}$.

Example 1. *Solve the equation by the factorization method :*

$$(b-c)x^2+(c-a)\,x+(a-b)=0.$$

Solution. By inspection we find that the left hand side vanishes when $x=1$.

$\therefore$ $x-1$ must be a factor of the left hand side.

$\therefore$ The given equation may be written as

$$(b-c)x^2-(b-c)x-(a-b)x+(a-b)=0,$$

or $\quad (b-c)\,x\,(x-1)-(a-b)\,(x-1)=0,$

or $\quad (x-1)\,\{(b-c)x-(a-b)\}=0.$

$\therefore$ Either $x-1=0$, *i e.*, $x=1$,

or $\quad (b-c)x-(a-b)=0,$

i.e., $\quad x=\dfrac{a-b}{b-c}.$

Hence the roots are 1 and $\dfrac{a-b}{b-c}$.

Example 2. *Solve the equation :*

$$2x^2-3x-9=0,$$

by completing the square.

Solution. $2x^2-3x-9=0,$

$$\Leftrightarrow \quad x^2-\frac{3}{2}\,x-\frac{9}{2}=0, \text{ (Dividing throughout by 2),}$$

where $a \neq 0$, and $a, b, c \in \mathbf{R}$ is called a quadratic equation with real coefficients. If $a, b, c \in \mathbf{C}$, then (1) is said to be a quadratic equation with complex coefficients.

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\Leftrightarrow\quad & x-1=0 \text{ or } 2x-3=0,\\
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\Leftrightarrow\quad & x^2+2\cdot\frac{b}{2a}x=-\frac{c}{a},\\
\Leftrightarrow\quad & x^2+2\cdot\frac{b}{2a}x+\left(\frac{b}{2a}\right)^2=\left(\frac{b}{2a}\right)^2-\frac{c}{a},\\
\Leftrightarrow\quad & \left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2},\\
\Leftrightarrow\quad & x+\frac{b}{2a}=\pm\frac{\sqrt{(b^2-4ac)}}{2a},\\
\Leftrightarrow\quad & x=-\frac{b}{2a}\pm\frac{\sqrt{(b^2-4ac)}}{2a},
\end{aligned}$$

$$\Leftrightarrow \qquad x=\frac{-b\pm\sqrt{(b^2-4ac)}}{2a}.$$

Remarks. 1. The above method is called 'the method of completing the square' because we have added $b^2/4a^2$ to both sides in order to express the left hand side as a perfect square.

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or $$(b-c)\,x\,(x-1)-(a-b)\,(x-1)=0,$$

or $$(x-1)\,\{(b-c)x-(a-b)\}=0.$$

$\therefore$ Either $x-1=0$, *i.e.*, $x=1$,

or $$(b-c)x-(a-b)=0,$$

i.e., $$x=\frac{a-b}{b-c}.$$

Hence the roots are 1 and $\dfrac{a-b}{b-c}$.

Example 2. *Solve the equation :*

$$2x^2-3x-9=0,$$

by completing the square.

Solution. $2x^2-3x-9=0$,

$$\Leftrightarrow \qquad x^2-\frac{3}{2}\,x-\frac{9}{2}=0, \text{ (Dividing throughout by 2),}$$

$$\Leftrightarrow \quad x^2-2.x.\frac{3}{4}+\left(\frac{3}{4}\right)^2=\left(\frac{3}{4}\right)^2+\frac{9}{2},$$

$$\Leftrightarrow \quad \left(x-\frac{3}{4}\right)^2=\frac{81}{16},$$

$$\Leftrightarrow \quad x-\frac{3}{4}=\pm\frac{9}{4},$$

$$\Leftrightarrow \quad x=\frac{3}{4}\pm\frac{9}{4}=3 \text{ or } -\frac{3}{2}.$$

Example 3. *Solve the equation*

$$x^2+3x+4=0$$

by the formula method.

Solution. Comparing the given equation with $ax^2+bx+c=0$ we find that $a=1$, $b=3$, $c=4$.

$$\therefore \quad x=\frac{-b\pm\sqrt{(b^2-4ac)}}{2a}$$

$$=\frac{-3\pm\sqrt{3^2-4.1.4}}{2},$$

$$=\frac{-3\pm\sqrt{-7}}{2}.$$

EXERCISE 4 (*a*)

1. Which of the following are identities ?
   (*i*) $(x-1)^2+2(x-1)+1=x^2$;
   (*ii*) $(x+1)^3-x^3=x^2+x+1$;
   (*iii*) $x^4-1=(x-1)(x+1)(x^2+1)$.
2. Solve by the factorization method :
   (*i*) $x^2-3x+2=0$; (*ii*) $x^2-4x+3=0$;
   (*iii*) $x^2+2x-15=0$; (*iv*) $6x^2-5x+1=0$.
3. Solve the following equations by the method of completing the square :
   (*i*) $x^2-6x+8=0$; (*ii*) $x^2-8x+15=0$;
   (*iii*) $16x^2-24x+5=0$; (*iv*) $x^2-9x+4=0$.
4. Solve the following equations by using the formula :
   (*i*) $x^2-6x+5=0$; (*ii*) $2x^2-3x+1=0$;
   (*iii*) $3x^2-5x+1=0$; (*iv*) $4x^2-7x+13=0$.

Solve the following :

5. $x^2-9x+8=0$.
6. $x^2-5x-14=0$.
7. $3x^2+x-24=0$.

8. $x^2-(a+b)x+ab=0$.
9. $(a-b)x^2+(b-c)x+(c-a)=0$.
10. $a(b-c)x^2+b(c-a)x+c(a-b)=0$.
11. $(4-2\sqrt{3})x^2+2(\sqrt{3}-1)x-15=0$.
12. $(7-4\sqrt{3})x^2-(2-\sqrt{3})x-2=0$.

4 3. EQUATIONS REDUCIBLE TO QUADRATIC EQUATIONS

There are certain types of equations which, though not quadratic, can be reduced to quadratic form by making suitable substitutions, performing algebraic operations such as squaring, or using certain artifices. In the present section we shall deal with equations which can be solved in this manner.

Example 4. *Solve* : $x^6-9x^3+8=0$.

Solution. Putting $x^3=y$, we have

$$y^2-9y+8=0,$$

or $$(y-1)(y-8)=0.$$

$\therefore$ $$y=1 \text{ or } 8.$$

When $y=1$, $x^3=1$, so that $x^3-1=0$,

or $$(x-1)(x^2+x+1)=0,$$

$\therefore$ $$x=1, \text{ or } \frac{-1\pm\sqrt{-3}}{2}.$$

When $y=8$, $x^3=8$, so that $x^3-8=0$,

or $$(x-2)(x^2+2x+4)=0,$$

$\therefore$ $$x=2, \text{ or } \frac{-2\pm\sqrt{4-16}}{2}, \text{ i.e., } -1\pm\sqrt{-3}.$$

Hence the roots are 1, 2, $\frac{-1\pm\sqrt{-3}}{2}$, $-1\pm\sqrt{-3}$.

Example 5. *Solve* : $3^{2x}-10.3^x+9=0$.

Solution. Putting $3^x=y$, we have

$$y^2-10y+9=0,$$

or $$(y-1)(y-9)=0.$$

$\therefore$ $$y=1 \text{ or } 9.$$

When $y=1$, $3^x=1=3^0$, so that $x=0$.

When $y=9$, $3^x=9=3^2$, so that $x=2$.

Hence the roots are 0 and 2.

Example 6. *Solve* :

$$\sqrt{\frac{2x+1}{2x-1}}-\sqrt{\frac{2x-1}{2x+1}}=2\frac{2}{3}.$$

Solution. Putting $\sqrt{\dfrac{2x+1}{2x-1}}=y$, we have

$$y-\frac{1}{y}=\frac{8}{3},$$

or $$3y^2-8y-3=0,$$

or $$(3y+1)(y-3)=0.$$

$\therefore$ $$y=-\tfrac{1}{3} \text{ or } 3.$$

When $y=-\dfrac{1}{3}$, $$\sqrt{\frac{2x+1}{2x-1}}=-\frac{1}{3}, \qquad \ldots(i)$$

or $$\frac{2x+1}{2x-1}=\frac{1}{9}, \qquad \ldots(ii)$$

or $$18x+9=2x-1, \qquad \ldots(iii)$$

or $$16x=-10, \qquad \ldots(iv)$$

or $$x=-\frac{5}{8}. \qquad \ldots(v)$$

When $y=3$, $\sqrt{\dfrac{2x+1}{2x-1}}=3$,

or $$\frac{2x+1}{2x-1}=9, \text{ i.e., } 2x+1=18x-9,$$

or $$16x=10, \text{ so that } x=\frac{5}{8}.$$

By actual substitution we find that whereas $x=\dfrac{5}{8}$ satisfies the given equation, $x=-\dfrac{5}{8}$ does not satisfy it. Therefore, $-\dfrac{5}{8}$ is not a root. Hence the only root is $\dfrac{5}{8}$.

Note. $-\dfrac{5}{8}$ is a number which we have obtained as a value of x by solving the given equation but which does not satisfy it. $-\dfrac{5}{8}$ has entered the solution in the irreversible process of squaring (i); for proceeding backwards [from (v) we find by actual verification that whereas $x=-\dfrac{5}{8}$ satisfies (v), (iv), (iii) and (ii), it does not satisfy (i). The process transforming (i) to (ii) is an irreversible one. We thus find that whenever we perform any irreversible operation such as squaring, we should verify by actual substitution as to whether the values obtained as a result of such an

operation actually satisfy such an equation or not. The value which does not satisfy the equation should be rejected.

Example 7. *Solve* $(x-1)(x-3)(x-5)(x-7)=9$.

Solution. The given equation may be written as

$$(x-1)(x-7)(x-3)(x-5)=9,$$

or $$(x^2-8x+7)(x^2-8x+15)=9. \quad ...(i)$$

Putting $x^2-8x=a$, (i) becomes

$$(a+7)(a+15)=9,$$

or $$a^2+22a+105=9,$$

or $$a^2+22a+96=0,$$

or $$(a+16)(a+6)=0.$$

$\therefore$ $$a=-16 \text{ or } -6.$$

When $a=-16$, $x^2-8x=-16$,

or $$x^2-8x+16=0,$$

i.e, $$(x-4)^2=0, \text{ or } x=4, 4.$$

When $a=-6$, $x^2-8x=-6$,

or $$x^2-8x+6=0,$$

i.e, $$x=\frac{8\pm\sqrt{64-24}}{2}=4\pm\sqrt{10}.$$

Hence the roots are 4, 4, $4\pm\sqrt{10}$.

Example 8. *Solve* $x^4+x^3-4x^2+x+1=0$.

Solution. In this equation the coefficients of the terms equidistant from the beginning and the end are equal. We shall solve it by dividing throughout by x^2.

Dividing throughout by x^2, we have

$$x^2+x-4+\frac{1}{x}+\frac{1}{x^2}=0,$$

or $$x^2+\frac{1}{x^2}-4+x+\frac{1}{x}=0,$$

or $$\left(x+\frac{1}{x}\right)^2+\left(x+\frac{1}{x}\right)-6=0.$$

Putting $x+\frac{1}{x}=y$, we have

$$y^2+y-6=0,$$

or $$(y-2)(y+3)=0.$$

$\therefore$ $$y=2 \text{ or } -3.$$

When $y=2$, $x+\frac{1}{x}=2$, *i.e.*, $x^2-2x+1=0$,

i.e, $$x=1, 1.$$

When $y=-3$, $x+\frac{1}{x}=-3$, *i.e.*, $x^2+3x+1=0$,

or $$x=\frac{-3\pm\sqrt{5}}{2}.$$

Hence the roots are 1, 1 and $\frac{-3\pm\sqrt{5}}{2}$.

Example 9. *Solve* :

$$x^2+2x+2\sqrt{(x^2+2x+3)}=12.$$

Solution Putting $\sqrt{(x^2+2x+3)}=y$, we have

$$y^2-3+2y=12,$$

or $$y^2+2y-15=0,$$

or $$(y+5)(y-3)=0.$$

$\therefore$ $$y=-5 \text{ or } 3.$$

When $y=-5$, $\sqrt{(x^2+2x+3)}=-5$ or $x^2+2x+3=25$,

i.e., $$x^2+2x-22=0.$$

Therefore, $x=\frac{-2\pm\sqrt{4+88}}{2}=-1\pm\sqrt{23}.$

When $y=3$, $\sqrt{(x^2+2x+3)}=3$ or $x^2+2x+3=9$,

i.e., $$x^2+2x-6=0.$$

Therefore $x=\frac{-2\pm\sqrt{4+24}}{2}=-1\pm\sqrt{7}.$

Let us check up by substitution whether $x=-1\pm\sqrt{23}$ and $x=-1\pm\sqrt{7}$ satisfy the given equation. First, let us consider $x=-1\pm\sqrt{23}$. Since $x=-1\pm\sqrt{23} \Rightarrow (x+1)^2=23 \Rightarrow x^2+2x=22$, therefore, by substituting $x^2+2x=22$ in the given equation, we find that L.H.S.$=22+2\sqrt{22+3}=22+10=32$, which is not equal to the right hand side. Therefore $x=-1\pm\sqrt{23}$ do not satisfy the given equation.

Next let us consider $x=-1\pm\sqrt{7}$. Since $x=-1\pm\sqrt{7} \Rightarrow (x+1)^2=7 \Rightarrow x^2+2x=6$, therefore, by substituting $x^2+2x=6$ in the given equation, we find that L.H.S.$=6+2\sqrt{6+3}=6+2.3=12$. Therefore, $x=-1\pm\sqrt{7}$ satisfy the given equation.

Hence the roots are $-1\pm\sqrt{7}$.

Example 10. *Solve* $\sqrt{x+5}+\sqrt{x+12}=\sqrt{2x+41}$.

Solution. Squaring both sides of the given equation, we have

$$(x+5)+(x+12)+2\sqrt{\{(x+5)(x+12)\}}=2x+41,$$

or $$2\sqrt{(x^2+17x+60)}=24.$$

Squaring again, we have

$$x^2+17x+60=144,$$

or $$x^2+17x-84=0,$$

or $$(x-4)(x+21)=0,$$

or $$x=4 \text{ or } -21.$$

By actual substitution we find that whereas $x=4$ satisfies the given equation, $x=-21$ does not satisfy it.

Hence the only root is 4.

Example 11. *Solve* $\sqrt{x+4}-\sqrt{x-1}=5.$

Solution. The given equation may be written as

$$\sqrt{x+4}=5+\sqrt{x-1}. \quad \text{...}(i)$$

Squaring both sides of (i), we have

$$x+4=25+(x-1)+10\sqrt{x-1},$$

or $$-20=10\sqrt{x-1},$$

or $$-2=\sqrt{x-1}.$$

Squaring again, we have

$$4=x-1,$$

or $$x=5.$$

By actual substitution we find that $x=5$ does not satisfy the given equation.

Hence the given equation has no root.

Example 12. *Solve* $\sqrt{(x^2-16)}-(x-4)=\sqrt{(x^2-5x+4)}.$

Solution. The given equation may be written as

$$\sqrt{\{(x+4)(x-4)\}}-\sqrt{\{(x-4)\}^2}-\sqrt{\{(x-4)(x-1)\}}=0,$$

or $$\sqrt{(x-4)}\,\{\sqrt{(x+4)}-\sqrt{(x-4)}-\sqrt{(x-1)}\}=0.$$

$\therefore$ Either $\sqrt{x-4}=0,$

or $$\sqrt{x+4}-\sqrt{x-4}-\sqrt{x-1}=0.$$

When $\sqrt{x-4}=0,\ x=4.$

When $\sqrt{x+4}-\sqrt{x-4}-\sqrt{x-1}=0,$

we have $\sqrt{x+4}-\sqrt{x-4}=\sqrt{x-1}.$

Squaring both sides, we have

$$(x+4)+(x-4)-2\sqrt{(x^2-16)}=x-1,$$

or $$x+1=2\sqrt{(x^2-16)}.$$

Squaring again we have

$$x^2+2x+1=4(x^2-16),$$

or $$3x^2-2x-65=0,$$

or $$(x-5)(3x+13)=0.$$

$$\therefore\quad x=5 \text{ or } -\frac{13}{3}.$$

By actual substitution we find that whereas $x=4, 5$ satisfy the given equation, $x=-\frac{13}{3}$ does not satisfy it.

Hence the roots are 4 and 5.

Note. In the above example we found that $\sqrt{(x-4)}$ was a common factor throughout. We should always first of all examine if there is any such common factor.

Example 13. *Solve* $\sqrt{(5x^2-6x+8)}-\sqrt{(5x^2-6x-7)}=1.$

(*Roorkee Entrance 1984*)

Solution. Let $\sqrt{(5x^2-6x+8)}=a,\ \sqrt{(5x^2-6x-7)}=b$ ...(*i*)

The given equation then becomes

$$a-b=1 \quad ...(ii)$$

Also, $a^2-b^2=(5x^2-6x+8)-(5x^2-6x-7),$

$$=15. \quad ...(iii)$$

Dividing both sides of (*iii*) by (*ii*), we have

$$a+b=15. \quad ..(iv)$$

Adding (*ii*) and (*iv*), we have

$$2a=16,$$

or $$a=8.$$

$$\therefore\quad \sqrt{(5x^2-6x+8)}=8$$

Squaring both sides, we have

$$5x^2-6x+8=64,$$

or $$5x^2-6x-56=0.$$

$$\therefore\quad x=\frac{6\pm\sqrt{36+1120}}{10}=\frac{6\pm 34}{10}=4,\ -\frac{14}{5}.$$

By actual substitution we find that 4, $-\frac{14}{5}$ both satisfy the given equation.

Hence the roots are 4 and $-\frac{14}{5}$.

Example 14. *Solve* $\sqrt{3x^2-7x-30}-\sqrt{2x^2-7x-5}=x-5$.

Solution. Let $\sqrt{3x^2-7x-30}=a,\ \sqrt{2x^2-7x-5}=b$ ...(*i*)

The given equation then becomes

$$a-b=x-5. \qquad ...(ii)$$

Also, $a^2-b^2=(3x^2-7x-30)-(2x^2-7x-5)$,

$$=x^2-25. \qquad ...(iii)$$

Dividing both sides of (*iii*) by (*ii*), we have

$$a+b=\frac{x^2-25}{x-5}. \qquad ...(iv)$$

Now two different cases arise :

Case I : $x-5=0$, so that $x=5$.

By actual substitution we find that $x=5$ is indeed a root of the given equation.

Case II : $x-5\neq0$. From (*iv*), we have

$$a+b=x+5. \qquad ...(v)$$

Adding corresponding sides of (*ii*) and (*v*), we have

$$2a=2x,$$

or $$a=x,$$

or $$\sqrt{3x^2-7x-30}=x.$$

Squaring both sides, we have

$$3x^2-7x-30=x^2,$$

or $$2x^2-7x-30=0,$$

or $$(x-6)(2x+5)=0.$$

$\therefore\ x=6$ or $-\frac{5}{2}$.

By actual substitution we find that $x=6$ and $-\frac{5}{2}$ both satisfy the given equation.

Hence the roots are 5, 6, $-\frac{5}{2}$.

EXERCISE 4 (*b*)

Solve :

1. $x^4-7x^2+12=0$.
2. $4^{1+x}+4^{1-x}=10$.
3. $8\sqrt{\frac{x}{x+3}}-\sqrt{\frac{x+3}{x}}=2$.

4. $\sqrt{\left(\frac{3x^2}{1+x^2}\right)}+\sqrt{\left(\frac{1+x^2}{3x^2}\right)}=2.$

5. $(x+1)(x+2)(x+3)(x+4)+1=0.$

6. $x(x+1)^2(x+2)=72.$

7. $16x(x+1)(x+2)(x+3)=9.$

8. $(x+2)(3x+4)(3x+7)(x+3)=2600.$

9. $x^4+8x^3+17x^2-8x+1=0.$

10. $x^4+6x^3-5x^2+6x+1=0.$

11. $2x^4+x^3-17x^2+x+2=0.$

12. $2x^2-3x+2\sqrt{(2x^2-5x+13)}=2x-5.$

13. $3x^2+15x-2=2\sqrt{(x^2+5x+1)}.$

14. $3x^2-7+3\sqrt{(3x^2-16x+21)}=16x.$

15. $\sqrt{x+4}-\sqrt{x-4}=4.$

16. $\sqrt{x+7}+\sqrt{4x+1}=3\sqrt{x+2}.$

17. $(x^2-4x+3)+\sqrt{(x^2-9x+18)}=5\sqrt{x-3}$

18. $\sqrt{(x^2-3x+36)}-\sqrt{(x^2-3x+9)}=3.$

19. $\sqrt{(3x^2-2x+9)}+\sqrt{(3x^2-2x-4)}=13.$

20. $\sqrt{(3x^2+7x+2)}-\sqrt{(2x^2+7x+11)}=x-3.$

4·4. RELATION BETWEEN THE ROOTS AND COEFFICIENTS OF AN EQUATION

Let α and β be the roots of the quadratic equation

$$ax^2+bx+c=0. \quad \ldots(i)$$

Since α and β are the roots of (i), therefore, $(x-\alpha)$ and $(x-\beta)$ must both be factors of ax^2+bx+c. Again, since ax^2+bx+c is of the second degree in x, the only other factor must be a constant.

$$\therefore\ ax^2+bx+c=a(x-\alpha)(x-\beta).$$

Dividing throughout by a ($\neq 0$) we have,

$$x^2+\frac{b}{a}x+\frac{c}{a}=(x-\alpha)(x-\beta)=x^2-(\alpha+\beta)x+\alpha\beta. \quad \ldots(ii)$$

Comparing the coefficients of x and the constant terms on both sides of the identity (ii), we have

$$\frac{b}{a}=-(\alpha+\beta),\ \frac{c}{a}=\alpha\beta.$$

Hence $\alpha+\beta=-\dfrac{b}{a},\ \alpha\beta=\dfrac{c}{a},$ $\quad\ldots$(A)

i.e.,
$$\boxed{\begin{aligned}\text{Sum of the roots}&=-\frac{\text{coefficient of } x}{\text{coefficient of } x^2}\\ \text{Product of the roots}&=\frac{\text{constant term}}{\text{coefficient of } x^2}\end{aligned}} \qquad \ldots(B)$$

which are the required relations.

Aliter. Let α, β be the roots of the quadratic equation

$$ax^2+bx+c=0.$$

$$\therefore \quad \alpha=\frac{-b+\sqrt{(b^2-4ac)}}{2a},\ \beta=\frac{-b-\sqrt{(b^2-4ac)}}{2a} \qquad \ldots(i)$$

Adding the relations (i), we have

$$\alpha+\beta=-\frac{b}{a} \qquad \ldots(ii)$$

Multiplying the corresponding sides of the relations (i), we have

$$\alpha\beta=\frac{b+\sqrt{(b^2-4ac)}}{2a}\times\frac{-b-\sqrt{(b^2-4ac)}}{2a}.$$

$$=\frac{(-b)-(b^2-4ac)}{4a^2}=\frac{c}{a}.$$

Hence $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$.

Example 15. *If one root of the quadratic equation $ax^2+bx+c=0$ is equal to the nth power of the other, then show that*

$$(ac^n)^{\frac{1}{n+1}}+(a^nc)^{\frac{1}{n+1}}+b=c. \qquad (I.I.T.\ J.E.E.,\ 1983)$$

Solution. Let the roots of the equation

$$ax^2+bx+c=0$$

be α, β. Then

$$\alpha+\beta=-\frac{b}{a} \qquad \ldots(1)$$

$$\alpha\beta=\frac{c}{a}. \qquad \ldots(2)$$

Since one roots is the nth power of the other, we have

$$\alpha=\beta^n. \qquad \ldots(3)$$

We have to eliminate α and β from relations (1)—(3). From (2) and (3), we have

$$\beta^{n+1}=\frac{c}{a},$$

or
$$\beta=\left(\frac{c}{a}\right)^{\frac{1}{n+1}},$$

and
$$\alpha=\beta^{n}=\left(\frac{c}{a}\right)^{\frac{n}{n+1}}. \qquad \ldots(4)$$

Substituting the values of α and β in (1), we have

$$\left(\frac{c}{a}\right)^{\frac{n}{n+1}}+\left(\frac{c}{a}\right)^{\frac{1}{n+1}}=-\frac{b}{a},$$

or
$$(ac^{n})^{\frac{1}{n+1}}+(a^{n}c)^{\frac{1}{n+1}}+b=0.$$

EXERCISE 4 (*c*)

1. Find the condition that the roots of the equation
$$lx^2+2mx+n=0$$
may be equal in magnitude but opposite in sign.

2. Find the condition that one root of the equation
$$ax^2+bx+c=0$$
may be double the other.

3. Find the condition that one root of the equation
$$ax^2+bx+c=0$$
may be four times the other.

4. What is the value of m in the equation
$$2x^2-10x+m=0,$$
when one root is 2/3 of the other ?

5. Find the condition that the roots of the equation
$$lx^2+2mx+n=0$$
may be reciprocals of each other.

6. Find the condition that one root of the equation
$$px^2+qx+r=0$$
may be square of the other.

7. If k be the ratio of the roots of the equation $x^2-px+q=0$, show that
$$\frac{k^2+1}{k}=\frac{p^2-2q}{q}.$$

8. If the roots of the equation $x^2-px+q=0$ differ from each other by unity, prove that
$$p^2=4q+1$$
9. If the roots of the equation $ax^2+bx+c=0$ be such that their sum is equal to the sum of their squares, show that
$$2ac=ab+b^2.$$
10. If the roots of the equation $ax^2+cx+c=0$ be in the ratio $p:q$, show that
$$\sqrt{\frac{p}{q}}+\sqrt{\frac{q}{p}}+\sqrt{\frac{c}{a}}=0.$$

4·5. NATURE OF THE ROOTS OF THE QUADRATIC EQUATION $ax^2+bx+c=0$, WHERE a, b, c ARE REAL NUMBERS

The roots of the equation $ax^2+bx+c=0$ are
$$\alpha=\frac{-b+\sqrt{(b^2-4ac)}}{2a},\ \beta=\frac{-b-\sqrt{(b^2-4ac)}}{2a}. \quad \cdots(A)$$

The following different cases arise :

(*i*) $\mathbf{b^2-4ac>0}$. When $b^2-4ac>0$, α, β are both real and different from each other. Hence the roots are real and unequal.

Conversely, if α, β are real and unequal, $\sqrt{(b^2-4ac)}$ must be real and different from zero, *i e.*, $b^2-4ac>0$.

As a special case, if a, b, c be rational numbers, such that b^2-4ac is the square of a rational number, then α and β are rational Hence the roots are rational.

(*ii*) $\mathbf{b^2-4ac=0}$. When $b^2-4ac=0$, α and β are real and equal to each other. Hence the roots are real and equal.

Conversely, if α, β are equal, we must have $b^2-4ac=0$.

(*iii*) $\mathbf{b^2-4ac<0}$. When $b^2-4ac<0$, α and β are both imaginary. Hence the roots are imaginary.

Conversely, if α, β are imaginary, $\sqrt{(b^2-4ac)}$ must be an imaginary number, *i e.*, $b^2-4ac<0$.

Thus we have the following results :

The roots of the equation $ax^2+bx+c=0$ (a, b, c being real numbers) are

(*i*) *real and unequal if and only if $b^2-4ac>0$; real, rational and unequal whenever a, b, c are rational and b^2-4ac is a perfect square ;*

(*ii*) *real and equal if and only if $b^2-4ac=0$;*

(*iii*) *imaginary if and only if $b^2-4ac<0$.*

Cor. 1. From (A) we find that when a, b, c are rational numbers, then if α is irrational, β is also irrational, and if β is irrational, α is also irrational ; *i.e.*, α and β are either both rational or

both irrational, and are conjugate numbers in the latter case. Hence we have the following result :

In a quadratic equation with rational coefficients either no root is irrational or both roots are irrational.

Cor. 2. From (A) we find that when a, b, c are any real numbers whatever, then if any one of α or β is complex, so also is the other *i.e.*, α and β are either both real or both complex, and are conjugate numbers in the latter case. Hence we have the following result :

In a quadratic equation with real coefficients, either no root is imaginary or both roots are imaginary.

Note. From the above discussion we find that the nature of roots of the quadratic equation $ax^2+bx+c=0$ depends on the expression b^2-4ac, which is generally called the **discriminant** of the equation.

Example 16. *Examine the nature of the roots of the following equations :*

(*i*) $x^2-5x-3=0$,

(*ii*) $2x^2-7x+3=0$,

(*iii*) $x^2-6x+9=0$,

(*iv*) $x^2-x+1=0$.

Solution. (*i*) The discriminant$=(-5)^2-4(1)(-3)=37$. Since the discriminant is positive, but is not the square of a rational number, the roots are real, unequal and irrational.

(*ii*) The discriminant$=7^2-4.2.3=25=5^2$. Since the discriminant is non-zero and the square of a rational number, the roots are rational and unequal.

(*iii*) The discriminant$=6^2-4.1.9.=0$. Since the discriminant is zero, the roots are real and equal.

(*iv*) The discriminant$=(-1)^2-4.1.1.=-3$. Since the discriminant is negative, the roots are imaginary.

Example 17. *For what value of m will the equation*

$$(m+1)x^2+2(m+3)x+m+8=0 \text{ have equal roots ?}$$

Solution. Discriminant $=4(m+3)^2-4(m+1)(m+8)$,

$$=4(-3m+1).$$

If the given equation has equal roots, the discriminant must be zero.

$$\therefore \quad -3m+1=0 \text{ or } m=\frac{1}{3}.$$

Hence $\quad m=\frac{1}{3}.$

Example 18. *Give reasons for the following :*

(*i*) *The discriminant of the equation $z^2-iz-1=0$ is positive, but still is has imaginary roots.*

(*ii*) *The discriminant of the equation $x^2-2\sqrt{3}x-1=0$ is a perfect square, but still it has irrational roots.*

Solution. (*i*) Here the coefficient of z is imaginary. This explains as to why the roots are imaginary even though the discriminant is positive.

(*ii*) Here the coefficient of x is irrational. This explains as to why the roots are irrational even though the discriminant is a perfect square.

Note. The examples show that the nature of the roots depends not only on the sign of b^2-4ac, but also on the nature of the coefficients **a, b, c**.

Example 19. *Show that the roots of the equation*

$$(x-a)(x-b)=h^2$$

are always real, a, b, h being real numbers.

Solution. The given equation is

$$(x-a)(x-b)-h^2=0,$$

or $$x^2-(a+b)x+ab-h^2=0.$$

The discriminant$=(a+b)^2-4(ab-h^2)$,

$$=(a-b)^2+4h^2,$$

which is never negative.

Hence the roots of the given equation are always real.

Example 20. *If the roots of the equation $(a^2+b^2)+x^2$ $2(ac+bd)x+c^2+d^2=0$ be real, show that they are also equal, a, b, c, d, being real numbers.*

Solution. The discriminant$=4(ac+bd)^2-4(a^2+b^2)(c^2+d^2)$,

$$=4(2abcd-a^2d^2-b^2c^2).$$

$$=-4(ad-bc)^2.$$

If the roots of the given equation are real, the discriminant must be greater than or equal to zero.

$\therefore$ We must have $-4(ad-bc)^2 \geqslant 0$

or $$(ad-bc)^2 \leqslant 0 \qquad ...(i)$$

Since $(ad-bc)^2$ is the square of a real number,

$\therefore$ $$(ad-bc)^2 \geqslant 0 \qquad ...(ii)$$

From (*i*) and (*ii*), we have

$$(ad-bc)^2=0,$$

i.e., the discriminant is zero.

Hence the roots of the given equation are equal.

Example 21. *Show that if the roots of the equation $ax^2+2bx+c=0$ are real and unequal, then those of*

$$(a^2-ac+2b^2)\,x^2+2b(a+c)x+c^2-ac+2b^2=0$$

are either imaginary, or real and equal, a, b, c being real numbers.

Solution. The discriminant D of the equation

$$(a^2-ac+2b^2)x^2+2b(a+c)x+c^2-ac+2b^2=0 \qquad \cdots(i)$$

is given by
$$\begin{aligned} D&=4b^2(a+c)^2-4(a^2-ac+2b^2)(c^2-ac+2b^2).\\ &=4\{b^2(a+c)^2+ac(a-c)^2-2b^2(a-c)^2-4b^4\},\\ &=4\{4b^2ac+ac(a-c)^2-b^2(a-c)^2-4b^4\},\\ &=4(ac-b^2)\{(a-c)^2+4b^2\}. \qquad \cdots(ii)\end{aligned}$$

If the roots of the equation $ax^2+2bx+c=0$ are real, we must have

$$4b^2-4ac\geqslant 0,$$

i.e.,
$$b^2-ac\geqslant 0.$$

When $b^2-ac\geqslant 0$, from (*ii*) we find that $D\leqslant 0$, so that the roots of the equation (*i*) are either imaginary, or real and equal.

EXERCISE 4 (*d*)

1. Examine the nature of the roots of the following equations :

   (*i*) $x^2-7x+3=0$ (*ii*) $2x^2-5x+7=0$

   (*iii*) $x^2-12x+36=0$ (*iv*) $x^2-x-12=0$.

2. Find the values of m for which the following equations have equal roots :

   (*a*) $(m+1)x^2+2(m+3)x+2m+3=0$

   (*b*) $x^2-15-m(2x-8)=0$.

3. For what values of m will the equation

   $x^2-2(1+3m)x+7(3+2m)=0$ have equal roots ?

4. Prove that the roots of the equation

   $$(a-b+c)\,x^2+4(a-b)\,x+(a-b-c)=0$$

   are real, a, b, c being real numbers.

5. If a, b, c be real numbers, prove that the roots of the equation

   $$(x-a)(x-b)+(x-b)(x-c)+(x-c)(x-a)=0$$

   are always real and they cannot be equal unless $a=b=c$. a, b, c being real numbers.

   [**Hint** : Discriminant$=2\{(a-b)^2+(b-c)^2+(c-a)^2\}$.]

6. If the roots of the equation $x^2+a^2=8x+6a$ be real, show that a must lie between -2 and 8.

7. Prove that the roots of $(a+c-b)x^2+2cx+(c+b-a)=0$ are rational, a, b, c being rational.

8. If $a+b+c=0$, show that the roots of the equation $ax^2+bx+c=0$ will be rational, a, b, c being rational. Hence show that the roots of $(p+q)x^2-2px+(p-q)=0$ are rational, p, q being rational.

9. If the roots of the equation $x^2-2cx+ab=0$ be real and unequal, them prove that the roots of $x^2-2(a+b)x+a^2+b^2+2c^2=0$ will be imaginary, a, b, c being all real.

10. If a, b are rational, prove that the equation
$$2ax^2+(2a+b)x+b=0$$
has rational roots.

11. Show that $fa\neq b$, then the roots of the equation
$$2(a^2+b^2)x^2+2(a+b)x+1=0$$
are imaginary.

4·6. FORMATION OF QUADRATIC EQUATIONS WITH GIVEN ROOTS

We have already studied as to how to solve a given quadratic equation. We shall now consider the converse problem, *i.e.*, to find the quadratic equation whose roots are given.

Suppose we are required to find the equation whose roots are α and β.

Let the required equation be
$$ax^2+bx+c=0 \text{ (where } a\neq 0). \qquad ...(i)$$
Since α, β are the roots of (i), $(x-\alpha)$ and $(x-\beta)$ must both be factors of ax^2+bx+c.

Again, since ax^2+bx+c is of the second degree in x, the only other factor must be a constant.

$$\therefore \qquad ax^2+bx+c\equiv a(x-\alpha)(x-\beta)$$

The required equation is
$$a(x-\alpha)(x-\beta)=0.$$
i.e., $\quad (x-\alpha)(x-\beta)=0$, since $a\neq 0$. $\qquad ...(ii)$

Now (ii) may also be written as
$$x^2-(\alpha+\beta)x+\alpha\beta=0, \qquad ...(iii)$$
or $\quad x^2-x(\text{sum of the roots})+\text{product of the roots}=0. \quad ...(iv)$

Remark. From the form of equation (iii) we find that there is only one quadratic equation (apart from a constant factor) having a given pair of numbers α and β as its roots. We are, therefore, justified in the use of the phrase 'the quadratic equation whose roots are α and β'.

Example 22. *Form the quadratic equation whose roots are 2 and 3.*

Solution. Since 2, 3 are roots of the equation, therefore, $(x-2)$ and $(x-3)$ must be both factors of the left hand side of the equation.

The required equation is

$$(x-2)(x-3)=0,$$

i.e., $$x^2-5x+6=0.$$

Hence the required equation is $x^2-5x+6=0$.

Example 23. *Form the quadratic equation whose roots are :*

$$\frac{a-b}{a+b}, \frac{a+b}{a-b}.$$

Solution. Let $\alpha=\dfrac{a-b}{a+b}$, $\beta=\dfrac{a+b}{a-b}$.

$$\therefore \qquad \alpha+\beta=\frac{a-b}{a+b}+\frac{a+b}{a-b}=\frac{2(a^2+b^2)}{a^2-b^2},$$

$$\alpha\beta=\frac{a-b}{a+b}\cdot\frac{a+b}{a-b}=1.$$

The required equation (having α and β as its roots) is

$$x^2-(\alpha+\beta)+\alpha\beta=0,$$

i.e., $$x^2-\frac{2(a^2+b^2)}{a^2-b^2}x+1=0.$$

Example 24. *Form the quadratic equation with rational co-efficients one of whose roots is $2+\sqrt{3}$.*

Solution. Since the required quadratic must have rational coefficients, if $2+\sqrt{3}$ is one root, the other root must be $2-\sqrt{3}$.

$\therefore$ Sum of the roots$=(2+\sqrt{3})+(2-\sqrt{3})=4$.

Product of the roots$=(2+\sqrt{3})(2-\sqrt{3})=1$.

Hence the required equation is

$$x^2-4x+1=0$$

Aliter. Let $x=2+\sqrt{3}$.

By transposition and squaring, we have

$$(x-2)^2=3,$$

i.e., $$x^2-4x+1=0,$$

which is the required equation.

Example 25. *Form the quadratic equation with real coefficients having $2+i\sqrt{3}$ as one of its roots.*

Solution. Since the required quadratic equation must have real coefficients, if $2+i\sqrt{3}$ is one root, the other root must be its complex conjugate, *i.e.,* $2-i\sqrt{3}$.

$\therefore$ Sum of the roots$=(2+i\sqrt{3})+(2-i\sqrt{3})=4$.

Product of the roots$=(2+i\sqrt{3})(2-i\sqrt{3})=7$.

Hence the required equation is

$$x^2-4x+7=0.$$

Aliter. Let $x=2+i\sqrt{3}$.

By transposition and squaring, we have

$$(x-2)^2=-3,$$

i.e., $$x^2-4x+7=0,$$

which is the required equation.

Example 26. *Find the value of* $x^4-9x^3+21x^2-15x+3$ *when* $x=3+\sqrt{8}$.

Solution. The quadratic equation with rational coefficients and having $3+\sqrt{8}$ as one of its roots is

$$(x-3)^2=(\sqrt{8})^2,$$

i.e., $$x^2-6x+1=0.$$

Dividing the given polynomial by x^2-6x+1, we have the identity

$$x^3-9x^4+21x^2-15x+3\equiv(x^2-6x+1)(x^2-3x+2)+1.$$

Since $3+\sqrt{8}$ is a root of $x^2-6x+1=0$, we find that the R.H.S.$=1$, when $x=3+\sqrt{8}$.

Hence the required value is 1.

EXERCISE 4 (*e*)

Form the quadratic equations whose roots are :

1. $2, -3$.
2. $4, -\frac{12}{5}$.
3. $2+\sqrt{3}, 2-\sqrt{3}$.
4. $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}, \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$.
5. $4+\sqrt{-7}, 4-\sqrt{-7}$.
6. $2-\sqrt{-3}, 2+\sqrt{-3}$.
7. $\frac{4+\sqrt{-5}}{4-\sqrt{-5}}, \frac{4-\sqrt{-5}}{4+\sqrt{-5}}$
8. $a+ib, a-ib$.
9. $i(a-b), -i(a-b)$.
10. $\frac{a-ib}{a+ib}, -\frac{a+ib}{a-ib}$.
11. a^2-b^2, a^2+b^2.
12. Form the quadratic equation with rational coefficients one of whose roots is $2+\sqrt{5}$.
13. Form the quadratic equation with rational coefficients one of whose roots is $\frac{1}{2+\sqrt{5}}$.

14. Form the quadratic equation with real coefficients one of whose roots is $2+i\sqrt{3}$,

15. Form the quadratic equation with real coefficients one of whose roots is $\frac{1}{2+i\sqrt{3}}$.

16. Find the value of $2x^3-9x^2-10x+13$, when $x=3+\sqrt{5}$.

17. Find the value of $2x^4+5x^3+7x^2-x+37$, when $x=-2-i\sqrt{3}$.

18. Find the value of $2x^4-2x^3-3x^2+8x+5$, when

$$x=\frac{3+i\sqrt{5}}{2}.$$

4 7. SYMMETRIC FUNCTIONS OF THE ROOTS OF A QUADRATIC EQUATION

An expression involving two quantities α, β is said to be symmetric in α, β if it remains unchanged when we interchange α and β. The expressions $\alpha+\beta$ and $\alpha\beta$ are, therefore, symmetric functions of α and β. (Observe that $\alpha+\beta$ becomes $\beta+\alpha$ by interchanging α and β, and this is the same as $\alpha+\beta$. Similarly, by interchanging α and β, the expression $\alpha\beta$ becomes $\beta\alpha$ which is the same as $\alpha\beta$.) $\alpha^2+\beta^2$, $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$ are also symmetric functions of α, β. However, $\alpha-\beta$ and $\alpha^2+2\beta^2+\alpha$ are not symmetric functions of α and β. (In fact by interchanging α and β, $\alpha-\beta$ becomes $\beta-\alpha$ which is not equal to $\alpha-\beta$; also, $\alpha^2+2\beta^2+\alpha$ becomes $\beta^2+2\alpha^2+\beta$ which is not the same as $\alpha^2+2\beta^2+\alpha$.)

Every symmetric function of α and β can be expressed in terms of $\alpha+\beta$ and $\alpha\beta$. Since we knowthe values of $\alpha+\beta$ and $\alpha\beta$ in terms of the coefficients of the equation having α, β as its roots, we can find the value of every symmetric function of the roots of a quadratic equation in terms of its coefficients. The following examples will illustrate the method.

Example 27. *If α, β be the roots of the equation $x^2-2x-1=0$, find the value of (i) $\alpha^2+\beta^2$, (ii) $(\alpha-\beta)^2$, (iii) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$.*

Solution. Since α, β are the roots of $x^2-2x-1=0$, therefore,

$$\alpha+\beta=2,\ \alpha\beta=-1.$$

Now (i) $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=2^2-2(-1)=6.$

(ii) $(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta=2^2-4(-1)=8.$

(iii) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}=\frac{2^2-2(-1)}{-1}=-6.$

Example 28. *If α, β be the roots of the equation $x^2-px+q=0$, find the value of $\left(\alpha+\frac{1}{\beta}\right)^2+\left(\beta+\frac{1}{\alpha}\right)^2$.*

Solution. Since α, β are the roots of $x^2-px+q=0$, therefore, $\alpha+\beta=p$, $\alpha\beta=q$.

Now $\left(\alpha+\frac{1}{\beta}\right)^2+\left(\beta+\frac{1}{\alpha}\right)^2=\frac{(\alpha\beta+1)^2}{\beta^2}+\frac{(\alpha\beta+1)^2}{\alpha^2}$,

$$=(\alpha\beta+1)^2\left(\frac{1}{\beta^2}+\frac{1}{\alpha^2}\right),$$

$$=\frac{(\alpha\beta+1)^2(\alpha^2+\beta^2)}{\alpha^2\beta^2},$$

$$=\frac{(\alpha\beta+1)^2\{(\alpha+\beta)^2-2\alpha\beta\}}{(\alpha\beta)^2},$$

$$=\frac{(q+1)^2(p^2-2q)}{q^2}.$$

Example 29. *If α, β be the roots of the equation*
$$ax^2+bx+c=0,$$
find the value of $(\alpha+k)(\beta+k)$.

Solution. Since α, β are the roots of $ax^2+bx+c=0$,

therefore, $\alpha+\beta=-\frac{b}{a}$,

$$\alpha\beta=\frac{c}{a}.$$

Now $(\alpha+k)(\beta+k)=\alpha\beta+k(\alpha+\beta)+k^2$,

$$=\frac{c}{a}-k\,\frac{b}{a}+k^2,$$

$$=\frac{ak^2-bk+c}{a}.$$

EXERCISE 4 (*f*)

1. If α, β be the roots of the equation $x^2-3x+1=0$, find the value of (*i*) $\alpha^2+\beta^2$, (*ii*) $\alpha^3+\beta^3$, (*iii*) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$, (*iv*) $\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}$ (*v*) $(\alpha+1)(\beta+1)$.
2. If α, β be the roots of the equation $x^2-px+q=0$, find the value of $\alpha^3+\beta^3$.
3. If α, β be the roots of the equation $px^2+qx+r=0$, find the value of $(p\alpha+q)^{-2}+(p\beta+q)^{-2}$.
4. If α, β be the roots of the equation $ax^2+bx+c=0$, find the value of $\alpha^4+\alpha^2\beta^2+\beta^4$.

5. If α, β be the roots of the equation $ax^2+bx+c=0$, find the value of $\left(\frac{\alpha}{\beta}-\frac{\beta}{\alpha}\right)^2$.

6. If α, β be the roots of $x^2-4x+1=0$, find the value of
$$\frac{2-\alpha}{4+\alpha}+\frac{2-\beta}{4+\beta}.$$

7. If p, q be the roots of $2x^2-6x+3=0$, find the value of
$$p^3+q^3-3pq(p^2+q^2)-3pq(p+q).$$

8. If α, β be the roots of the equation $x^2+qx+r=0$, find the value of

(i) $\frac{(q+\alpha)(q+\beta)}{(r-\alpha^2)(r-\beta^2)}$, (ii) $\alpha^3\beta+\beta^3\alpha$.

9. If α, β be the roots of the equation
$$x^2-(1+a^2)x+\tfrac{1}{2}(1+a^2+a^4)=0,$$
prove that $\alpha^2+\beta^2=a^2$.

10. Form the equation whose roots are the squares of the roots of the equation $x^2+x+1=0$.

11. If α, β be the roots of the equation $x^2-2x+5=0$, find the equation whose roots are α^2, β^2.

12. If α, β be the roots of the equation $x^2-2x+3=0$, form the equation whose roots are $\frac{1}{\alpha}$, $\frac{1}{\beta}$.

13. If α, β be the roots of the equation $2x^2+3x-6=0$, form the equation whose roots are α^2+2, β^2+2.

14. If α, β be the roots of the equation $x^2-x+1=0$, form the equation whose roots are α^3, β^3.

15. If α, β be the roots of the equation $2x^2-3x+5=0$, form the equation whose roots are $\frac{\alpha}{\alpha^2+\beta^2}$, $\frac{\beta}{\alpha^2+\beta^2}$.

16. If x_1, x_2 be the roots of the equation $ax^2+bx+c=0$, form an equation whose roots are $(x_1-x_2)^2$, $(x_1+x_2)^2$.

17. If α, β be the roots of the equation $x^2-px+q=0$, form the equation whose roots are $2\alpha-\beta$, $2\beta-\alpha$.

18. If α, β be the roots of the equation $3x^2+2x+1=0$, form the equation whose roots are $\frac{1-\alpha}{1+\alpha}$, $\frac{1-\beta}{1+\beta}$.

19. If α, β are the roots of the equation $x^2-ax+b=0$, form the equation whose roots are $\frac{1}{\alpha+\beta}$, $\frac{1}{\alpha\beta}$.

20. If α, β be the roots of $x^2-3x+1=0$, form the equation whose roots are $\frac{\alpha}{1+\beta}, \frac{\beta}{1+\alpha}$.

21. If α, β be the roots of $ax^2+bx+c=0$, form the equation whose root are $\alpha+\frac{1}{\alpha}, \beta+\frac{1}{\beta}$.

22. Form the equation whose roots exceed by p the roots of the equation $ax^2+bx+c=0$.

23. If α, β be the roots of the equation $x^2-px+q=0$, form the equation whose roots are $\frac{1}{a\alpha+b}, \frac{1}{a\beta+b}$.

TEST YOUR UNDERSTANDING IV

In each of the following problems, four alternatives are given out of which only one is correct. Place a tick mark (√) against the correct alternative :

1. The roots of the equation $x^2+x-110=0$ are
   (*a*) 10, 11 (*b*) 10, -11 (*c*) -10, 11 (*d*) -10, -11.
2. If 3 is a root of the equation $x^2-8x+k=0$, the value of k is
   (*a*) 9 (*b*) -15 (*c*) 15 (*d*) 24.
3. If α, β be the roots of the equation $x^2-2x+3=0$, the value of $\alpha^2+\beta^2$ is
   (*a*) -4 (*b*) 4 (*c*) 2 (*d*) -2.
4. If the sum of the roots of the equation $x^2+px+q=0$ is zero, the value of p is
   (*a*) 3 (*b*) -1 (*c*) 0 (*d*) 2.
5. The roots of the equation $x^2-7x+8=0$ are
   (*a*) rational (*b*) imaginary (*c*) irrational (*d*) equal.
6. If α is one root of the equation $x^2+x+1=0$, the other root is
   (*a*) $-\alpha$ (*b*) 2α (*c*) $1/\alpha$ (*d*) $1-\alpha$.
7. If the roots of the equation $x^2+mx+36=0$ are equal, then a possible value of m is
   (*a*) 10 (*b*) 12 (*c*) 9 (*d*) 36.
8. If α, β be the roots of the equation $x^2-8x+15=0$, then a possible value of $\alpha+2\beta$ equals
   (*a*) 13 (*b*) 14 (*c*) 15 (*d*) 16.
9. The product of the roots of the equation $x^2+x+m^2-1=0$ is 8. The possible values of m are
   (*a*) -1, 1 (*b*) -9, 9 (*c*) -3, 3 (*d*) -8, 8.

10. The quadratic equation with real coefficients having $2-3i$ as one of its roots is

(a) $x^2+4x-5=0$ (b) $x^2-4x+5=0$

(c) $x^2-4x-13=0$ (d) $x^2-4x+13=0$.

REVIEW EXERCISE IV

Solve the following equatians :

1. $(b+c-2a)x^2+(c+a-2b)x+(a+b-2c)=0$.
2. $(3-2\sqrt{2})x^2-(\sqrt{2}-1)x-12=0$.
3. $\sqrt{\dfrac{x}{x-1}}+\sqrt{\dfrac{x-1}{x}}=2\dfrac{1}{6}$.
4. $3x^{\frac{1}{2n}}-x^{\frac{1}{n}}-2=0$.
5. $(x-3)(x-2)(x-6)(x-4)=12x^2$.
6. $x^5-13x^4+36x^3+36x^2-13x+1=0$.
7. $x^6-x^5+x^4-x^2+x-1=0$.
8. $x^5-5x^3+5x^2-1=0$.
9. $\sqrt{2x+8}+\sqrt{x+5}=7$.
10. $\sqrt{(2x^2-7x+1)}-\sqrt{(2x^2-9x+4)}=-1$.
11. $(5+2\sqrt{6})^{x^2-3}+(5-2\sqrt{6})^{x^2-3}=10$. (*I.I.T., J E E*, 1985)
12. Form the equation whose roots are $\dfrac{1}{2+\sqrt{3}}$ and $\dfrac{1}{2-\sqrt{3}}$.
13. Form the equation whose roots are $\dfrac{-i\pm\sqrt{3}}{2}$.
14. If the roots of the equation
$$(x-a)(x-b)-k=0$$
are c and d, prove that the roots of the equation
$(x-c)(x-d)+k=0$ are a and b.
15. If α, β be the roots of the equation $x^2+x+1=0$, form the equation whose roots are α/β and β/α.
16. Find the condition that one root of the equation $ax^2+bx+c=0$ may be four times the square of the other.
17. If one root of the equation $x^2+x+1=0$ is α, prove that the other root is α^2.
18. If one root of the equation $4x^2+2x-1=0$ is α, prove that the other root is $4\alpha^3-3\alpha$.
19. Show that the roots of the equation
$$\frac{ax-b}{bx-a}=\frac{px-q}{qx-p}$$
are equal in magnitude but opposite in sign.

20. For what values of m will the equation

$$(3+2m)x^2-2(1+3m)x+7=0$$

have equal roots ?

SUMMARY

1. The roots of the quadratic equation $ax^2+bx+c=0$ (where $a\neq 0$) are given by,

$$x=\frac{-b\pm\sqrt{(b^2-4ac)}}{2a}.$$

2. If α, β be the roots of the quadratic equation $ax^2+bx+c=0$, (where $a\neq 0$), then

$$\alpha+\beta=-b/a,\ \alpha\beta=c/a.$$

3. The roots of the quadratic equation $ax^2+bx+c=0$, where $a, b, c \in \mathbf{R}$, are real and unequal if $b^2-4ac>0$, real and equal if $b^2-4ac=0$, and imaginary if $b^2-4ac<0$.
4. The roots of the quadratic equation $ax^2+bx+c=0$, where $a, b, c \in \mathbf{Q}$, are rational if b^2-4ac is the square of a rational number.
5. If one root of the quadratic equation $ax^2+bx+c=0$ where $a, b, c \in \mathbf{Q}$, is $p+\sqrt{\ }$ not being the square of a rational number, the other root is $p-\sqrt{\ }$.
6. The roots of the quadratic equation $ax^2+bx+c=0$ where $a, b, c \in \mathbf{R}$, and $b^2-4ac<0$ are conjugate complex numbers.

HISTORICAL NOTE

Interest in the study of quadratic equations goes back to Babylonians some 4000 years ago. Clay-tablets, known as *Yale tablets* and dating back to 1600 B.C. contain several unsolved problems involving quadratic equations. The Greeks also made use of quadratic equations in solving geometrical problems.

Hindu mathematicians made notable contributions to the solution of quadratic equations. *Aryabhata* (born in 476 A.D.) gave a rule to solve geometric series which required solution of quadratic equations. *Brahmagupta* (born in 598 A.D.) in his well-known work 'Brahma Sphuta Sidhanta' gave a rule for solving the quadratic equation $x^2+px-q=0$ which resembles the modern formula and gives one root correctly. *Sridhara* (11th Century A.D.) was the first Hindu mathematician to give the 'method of completing the square' for solving quadratic equations. The formula given by him is substantially the modern formula.

The Arab mathematicians *Al-khowarizmi* (around 805 A.D.) and *Omar Khayyam* (around 1100 A.D.) gave rules for solving quadratic equations.

The first systematic and detailed study of quadratic equations is found in the work of *Thomas Hariot* (born in 1560 A.D.). His treatise *Artis Analyticae Praxis* published posthumously in 1631 A.D., contains discussion of solution of quadratic equations, formation of quadratic equations with given roots, relation between the roots and coefficients of an equation, and transformation of a quadratic equation into another.

□□

CARL FRIEDRICH GAUSS (1777-1855)

Carl Friedrich Gauss, the prince of mathematicians, was born at Brunswick on April 30, 1777 in a poor family. He had shown signs of greatness while he was still at school. As the story goes, oneday, in order to keep the class occupied, the teacher asked all the children to add up all the numbers from 1 to 100, with instructions that each one should place the slate as soon as he had completed the task. Carl placed his slate on the table almost immediately, saying," There it is." The teacher looked at him scornfully while the others kept on working diligently. When the teacher checked the results, Gauss's was the only one to have the correct answer, 5050, without any calculations whatever. The ten-year old boy had obviously calculated the sum of the A.P. $1+2+3+...+100$. Such was Gauss. When he was only nineteen, he proved that a seventeen-sided regular polygon can be constructed by means of a ruler and compass. He also proved the remarkable result that every polynomial can be resolved into real factors of the first or second degree. The complex plane is often called the Gaussian plane. In 1807 Gauss was made professor and director of the obser-vatory at the University of Gottingen, a position which he held until his death on February 23, 1855. In his honour the king of Hanover ordered a medal which hailed him as the prince of mathematicians. In 1880, a memorial column whose base was a regular polygon of 17 sides was erected in his memory in Brunswick.

Gauss was one of the greatest mathematicians of all times. He is often compared with Archimedes and Newton.

CHAPTER 5

Sequences and Series

[*Throughout the present chapter the letters a, b, c, d,* $t_1, \cdots\cdots, t_n$ *stand for real numbers and n stands for a natural number, unless stated otherwise.*]

5·1. SEQUENCES

Any succession of numbers $t_1, t_2, \cdots\cdots, t_n$ such that to each positive integer $m \leqslant n$, there corresponds a number t_m, is called a **finite sequence**. Also, a succession of numbers $t_1, t_2, \ldots\ldots t_n, \cdots\cdots$ such that to each positive integer n there corresponds a number t_n, is called an **infinite sequence**. The number of terms is finite in the case of a finite sequence, but is not so in the case of an infinite sequence. A characteristic difference between finite and infinite sequences is that whereas every finite sequence has a last term, an infinite sequence has no last term.

Consider the following examples :

(*i*) 1, 3, 5, 7.

(*ii*) 7, 5, 3, 1.

(*iii*) 2, 4, 8, 16, 32, 64.

(*iv*) $1, \frac{1}{2}, \frac{1}{3}, \cdots\cdots, \frac{1}{10}$.

(*v*) $1, 2, 3, 4, 5, 6, \cdots\cdots, n, \cdots$

(*vi*) $x, x^2, x^3, \ldots\ldots x^n, \cdots\cdots$, where x is any fixed real number.

(*vii*) $1, 0, 1, 0, 1, 0, 1, 0, \cdots\cdots$

In the above examples the first four sequences are all finite sequences. The last three sequences are all infinite sequences, none of them possessing a last term.

It is not necessary that all the terms of a sequence be distinct. For example, the sequence (*vii*) is an infinite sequence whose members take only two distinct values, namely 0 and 1.

The order which the various members of a sequence occupy is very important. Though the sequences (*i*) and (*ii*) consist of the same set of numbers 1, 3, 5, 7, yet the two sequences are distinct because the various members do not occupy the same place in the two sequences. If all the members of a sequence are real numbers,

the sequence is said to be a *real sequence*. Throughout the remainder of this book we shall deal with real sequences only and use the word sequence so as to mean a real sequence only.

5·2. ARITHMETIC PROGRESSION

A sequence is said to be an **Arithmetic Sequence** or **Arithmetic Progression** (written briefly as A.P.) if the difference between any term and the one that precedes it is the same throughout. For example, the sequences (*i*), (*ii*) and (*v*) are arithmetic progressions. In (*i*) every term exceeds the preceding term by 2, in (*ii*) every term exceeds the preceding term by -2 and in (*v*) every term exceeds the preceding term by 1. The excess of any term of an A.P. over the preceding term is called the **common difference** of the A.P. In examples (*i*), (*ii*) and (*v*), the common differences are 2, -2 and 1 respectively.

In an A.P. any term may be obtained by adding the common difference to the term that precedes it. Therefore, if any one term and the common difference of an A.P. be known, every term can be written out, *i.e.*, the A.P. is then completely known. In particular, if the first term and the common difference is known, then the A.P. is completely known. The first term and the common difference of an A.P. are generally denoted by a and d respectively.

The A.P. whose first term is a and common difference is d is

$$a,\ a+d,\ a+2d,\ a+3d, \ldots\ldots$$

5·3. TO FIND THE nTH TERM OF THE A.P. WHOSE FIRST TERM AND COMMON DIFFERENCE ARE GIVEN

Let a be the first term and d the common difference.

The second term$=a+d=a+(2-1)\,d$.

the third term$=a+2d=a+(3-1)\,d$,

the fourth term$=a+3d=a+(4-1)\,d$.

From the above pattern we can guess that the nth term must be $a+(n-1)\,d$. We shall prove that this is indeed the case by mathematical induction.

Let T(n) be the statement '*The nth term, say t_n, of the A.P. having a as, its first term and d as the common difference is $a+(n-1)\,d$.*

Step 1. T(1) is true. For, $t_1=a+(1-1)\,d=a$, which is true.

Step 2. Let T(k) be true, *i.e.*, let $t_k=a+(k-1)\,d$.

Then $t_{k+1}=t_k+d=[a+(k-1)\,d]+d$,

$$=a+kd=a+(k+1-1)\,d,$$

showing that T $(k+1)$ is true.

By PFI it follows that $t_n=a+(n-1)\,d$ for every positive integer n.

Thus we find that *the nth term of the A.P. whose first term is a and common difference is d, is* $a+(n-1)\,d$.

Example 1. *Find the 10th term of the A.P. 2, 7, 12,......*

Solution. Here $a=2$, $d=5$, $n=10$.

$$t_{10}=2+(10-1).\,5=47.$$

Example 2. *Which term of the A.P. 21, 18, 15,......is −81 ?*

Solution. Let the nth term be −81.

The first term = 21,

common difference = −3.

$$t_n=21+(n-1)\,(-3)=24-3n.$$

$$\therefore \qquad -81=24-3n,$$

or $$3n=24+81=105,$$

or $$n=35.$$

Thus −81 is the 35th term of the given A.P.

Example 3. *If the third term of an A.P. is 5 and the seventh term is 9, find the 17th term.*

Solution. Let a be the first term and d the common difference. Then

$$t_3=a+2d=5, \qquad ...(i)$$

$$t_7=a+6d=9. \qquad ...(ii)$$

Subtracting (*i*) from (*ii*), we have

$$4d=4 \quad \text{or} \quad d=1. \qquad ...(iii)$$

Substituting the value of d in (*i*), we have

$$a+2=5 \quad \text{or} \quad a=3. \qquad ...(iv)$$

$$t_{17}=a+16d=3+16.1=19.$$

Hence the 17th term = 19.

EXERCISE 5 (*a*)

1. Find the 7th, 15th and $(r-1)$th terms of the following sequences :

   (*i*) 3, 6, 9, 12,...... (*ii*) −4, −8, −12, −16,......

   (*iii*) $\frac{13}{5}, \frac{7}{5}, \frac{1}{5}, -1,\cdots$ (*iv*) 17, 11, 5, −1,......

2. Which term of the A.P. 8, 12, 16, 20,......is 48 ?

3. Which term of the A.P. 3, 8, 13, 18,......is 78 ?

4. How many terms are there in the sequence 7, 13, 19,..., 205 ?

5. The 6th term of an A.P. is 16 and the 14th term is 32. Determine the 36th term.

6. The 4th term of an A.P. is -4 and the 8th term is 12. Find the first twelve terms of the A.P.

7. If the 3rd and 9th terms of an A.P. be 4 and -8 respectively, which term is zero ?

8. An A.P. consists of 60 terms. If the first and the last terms be 7 and 125 respectively, find the 31st term.

9. An A.P. consists of 50 terms of which the 3rd term is 12 and the last term is 106. Find the 29th term of the A.P.

10. The sum of the 4th and 8th terms of an A.P. is 24, and the sum of the 6th and 10th terms is 34. Find the first three terms of the A.P. Also find the rth term.

11. Prove that in an A.P., $t_p+t_{p+2n}=2t_{n+p}$.

12. Is 301 a term of the sequence 5, 11, 17, 23,...?

13. The pth term of an A.P. is q, and the qth term is p. Show that the rth term is $p+q-r$.

5·4. TWO IMPORTANT THEOREMS

Here below we state and prove two important theorems relating to Arithmetic Progressions.

Theorem 1. *If the same number is added to each term of an A.P., the resulting sequence is also an A.P.*

Proof. Let the given A.P. be $a, a+d, a+2d, \ldots$

Adding k to each term of the given A.P., it becomes

$$a+k, (a+d)+k, (a+2d)+k, \ldots$$

i.e., $$a+k, (a+k)+d, (a+k)+2d, \ldots$$

which is an A.P. having $a+k$ as the first term and d as the common difference.

In the above discussion, k may be positive or negative.

Theorem 2. *If every term of an A.P. be multiplied by the same number, the resulting sequence is also an A.P.*

Proof. Let the given A.P. be

$$a, a+d, a+2d, \cdots\cdots$$

Multiplying every term by k, we have the sequence

$$ka, k(a+d), k(a+2d), \cdots\cdots$$

i.e., $$ka, ka+kd, ka+2(kd), \cdots\cdots$$

which is an A.P. having ka as the first term and kd as the common difference.

Corollary. If $x, y, z, \cdots\cdots$ be in A.P., then $kx+s, ky+s, kz+s \cdots$ are also in A.P. For,

$x, y, z, \ldots\ldots$ being in A.P., by Theorem 2,

$kx, ky, kz, \ldots\ldots$

are also in A.P. Therefore, by Theorem 1, $kx+s, ky+s, kz+s, \ldots\ldots$ are also in A.P.

Example 4. *If $a(b+c)$, $b(c+a)$, $c(a+b)$ are in A.P., and a, b, c are all different from zero, show that a^{-1}, b^{-1} and c^{-1} are also in A.P.*

Solution. Since $a(b+c), b(c+a), c(a+b)$ are in A.P.,

$\therefore \quad -a(b+c), -b(c+a), -c(a+b)$ are also in A.P.

Adding $ab+bc+ca$ throughout, we have

$(ab+bc+ca)-a(b+c), (ab+bc+ca)-b(c+a), (ab+bc+ca)$ $-c(a+b)$ are also in A.P.

i.e., $\qquad bc, ca, ab$ are also in A.P.

Dividing throughout by abc, we have

a^{-1}, b^{-1}, c^{-1} are also in A.P.

Example 5. *If a, b, c are in A.P., show that $a^2(b+c), b^2(c+a)$, $c^2(a+b)$ are also in A.P.*

Solution. $a^2(b+c), b^2(c+a), c^2(a+b)$ are in A.P.

if and only if $b^2(c+a)-a^2(b+c)=c^2(a+b)-b^2(c+a)$,

i.e., if and only if $(b-a)(ab+bc+ca)=(c-b)(ab+bc+ca)$,

i.e., if and only if $b-a=c-b$,

i.e., if and only if a, b, c are in A.P.

Since a, b, c are given to be in A.P., therefore,

$a^2(b+c), b^2(c+a), c^2(a+b)$ are also in A.P.

Example 6. *If $\dfrac{1}{b+c}, \dfrac{1}{c+a}, \dfrac{1}{a+b}$ are in A.P., show that a^2, b^2, c^2 are also in A.P.*

Solution. Since $\dfrac{1}{b+c}, \dfrac{1}{c+a}, \dfrac{1}{a+b}$ are in A.P.,

$$\therefore \qquad \frac{1}{c+a}-\frac{1}{b+c}=\frac{1}{a+b}-\frac{1}{c+a},$$

$$\text{or} \qquad \frac{b-a}{(b+c)(c+a)}=\frac{c-b}{(a+b)(c+a)},$$

$$\text{or} \qquad b^2-a^2=c^2-b^2.$$

Hence a^2, b^2, c^2 are in A.P.

Aliter. Since $\dfrac{1}{b+c}, \dfrac{1}{c+a}, \dfrac{1}{a+b}$ are in A.P., multiplying throughout by $(b+c)(c+a)(a+b)$, we find that $(c+a)(a+b)$, $(b+c)(a+b)$, $(b+c)(c+a)$ are also in A.P.

i.e., $a^2+ab+bc+ca$, $b^2+ab+bc+ca$, $c^2+ab+bc+ca$ are also in A.P. Subtracting $ab+bc+ca$ from each term we find that a^2, b^2, c^2 are also in A.P.

Example 7. *Find three numbers in A.P. whose sum is 30 and whose product is 990*

Solution. Let the three numbers in A.P. be $a-d$, a, $a+d$.

Then $\quad (a-d)+a+(a+d)=30,$

i.e., $\quad 3a=30,$

or $\quad a=10, \quad \ldots(i)$

Also, $\quad a\,(a-d)\,(a+d)=990. \quad \ldots(ii)$

Substituting the value of a, we have

$$10\,(10-d)\,(10+d)=990,$$

or $\quad 100-d^2=99,$

or $\quad d^2=1,$

or $\quad d=\pm 1.$

Taking $d=1$, the numbers are $10-1$, 10, $10+1$, *i.e.*, 9, 10, 11.

Taking $d=-1$, the numbers are $10+1$, 10, $10-1$, *i.e.*, 11, 10, 9.

Hence the required numbers are 9, 10 and 11.

Note. We could have taken the three numbers to be a, $a+d$, $a+2d$, but as we have seen above, the working becomes simpler if we take the numbers to be $a-d$, a, $a+d$.

Example 8. *Find four numbers in A.P., whose sum is 20 and the sum of whose squares is 120.*

Solution. Let the numbers be $a-3d$, $a-d$, $a+d$, $a+3d$.

Then $\quad (a-3d)+(a-d)+(a+d)+(a+3d)=20 \quad \ldots(i)$

and $\quad (a-3d)^2+(a-d)^2+(a+d)^2+(a+3d)^2=120. \quad \ldots(ii)$

From (i), we have

$$4a=20, \text{ i.e., } a=5. \quad \ldots(iii)$$

From (ii), we have $4a^2+20d^2=120,$

or $\quad a^2+5d^2=30,$

or $\quad 5d^2=30-25,$

or $\quad d^2=1.$

$\therefore \quad d=\pm 1. \quad \ldots(iv)$

Taking $d=1$, the numbers are $5-3$, $5-1$, $5+1$, $5+3$, *i.e.*, 2, 4, 6, 8.

Taking $d=-1$, the numbers are $5+3$, $5+1$, $5-1$, $5-3$, *i.e.*, 8, 6, 4, 2.

Hence the required numbers are 2, 4, 6 and 8.

Note. We could have taken the four numbers in A.P. to be a, $a+d$, $a+2d$, $a+3d$, but as we have seen above, the working

becomes simplified if we take the numbers as $a-3d$, $a-d$, $a+d$, $a+3d$, which also form an A.P., but with $a-d$ as the first term and $2d$ as the common difference.

EXERCISE 5 (b)

1. If a, b, c are in A.P., prove that $(b+c)^2-a^2$, $(c+a)^2-b^2$ and $(a+b)^2-c^2$ are also in A.P.

2. If p^2, q^2, r^2 be in A.P., prove that $\frac{1}{q+r}$, $\frac{1}{r+p}$, and $\frac{1}{p+q}$ are also in A.P.

3. If a, b, c are in A.P., show that $\frac{ab+ac}{bc}$, $\frac{bc+ab}{ca}$ and $\frac{ca+cb}{ab}$ are in A.P.

4. If a, b, c be in A.P., show that $(b+c)^2-bc$, $(c+a)^2-ca$, $(a+b)^2-ab$ are also in A.P.

5. Find three numbers in A.P. whose sum is 18 and product is 192.

6. Find three numbers in A.P. whose sum is 24 and the sum of whose squares is 200.

7. Find three numbers in A.P. whose sum is 30 and the greatest of which is to the least as 3 : 1.

8. Divide 28 into four parts in A.P. such that the product of the extremes is to the product of the means as 5 : 6.

9. Find five numbers in A.P. whose sum is 25, and the sum of whose squares is 135.

10. Find four numbers in A.P. whose sum is 20 and the sum of whose squares is 120.

5·5. ARITHMETIC MEANS

If three numbers are in A.P., the middle one is called the **arithmetic mean** (A.M.) of the other two. For example, 4 is the arithmetic mean of 2 and 6; also, 5 is the arithmetic mean of 3 and 7.

If $n+2$ numbers $a, A_1, A_2, \ldots, A_n, b$ be in A.P., the numbers $A_1, A_2, \ldots, A_n$ are said to be the n arithmetic means between the numbers a and b. For example, 4, 6, 8, 10, 12 are five arithmetic means between 2 and 14.

5·5·1. To Find the Arithmetic Mean Between Two Given Numbers a and b

Let A be the arithmetic mean between a and b. Then a, A, b must be in A.P.

$\therefore \qquad A-a=b-A,$

or $\qquad A=\frac{1}{2}(a+b).$

Hence the A.M. of the numbers a and b is $\frac{1}{2}(a+b)$.

Example 9. *Find the arithmetic mean of 15 and 25.*

Solution. Let A be the arithmetic mean of 15 and 25. Then the numbers 15, A, 25 are in A.P.

$\therefore \qquad A-15=25-A,$

or $\qquad 2A=40,$

or $\qquad A=20.$

Thus the arithmetic mean of 15 and 25=20.

5·5·2. To Insert n Arithmetic Means Between the Numbers a and b

Let $A_1, A_2, \dots, A_n$ be the n arithmetic means between a and b. Then $a, A_1, A_2, \dots, A_n, b$ must be in A.P. If d be the common difference of this A.P., we have

$$t_{n+2}=a+\overline{(n+2}-1)d=b.$$

$$\therefore \qquad d=\frac{b-a}{n+1}.$$

$$\text{Now } A_1=t_2=a+d=a+\frac{b-a}{n+1}=\frac{na+b}{n+1},$$

$$A_2=t_3=a+2d=a+\frac{2(b-a)}{n+1}=\frac{(n-1)a+2b}{n+1},$$

$$A_3=t_4=a+3d=a+\frac{3(b-a)}{n+1}=\frac{(n-2)a+3b}{n+1},$$

$$\vdots \qquad \vdots$$

$$A_n=t_{n+1}=a+nd=a+\frac{n(b-a)}{n+1}=\frac{a+nb}{n+1}$$

Hence the required means are

$$\frac{na+b}{n+1}, \frac{(n-1)\,a+2b}{n+1}, \dots\dots, \text{ and } \frac{a+nb}{n+1}.$$

Example 10. *Insert 4 arithmetic means between 5 and 6.*

Solution. Let A_1, A_2, A_3, A_4 be the required arithmetic means. Then 5, A_1, A_2, A_3, A_4, 6 must be in A.P.

If d is the common difference of this A.P., we have

$$t_6=5+5d=6.$$

$$\therefore \qquad d=\frac{1}{5}.$$

$$\therefore \qquad A_1=5+d=5+\frac{1}{5}=\frac{26}{5},$$

$$A_2=5+2d=5+\frac{2}{5}=\frac{27}{5},$$

$$A_3=5+3d=5+\frac{3}{5}=\frac{28}{5},$$

$$A_4=5+4d=5+\frac{4}{5}=\frac{29}{5}.$$

Hence the required arithmetic means are

$$\frac{26}{5}, \frac{27}{5}, \frac{28}{5} \text{ and } \frac{29}{5}.$$

EXERCISE 5 (*c*)

1. Find the arithmetic mean of (*i*) -5 and 41,
(*ii*) $3p-2q$ and $3p+2q$, (*iii*) $(a+b)^2$ and $(a-b)^2$.
2. Insert three arithmetic means between 5 and 29.
3. Insert five arithmetic means between $4\frac{1}{2}$ and $1\frac{1}{2}$.
4. Insert 8 arithmetic means between $5r-4s$ and $5s-4r$.
5. Insert n arithmetic means between n^2 and 1.
6. There are n arithmetic means between 3 and 35 such that
$$6\text{th mean} : (n-3)\text{th mean}=5 : 9.$$
Find n.
7. There are p arithmetic means between 7 and 35 such that
$$(p-3)\text{th mean} : p\text{th mean}=11 : 24$$
Find p.
8. Find the value of n for which
$$\frac{a^n+b^n}{a^{n-1}+b^{n-1}}$$
is the arithmetic mean between a and b.

5.6. FINITE SERIES

The symbol $t_1+t_2+\cdots\cdots+t_n$ is called a finite series. It is said to be the series corresponding to the sequence $t_1, t_2, \cdots\cdots, t_n$. Thus, to each finite sequence there corresponds a finite series. The numbers $t_1, t_2, \cdots\cdots, t_n$, which are respectively the first, second, ···and nth terms of the sequence $t_1, t_2, \cdots\cdots, t_n$, are also called the first, second,······and nth terms respectively of the series $t_1+t_2+\cdots\cdots+t_n$. The sum of all the terms of a series is called the sum of the series. In other words, the sum of the series $t_1+t_2\cdots+t_n$ is the sum of the numbers $t_1, t_2, \cdots\cdots$ and t_n. The sum of the series $t_1+t_2+\cdots\cdots+t_m$ is generally denoted by the symbol $\sum_{n=1}^{m} t_n$. The Greek letter Σ is read

as sigma and stands for sum. When there is no danger of ambiguity, we may also write $\Sigma\, t_n$ in place of $\sum_{n=1}^{m} t_n$. If the sequence $t_1, t_2, \cdots\cdots, t_n$ be an A.P., the corresponding series $t_1+t_2+\cdots\cdots+t_n$ is said to be a series in A.P. Thus

(*i*) $1+3+5+\cdots\cdots+11$,

(*ii*) $1+2+3+\cdots\cdots+n$,

(*iii*) $(-5)+(-3)+(-1)+1+3+5+7$,

(*iv*) $a+(a+d)+(a+2d)+\cdots\cdots+(a+\overline{n-1}d)$

are all examples of series in A.P.

The series (*iii*) is generally written as

$$-5-3-1+1+3+5+7.$$

The *n*th term of the series

$$a+(a+d)+(a+2d)\cdots$$

in A.P. is the same as the *n*th term of sequence

$$a, a+d, a+2d, \cdots\cdots$$

and is, therefore, $a+\overline{n-1}\, d$.

5·7. TO FIND THE SUM OF n TERMS OF A GIVEN A. P.

Let a, d, l and n be respectively the first term, common difference, last term and number of terms of an A.P. The A.P. is then

$$a+(a+d)+\cdots+l.$$

If S_n denotes the sum of n terms of the A P., we have

$$S_n=a+(a+d)+\cdots+(l-d)+l. \qquad \ldots(i)$$

Writing the series (*i*) in the reverse order, we have

$$S_n=l+(l-d)+\cdots\cdots+(a+d)+a. \qquad \ldots(ii)$$

Adding (*i*) and (*ii*) we have

$$2S_n=(a+l)+(a+l)+\cdots\cdots \text{to } n \text{ terms},$$
$$=n(a+l).$$

$$\therefore \quad S_n=\frac{n}{2}(a+l). \qquad \ldots(iii)$$

Also, $\quad l=a+(n-1)d. \qquad \ldots(iv)$

From (*iii*), we have

$$S_n=\frac{n}{2}\{a+a+\overline{n-1}d\}.$$

Hence $\boxed{S_n=\dfrac{n}{2}\{2a+\overline{n-1}d\}.} \qquad \ldots(v)$

Example 11. *Find the sum of 10 terms of the A.P.*

$$4+6+8+\ldots\ldots$$

Solution. Here $a=4$, $d=2$, $n=10$.

$$S_{10}=\frac{10}{2}\{2.4+(10-1)2\}=130.$$

Example 12. *Find the sum of 20 terms of an A.P. whose first term is 3 and the last term is 57.*

Solution. Here $a=3, l=57, n=20$.

Since $$S_n=\frac{n}{2}(a+l),$$

$\therefore$ $$S_{20}=\frac{20}{2}(3+57)=600.$$

Hence the sum of 20 terms is 600.

Example 12. *How many terms of the series $18+15+\cdots$ when added together will amount to 45 ? Explain the double answer.*

Solution. The given series is an A.P. whose first term$=18$ and common difference$=-3$.

Let the sum of n terms be 45.

$$S_n=\frac{n}{2}[2\cdot18+(n-1)(-3)].$$

$$=\frac{n}{2}(39-3n).$$

$\therefore$ $$\frac{n}{2}(39-3n)=45,$$

or $$n^2-13n+30=0,$$

or $$(n-3)(n-10)=0.$$

$\therefore$ $$n=3 \text{ or } n=10.$$

Hence the sum of 3 terms$=$the sum of 10 terms$=45$.

Here the common difference is negative. Therefore, some terms of the A.P. are negative. The sum of terms from the 4th to the 10th is zero. It is for this reason that we get a double answer.

Example 14. *How many terms of the A.P. $-9,-6,-3\ldots$, must be added together so that the sum may be 66 ?*

Solution. Let 66 be the sum of n terms.

Here $a=-9$, $d=3$.

$$S_n=\frac{n}{2}[2(-9)+(n-1)\,.\,3]=\frac{n}{2}(3n-21).$$

$\therefore$ $$66=\frac{n}{2}(3n-21).$$

or $$n^2-7n-44=0,$$

or $$(n-11)(n+4)=0.$$

$\therefore$ $$n=11 \text{ or } -4.$$

Rejecting the negative value of n, we have $n=11$.

Hence the sum of 11 terms of the given A.P. is 66.

Note. The A.P. in the above problem is $-9, -6, -3, 0, 3, 6, 9, 12, 15, 18, 21$. If we begin with the last term and count four terms backwards the sum is also 66. Thus we see that though $n=-4$ is not an answer to the given problem, it can be given an intelligent interpretation.

Example 15. *How many terms of the series $26+21+16+\cdots$ are together equal to 80 ?*

Solution. Let n terms of the given series be together equal to 80.

The given series is an A.P. whose first term=26 and common difference=-5.

$$S_n=\frac{n}{2}[2.26+(n-1)(-5)]=\frac{n}{2}(57-5n).$$

$\therefore$ $$80=\frac{n}{2}(57-5n),$$

or $$5n^2-57n+160=0,$$

or $$(n-5)(5n-32)=0.$$

$\therefore$ $$n=5 \text{ or } \frac{32}{5}.$$

Rejecting the fractional value of n, we have $n=5$.

Hence the sum of 5 terms is 80.

Note. The value $\frac{32}{5}$ of n indicates that the sum of 6 terms of the given series is greater than 80 and the sum of 7 terms is less than 80.

EXERCISE 5 (*d*)

1. Sum the following series :

(*i*) $6\frac{1}{2}+7\frac{3}{4}+9+\cdots\cdots$ to 25 terms.

(*ii*) $-3+3+9+15+\cdots\cdots$ to 20 terms.

(*iii*) $\frac{7}{11}+\frac{13}{22}+\frac{6}{11}+\cdots\cdots$ to 29 terms.

(*iv*) $\frac{8}{7}+1+\frac{6}{7}+\frac{5}{7}+\cdots\cdots$ to 15 terms.

(v) $15+14\frac{1}{3}+13\frac{2}{3}+\ldots\ldots$ to 40 terms.

2. Find the sum of the following series which are in A.P. :

(i) $\frac{1}{n}+\left(1+\frac{1}{n}\right)+\left(2+\frac{1}{n}\right)+\ldots\ldots$ to n terms.

(ii) $(n-5)+(n-7)+(n-9)+\ldots\ldots$ to p terms.

(iii) $\frac{x-y}{x+y}+\frac{3x-2y}{x+y}+\frac{5x-3y}{x+y}+\ldots\ldots$ to n terms.

3. Sum the following series :

(i) $1+7+3+10+5+13+7+16+\ldots\ldots$ to 30 terms.

(ii) $3+4+8+9+13+14+18+19+\ldots\ldots$ to 20 terms.

(iii) $1+4-7+10+13-16+19+22-25+\ldots\ldots$ to $3p$ terms.

4. How many terms of the series $5+7+9+\ldots$ are together equal to 77 ?

5. How many terms of the A.P. 18, 16, 14, 12, ... are together equal to 78 ? Explain the double answer.

6. The sum of n terms of the series $2+5+8+\ldots$ is 950. Find n.

7. How many terms of the A.P. $15\frac{1}{4}$, $15\frac{1}{2}$, $15\frac{1}{4}$, ... are together equal to 504 ? Explain the double answer.

8. Find the sum of all natural numbers which are less than 100 and are multiples of 5.

9. Find the sum of all natural numbers which are multiples of 7 and lie between 200 and 400.

10. Find the sum of all natural numbers which lie between 200 and 600 and are multiples of 3 or 7 or both.

11. Find the sum of n terms of a sequence whose (i) rth term is $7r+5$ (ii) qth term is $aq+b$ (ii) $(p+3)$th term is $3p+4$.

[**Hint** : (i) Putting $r=1, 2, 3, 4, \ldots$, we find that the given sequence is 12, 19, 26, ... which is an A.P.]

12. If the sum of n terms of a series is $3n^2+6n$, find the first term and the common difference.

13. There are two series in A.P both having 3 as the first term. The common difference of the first series is twice as great as that of the second and the sum of the first ten terms of the first is greater than the sum of the same number of terms of the second by 9. Find the two series.

14. The ratio of the sum of n terms of two A.P.'s is $(3n+4)$: $(5n+6)$. Find the ratio of their 7th terms.

15. The ratio of the sum of n terms of two A.P.'s is $(2n+3):(3n+4)$. Find the ratio of their pth terms.

5·8. GEOMETRIC PROGRESSION

A sequence is said to be a *geometric progression* or G.P. (and the terms of a sequence are said to be in geometric progression) if the ratio of any term to the preceding term is the same throughout. For example

(*i*) $2, 4, 8, 16, 32, \ldots$

(*ii*) $3, -6, 12, -24, 48, \ldots$

(*iii*) $\frac{1}{4}, \frac{1}{12}, \frac{1}{36}, \frac{1}{108}, \frac{1}{324}, \ldots$

(*iv*) $\frac{1}{5}, \frac{1}{30}, \frac{1}{180}, \frac{1}{1080}, \frac{1}{6480}, \ldots$

(*v*) $x, x^2, x^3, x^4, x^5, \ldots$ (where x is any fixed real number) are all geometric progressions. The ratio of any term of (*i*) to the preceding term is 2. The corresponding ratios in (*ii*), (*iii*), (*iv*), and (*v*) are $-2, \frac{1}{3}, \frac{1}{6}$ and x respectively. The ratio of any term of a G. P. to the preceding term is called the **common ratio** of the G.P. Thus, in the above examples the common ratios are $2, -2, \frac{1}{3}, \frac{1}{6}$ and x respectively.

In a G. P. any term may be obtained by multiplying the preceding term by the common ratio of the G.P. Therefore, if any one term and the common ratio of a G.P. be known, any term can be written out *i.e.*, the G.P. is then completely known. In particular if the first term and the common ratio are known, the G. P. is completely known. The first term and the common ratio of a G. P. are generally denoted by a and r respectively. The G. P. whose first term is a and the common ratio is r is $a, ar, ar^2, \ldots$

5·9. TO FIND THE nth TERM OF THE G.P. WHOSE FIRST TERM AND THE COMMON RATIO ARE GIVEN

Let a be the first term and r the common ratio.

The second term $=ar=ar^{2-1}$.

The third term $=ar^2=ar^{3-1}$.

The fourth term $=ar^3=ar^{4-1}$.

From the above pattern we can guess that the nth term must be ar^{n-1}. We shall prove by mathematical induction that this is indeed the case.

Let T(n) be the statement, 'the nth term of the G. P. having a as its first term and r as its common ratio is ar^{n-1}'.

Step 1. T(1) is true. For, $t_1=a=ar^0=ar^{1-1}$.

Step 2. Let T(k) be true, *i.e.* let, $t_k=ar^{k-1}$.

Then $t_{k+1}=rt_k=r(ar^{k-1})=ar^{(k+1)-1}$, showing that T $(k+1)$ is true.

By PFI it follows that $t_n=ar^{n-1}$ for every positive integer n.

Thus the nth term of a G.P. whose first term is a and common ratio is r, is ar^{n-1}.

Example 16. *Find the 10th term of the G.P.* $\frac{1}{32}, \frac{1}{16}, \frac{1}{8}, \ldots$

Solution. Here $a=\frac{1}{32}, r=2$.

$$t_{10}=ar^9=\frac{1}{32}(2^9)=16.$$

Example 17. *Which term of the G.P. 5, 15, 45, ... is 3645 ?*

Solution. Let 3645, be the nth term.

The first term$=5$.

Common ratio$=3$.

$\therefore$ $$t_n=5.3^{n-1}$$

$\therefore$ $$5.3^{n-1}=3645,$$

or $$3^{n-1}=729=3^6$$

$\therefore$ $$n-1=6,$$

i.e., $$n=7.$$

Hence 3645 is the 7th term of the given G.P.

Example 18. *If the 3rd and the 7th terms of a G.P. be 15 and 135 respectively, find its first term and the common ratio.*

Solution. Let a be the first term and r the common ratio of the given G.P. Then

$$t_3=ar^2=15, \quad \ldots(i)$$

and $$t_7=ar^6=135. \quad \ldots(ii)$$

Dividing both sides of (ii) by those of (i), we have

$$r^4=9,$$

or $r=\pm\sqrt{3}$, considering real values of r only.

Substituting the value of r in (i), we have

$$3a=15,$$

i.e., $$a=5.$$

Hence the first term is 5 and the common ratio is $\pm\sqrt{3}$.

EXERCISE 5 (*e*)

1. Find the 10th term of the sequence 4, 12, 36, ...
2. Find the 5th term of the G.P. $2\frac{1}{2}$, 1...
3. Find the 8th term of the sequence $-\frac{1}{3}, 1, -3, 9, \cdots$
4. Find the pth term of the G. P. $\frac{3}{2}, -1, \frac{2}{3}, \cdots$
5. Find the nth term of the G.P. $x^{1/2}, x^{\frac{n+2}{2n}}, x^{\frac{n+4}{2n}}, \cdots$
6. Find the 15th term of the G.P. ·3, ·06, ·012, ...
7. The first two terms of a G.P are 125 and 25 ; find the 5th and 6th terms.
8. Which term of the G.P. 1, 3, 9,...is 243 ?
9. Which term of the G.P. $\frac{81}{2}, -27, 18, \cdots$ is $-\frac{64}{27}$?
10. Which term of the G.P. 27, 9, 3,...is $\frac{1}{81}$?
11. Find the G.P. whose 6th and 11th terms are 192 and 6144 respectively.
12. The pth and qth terms of a G.P. are x and y respectively. Find the nth term.
13. In a G.P., if the $(p+q)$th term$=m$, and the $(p-q)$th term$=n$, find the pth and qth terms.

5·10. SOME ASSORTED EXAMPLES

Here below we solve a few problems connected with geometric progressions.

Example 19. *If a, b, c, d be in G. P., prove that $(a+b)^2$, $(b+c)^2$ and $(c+d)^2$ are also in G.P.*

Solution. Since a, b, c, d are in G.P., we have

$$\frac{b}{a}=\frac{c}{b}=\frac{d}{c}=r,$$

i.e.,
$$b=ar,\ c=ar^2,\ d=ar^3.$$
$$(a+b)^2=(a+ar)^2=a^2(1+r)^2,$$
$$(b+c)^2=(ar+ar^2)^2=a^2r^2(1+r)^2,$$
$$(c+d)^2=(ar^2+ar^3)^2=a^2r^4(1+r)^2.$$

The sequence $(a+b)^2$, $(b+c)^2$, $(c+d)^2$ is the same as

$$a^2(1+r)^2,\ a^2r^2(1+r)^2,\ a^2r^4(1+r)^2,$$

which is a G.P. with common ratio r^2.

Hence the numbers $(a+b)^2$, $(b+c)^2$ and $(c+d)^2$ are in G.P.

Example 20. *Find three numbers in G.P. whose sum is 14 and the sum of whose squares is 84.*

Solution. Let the three numbers be $\frac{a}{r}$, a and ar. Then

$$\frac{a}{r}+a+ar=14, \qquad \text{...}(i)$$

and
$$\frac{a^2}{r^2}+a^2+a^2r^2=84. \qquad \text{...}(ii)$$

(i) and (ii) may also be written as

$$a(1+r+r^2)=14r, \qquad \text{...}(iii)$$

and
$$a^2(1+r^2+r^4)=84r^2, \qquad \text{...}(iv)$$

Eliminating a from (iii) and (iv), we have

$$\frac{(1+r+r^2)^2}{1+r^2+r^4}=\frac{196}{84},$$

or
$$\frac{1+r+r^2}{1-r+r^2}=\frac{7}{3},$$

or
$$3(1+r+r^2)=7(1-r+r^2),$$

or
$$2r^2-5r+2=0,$$

or
$$(2r-1)(r-2)=0.$$

$$\therefore \qquad r=\frac{1}{2}, \text{ or, } 2.$$

When $r=\frac{1}{2}$, from (i) we have $a=4$.

$\therefore$ The numbers are 8, 4, 2.

When $r=2$, from (i), we have $a=4$.

$\therefore$ The numbers are 2, 4, 8.

Hence the numbers are 2, 4 and 8.

Note. In the above example we could have taken the three numbers to be a, ar and ar^2, but the working becomes simplified when we take the numbers as a/r, a and ar which form a G.P. with a/r as the first term and r as the common ratio.

Example 21. *Divide 175 into four parts in G.P. such that the difference between the means may be to the difference between the extremes as 12 is to 37.*

Solution. Let the four numbers be a, ar, ar^2, ar^3. Then

$$a+ar+ar^2+ar^3=175$$

or $$a(1+r)(1+r^2)=175 \qquad ...(i)$$

Also $$\frac{cr^2-ar}{ar^3-a}=\frac{12}{37},$$

or $$\frac{r^2-r}{r^3-1}=\frac{12}{37},$$

or $12r^2+12r+12=37r$, since $r\neq 1$

or $12r^2-25r+12=0$,

or $(3r-4)(4r-3)=0$.

$\therefore$ $$r=\frac{4}{3} \text{ or } \frac{3}{4}.$$

Putting $r=\frac{4}{3}$ in (i), we have

$$a\left(1+\frac{4}{3}\right)\left(1+\frac{16}{9}\right)=175,$$

or $$a=27.$$

$\therefore$ the numbers are

$$27,\ 27\left(\frac{4}{3}\right),\ 27\left(\frac{4}{3}\right)^2,\ 27\left(\frac{4}{3}\right)^3,$$

i.e., 27, 36, 48, 64.

Similarly, if we take $r=\frac{3}{4}$, we find that the numbers are

64, 48, 36, 27.

Hence the four parts of 175 are 27, 36, 48 and 64.

Note. Sometimes it is convenient to take four numbers in G.P. as $\frac{a}{r^3}$, $\frac{a}{r}$, ar, and ar^3 which form a G.P. with common ratio r^2.

EXERCISE 5 (*f*)

1. If a, b, c, d be in G.P., show that $a-b$, $b-c$, $c-d$ are also in G.P.

2. If a, b, c, d be in G.P., show that $a^2+b^2+c^2$, $ab+bc+cd$, $b^2+c^2+d^2$ are in G.P.

3. If a, b, c, d be in G.P., show that $\frac{1}{a^2+b^2}$, $\frac{1}{b^2+c^2}$, $\frac{1}{c^2+d^2}$ are also in G.P.

4. If p^2+q^2, $pq+qr$, q^2+r^2 are in G.P., show that p, q, r are in G.P.

5. The sum of three numbers in G.P. is 42 and their product is 1728. Find the numbers.

6. Find three numbers in G.P. whose sum is 19 and whose product is 216.

7. Find three numbers in G.P. whose sum is 19 and the sum of whose squares is 133.

8. Find three numbers in G.P. whose product is 64 and the sum of whose products taken in pairs is 56.

9. The sum of four numbers in G.P. is 60 and the arithmetic mean of the first and the last is 18. Find the numbers.

5·10. GEOMETRIC MEANS

If three numbers are in G.P., the middle one is called the **geometric mean** (written briefly as G.M.) between the other two. For example, 4 is the geometric mean between 2 and 8 ; also 1 is the geometric mean between 2 and $\frac{1}{2}$.

If $n+2$ numbers a, $G_1, G_2, \cdots, G_n$, b be in G.P., the numbers $G_1, G_2, \cdots, G_n$ are said to be geometric means between a and b. For example, 6, 12, 24, 48, 96 are five geometric means between 3 and 192.

5·10·1. To find the Geometric Mean between Two Given Numbers a and b.

Let G be the geometric mean between a and b. Then a, G, b must be in G.P.

$$\therefore \qquad \frac{G}{a}=\frac{b}{G},$$

or $$G^2=ab,$$

or $$G=\sqrt{ab}. \qquad \cdots(i)$$

Hence the geometric mean between two numbers a and b is $\sqrt{ab}$.

Note. From (i) we find that the geometric mean between two numbers a and b is real if the numbers are of the same sign and imaginary if they are of opposite signs. Since in the present book we are dealing with real sequences only, we shall consider only real geometric means. If the geometric mean between two numbers happens to be imaginary, we shall simply say that the geometric mean does not exist, meaning thereby that no real geometric mean exists.

Example 22. *Find the geometric mean between 15 and 135.*

Solution. Let G be the required geometric mean.

Since G is the geometric mean between 15 and 135, therefore, 15, G, 135 must be in G.P.

$$\therefore \quad \frac{G}{15}=\frac{135}{G},$$

or $$G^2=15\times 135,$$

or $$G=\sqrt{15\times 135}=45.$$

Hence 45 is the required geometric mean.

5·10·2. To Interest n Geometric Means Between Two Given Numbers a and b.

Let $G_1, G_2, \ldots, G_n$ be the n geometric means between a and b. Then $a, G_1, G_2, \cdots, G_n, b$ must be in G.P. If r be the common ratio of this G.P., we have

$$t_{n+2}=ar^{n+1}=b.$$

$$\therefore \quad r=\left(\frac{b}{a}\right)^{\frac{1}{n+1}}, \qquad \ldots(i)$$

Now $$G_1=ar=a\left(\frac{b}{a}\right)^{\frac{1}{n+1}},$$

$$G_2=ar^2=a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}.$$

$$\vdots \qquad \vdots$$

$$G_n=ar^n=a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}.$$

Hence the required geometric means are

$$a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}, a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}, a\left(\frac{b}{a}\right)^{\frac{3}{n+1}}, \ldots\ldots \text{and } a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}.$$

Example 23. *Insert 7 geometric means between 3 and 1875.*

Solution. Let $G_1, G_2, \cdots, G_7$ be the required geometric means between 3 and 1875 then.

$$3, G_1, G_2, \cdots\cdots, G_7, 1875, \qquad \ldots(i)$$

are in G.P.

If r be the common ratio of (i), we have

$$t_9=3r^8=1875$$

or $$r^8=625.$$

$$\therefore \quad r=\pm\sqrt{5}.$$

When $r=\sqrt{5}$, the required means are $3(5)^{1/2}$, $3(5)$, $3(5)^{3/2}$, $3(5)^{4/2}$, $3(5)^{5/2}$, $3(5)^{6/2}$ and $3(5)^{7/2}$.

i.e., $3\sqrt{5}, 15, 15\sqrt{5}, 75, 75\sqrt{5}, 375$ and $375\sqrt{5}$ ...(*ii*)

When $r=-\sqrt{5}$, the required means are $3(-5^{1/2}), 3(-5^{1/2})^2, 3(-5^{1/2})^3, 3(-5^{1/2})^4, 3(-5^{1/2})^5, 3(-5^{1/2})^6$, and $3(-5^{1/2})^7$,

i.e., $-3\sqrt{5}, 15, -15\sqrt{5}, 75, -75\sqrt{5}, 375$ and $-375\sqrt{5}$.

Example 24. *Prove that the product of the n geometric means between two given numbers a and b is equal to the nth power of the geometric mean between a and b, where a and b are of the same sign.*

Solution. Let the n geometric means be $G_1, G_2, \cdots, G_n$.

$\therefore$ $a, G_1, G_2, \cdots, G_n, b,$ ...(*i*)

are in G.P.

If r be the common ratio of (*i*), we have

$$t_{n+2}=ar^{n+1}=b \text{ or } r=\left(\frac{b}{a}\right)^{\frac{1}{n+1}}. \quad \cdots(ii)$$

The n geometric means are

$$G_1=ar,\ G_2=ar^2, \cdots, G_n=ar^n.$$

$$G_1 G_2 \ldots G_n = ar \,.\, ar^2 \,.\, ar^3 \ldots ar^n.$$

$$=a^n r^{\frac{1}{2}n(n+1)}$$

$$=a^n \left(\frac{b}{a}\right)^{\frac{1}{2}n}, \text{ by } (ii)$$

$$=(\sqrt{ab})^n,$$

$$=G^n,$$

where $G\ (=\sqrt{ab})$ is the geometric mean between a and b.

The result holds only when a and b are of the same sign, for, otherwise, $\sqrt{ab}$ is not a real number, and therefore, no real geometric mean exists.

EXERCISE 5 (g)

1. Insert 3 geometric means between $\frac{4}{3}$ and $\frac{3}{4}$.
2. Insert 8 geometric means between 26 and $-\frac{13}{256}$.
3. Insert 5 geometric means between $3\frac{5}{9}$ and $40\frac{1}{2}$.
4. Insert 4 geometric means between $\frac{2}{3}$ and $-5\frac{1}{16}$.

5. Insert 3 geometric means between $\frac{1}{2}$ and 128.

6. If A and G respectively are the arithmetic and geometric means between two positive numbers, show that A>G.

7. If A and G respectively be the arithmetic and geometric means between two positive numbers, prove that the numbers are

$$A \pm \sqrt{(A+G)(A-G)}.$$

8. If $\dfrac{a^{n+1}+b^{n+1}}{a^n+b^n}$ be the geometric mean between a and b, find the value of n.

9. The sum of two numbers is 6 times their geometric mean. Show that the numbers are in the ratio $3-2\sqrt{2} : 3+2\sqrt{2}$.

10. The arithmetic mean of two positive numbers a and b is to their geometric mean as $m : n$. Show that

$$a : b = m-\sqrt{m^2-n^2} : m+\sqrt{m^2-n^2}.$$

5·11. FINITE GEOMETRIC SERIES

The series,

$$a+ar+ar^2+...+ar^{n-1},$$

corresponding to the finite G.P. $a, ar, ar^2, \cdots, ar^{n-1}$, is called a **finite geometric series** or a series in G.P. For example,

(*i*) $2+4+8+16+32$,

(*ii*) $\cdot1+\cdot01+\cdot001+\cdot0001$,

(*iii*) $\dfrac{1}{3}+\dfrac{1}{3^2}+\dfrac{1}{3^3}+\cdots+\dfrac{1}{3^n}$,

and (*iv*) $1-\dfrac{1}{5}+\dfrac{1}{5^2}-\dfrac{1}{5^3}+\cdots+\dfrac{(-1)^{n-1}}{5^{n-1}}$

are all finite series in G.P.

5·12. TO FIND THE SUM TO n TERMS OF A SERIES IN G.P.

Let a be the first term, r the common ratio, and S_n the sum to n terms of a series in G.P. Then

$$S_n = a+ar+ar^2+\cdots+ar^{n-1}, \quad \ldots(i)$$

$$rS_n = ar+ar^2+\cdots+ar^n \quad \ldots(ii)$$

Subtracting (*ii*) from (*i*), we have

$$S_n(1-r) = a-ar^n,$$

or $$S_n = \frac{a(1-r^n)}{1-r}, \text{ provided } r \neq 1. \quad \ldots(A)$$

The formula (A) may also be written as

$$S_n=\frac{a(r^n-1)}{r-1}\text{ , provided } r\neq 1. \qquad ...(B)$$

If l be the nth term of the given G.P., then $l=ar^{n-1}$, and we may write (A) as

$$S_n=\frac{a-lr}{1-r}\text{ , provided } r\neq 1. \qquad ...(C)$$

It is usual to use the formula (A) or (C) when $|r|<1$, and the formula (B) when $|r|>1$.

When $r=1$, each term is equal to a, and $S_n=na$.

Example 25. *Find the sum to 10 terms of the G.P.*

$$2+4+8+\cdots$$

Solution. Here $a=2$, $r=2$, $n=10$.

$$S_{10}=\frac{2(2^{10}-1)}{2-1}=2\,(2^{10}-1).$$

EXERCISE 5 (*h*)

Sum the following series :

1. $4+12+36+\cdots$to 10 terms.
2. $-\frac{1}{3}$, 1, -3, 9,$\cdots$ to 8 terms.
3. $\cdot 3+\cdot 06+\cdot 012+\ldots$ to 15 terms.
4. $\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\ldots+\frac{1}{3^n}$.
5. $1-\frac{1}{5}+\frac{1}{5^2}+\ldots+\frac{(-1)^{n-1}}{5^{n-1}}$.
6. The first term of a G.P. is 27. If the 8th term be $\frac{1}{81}$, what will be sum to 10 terms ?
7. The sum of the first six terms of a G.P. is equal to 65 times that of the first three terms. Find the common ratio.
8. The sum of the first 5 terms of a G.P. whose common ratio is $\cdot 2$ is $7\cdot 99744$. Find the first term.

5·13. CONVERGENT SEQUENCES

Consider the infinite sequence

(*i*) $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots, \frac{n+1}{n}\ldots$

Denoting the nth term of the above sequence by t_n, we have,

$$t_n=\frac{n+1}{n}=1+\frac{1}{n},$$

t_n differs from 1 by $\frac{1}{n}$. Since $\frac{1}{n}$ decreases as n increases, therefore, t_n becomes nearer and nearer to 1 as n increases. Also, the difference between t_n and 1 can be made as small as we please, *i.e.*, by properly choosing n, it can be made smaller than any given positive number howsoever small, *e.g.*,

$$t_n-1<\frac{1}{100}, \text{ if } n>100$$

$$t_n-1<\frac{1}{1000}, \text{ if } n>1000$$

$$t_n-1<10^{-10}, \text{ if } n>10^{10}.$$

Thus, howsoever small a positive number k may be, we can find a positive integer m, such that the difference between t_n and 1 is less than k, for $n>m$. ...(A)

Here $m=100$ when $k=\frac{1}{100}$, $m=1000$ when $k=\frac{1}{1000}$, $m=10^{10}$ when $k=10^{-10}$.

Next, let us consider the sequence

(*ii*) $1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n-1}{n}, \ldots$

For this sequence,

$$t_n=\frac{n-1}{n}=1-\frac{1}{n},$$

$$1-t_n=\frac{1}{n}.$$

As in the case of sequence (*i*), we find that in this case also, t_n becomes nearer and nearer to 1 as n increases.

Also, $$1-t_n<\frac{1}{100}, \text{ if } n>100,$$

$$1-t_n<\frac{1}{1000}, \text{ if } n>1000,$$

$$1-t_n<10^{-10}, \text{ if } n>10^{10},$$

Thus the statement (A) is true in this case also.

Next, let us consider the sequence

(*iii*) $0, \frac{3}{2}, \frac{2}{3}, \frac{5}{4}, \frac{4}{5}, \ldots, 1+\frac{(-1)^n}{n}, \ldots\ldots$

Hence $\quad t_n = 1 + \dfrac{(-1)^n}{n}$,

$$t_n - 1 = \frac{(-1)^n}{n},$$

$$|\, t_n - 1 \,| = \frac{1}{n}$$

As in the above examples (*i*) and (*ii*), we find that the difference between t_n and 1 can be made as small as we please and that the statement (A) is true in this case also.

Finally, consider the sequence

(*iv*) $\qquad \dfrac{1}{2}, \dfrac{1}{2^2}, \dfrac{1}{2^3}, \ldots, \dfrac{1}{2^n}, \ldots$

Since $\dfrac{1}{2^n}$ becomes nearer and nearer to zero as n increases, therefore, t_n (and therefore $t_n - 0$) becomes nearer and nearear to zero as n increases.

Also, by a proper choice of n, the difference between t_n and 0 can be made smaller than any small positive number whatsoever *e.g.*,

$$|\, t_n - 0 \,| < \frac{1}{100}, \text{ if } \frac{1}{2^n} < \frac{1}{10\text{C}}, \textit{ i.e.}, \text{ if } n > 6$$

$$|\, t_n - 0 \,| < \frac{1}{1000}, \text{ if } \frac{1}{2^n} < \frac{1}{1000}, \textit{ i.e.}, \text{ if } n > 9$$

$$|\, t_n - 0 \,| < \frac{1}{2^{100}}, \text{ if } \frac{1}{2^n} < \frac{1}{2^{100}}, \textit{ i.e.}, \text{ if } n > 100.$$

Thus, howsover small a positive number k may be, we can find a positive integer m such that the difference between r_n and 0 is less than k, for $n > m$.

Here $m=6$, when $k = \dfrac{1}{100}$; $m=9$, when $k = \dfrac{1}{1000}$; $m=100$ if $k = 2^{-100}$.

Such sequences, as we have considered above, are called *convergent sequences*. A sequence whose nth term is t_n is said to be **convergent** if there exists a number l such that having chosen a positive number k, howsoever small, we can find a positive integer m such that the difference between t_n and l is less than k for $n > m$. The number l is called the **limit** of the sequence and the sequence is said to converge (or tend) to l as n tends to infinity. Symbolically,

We write $\quad \lim\limits_{n \to \infty} t_n = l \qquad$ (or $\quad t_n \to l$ as $n \to \infty$)

In the above examples (*i*) to (*iv*), the limits of the four sequences considered are 1, 1, 1 and 0 respectively.

Not every infinite sequence is convergent. For example, none of the following sequences is convergent :

$$2, 3, 4, 5, \ldots\ldots$$
$$1, -3, 5, -7, \ldots\ldots$$
$$2, 4, 6, 8, \ldots\ldots$$
$$1, -1, 1, -1, \ldots\ldots$$

A sequence which does not converge is called a *non-convergent sequence*.

Notes. 1. The notion of convergence is available for infinite sequences only. It is meaningless to talk of the convergence or otherwise of a finite sequence.

2. Recall that the absolute value of a number x is written as $|x|$ and is read as mod x, *e.g.*, $|-5|=5$. The inequality $|x|<1$ means that x is numerically less than 1, *i.e.*, $-1<x<1$.

3. The absolute value of the defference between two numbers a and b is written as $|a-b|$ and is read as 'mod a minus b', *e g*., $|5-7|=|-2|=2$. $|(-5)-(-7)|=|-5+7|=|2|=2$.

The symbols $|a-b|$ and $a \sim b$ mean the same thing.

4. Using the above notation, the definition of the limit of a sequence may be written as follows :

A sequence (t_n) is said to converge to l, if given $k>0$ and arbitrarily small, it is possible to find a positive integer m, such that $|t_n-l|<k$, for all $n>m$.

5·14. INFINITE GEOMETRIC SEQUENCES

The infinite geometric sequence

$$x, x^2, x^3, \ldots, x^n, \ldots$$

whose nth term is x^n, is convergent when $|x|<1$ or $x=1$, and is non-convergent when $|x|>1$ or $x=-1$.

Also,

$$\lim_{n \to \infty} x^n = \begin{cases} 0, \text{ when } |x| < 1, \\ 1, \text{ when } \;\, x = 1. \end{cases}$$

In particular, the sequences

$$\frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3}, \ldots\ldots$$

$$-\frac{1}{4}, \left(-\frac{1}{4}\right)^2, \left(-\frac{1}{4}\right)^3, \ldots\ldots$$

are both convergent, but neither of the two sequences

$$5, 5^2, 5^3, \cdots$$

and $$-6, (-6)^2, (-6)^3, \ldots\ldots$$

is convergent.

5·15. INFINITE SERIES

The symbol

$$t_1+t_2+\cdots+t_n+\cdots$$

is called the infinite series corresponding to the infinite sequence $t_1, t_2, t_3, \ldots$. Thus, to each infinite sequence there corresponds an infinite series. For example,

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots,$$

$$1+2+3+4+\ldots\cdots,$$

and $$1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots$$

are the infinite series corresponding to the infinite sequences

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots,$$

$$1, 2, 3, 4, \cdots,$$

and $$1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots\ldots$$

respectively.

If we go on adding up the terms of an infinite series one by one, the process would never end; therefore, a *priori* it is not possible to talk of the sum of an infinite series. In some cases, however, it is possible to associate a number S with an infinite series and call it the sum of the series to infinity. When such is the case, the series is said to be convergent.

5·16. SUM OF AN INFINITE SERIES

Let $$t_1+t_2+t_3+\ldots+t_n+\cdots$$

be an infinite series. Denoting the sum to n terms of the above series by S_n, we have

$$S_n=t_1+t_2+\cdots+t_n.$$

$S_1, S_2, \cdots, S_n, \ldots$ is an infinite sequence called the sequence of partial sums of the given series. If this sequence is convergent and tends to S as n tends to infinity, we say that the given series is convergent and that its sum to infinity is S.

If the sequence $S_1, S_2, \ldots, S_n, \cdots$ is not convergent, the given series is said to be non-convergent and it is then not possible to talk of its sum to infinity.

Thus an infinite series is said to be convergent if and only if the sequence of its partial sums is convergent. Also, the limit of the sequence of partial sums is then called the *sum of the series to infinity*.

Note. The sum S of a convergent series is also called its limiting sum, for, as we have said above, it is the limit of the sum to n terms as n tends to infinity. As we go on taking a larger and larger number of terms of the series, the sum will become closer and closer to S but it may never be equal to S.

5·17. SUM TO INFINITY OF A GEOMETRIC SERIES

Let S_n denote the sum to n terms of the G.P.

$$a+ar+ar^2+\ldots\ldots$$

Then

$$S_n=\frac{a(1-r^n)}{1-r}=\frac{a}{1-r}-\frac{ar^n}{1-r}, \text{ provided } r\neq 1.$$

If $|r|<1$, $r^n\to 0$, as $n\to\infty$,

so that $S_n\to\frac{a}{1-r}$, as $n\to\infty$.

Therefore when $|r|<1$, the sequence of partial sums of the given series, *viz.*

$$S_1, S_2, \ldots\ldots S_n, \ldots\ldots\ldots$$

converges to $\frac{a}{1-r}$, so that the given G.P. is also convergent and its sum to infinity is $\frac{a}{1-r}$.

If $|r|>1$, or $r=-1$, r^n does not tend to any limit as $n\to\infty$.

The sequence

$$S_1, S_2, \ldots\ldots\ldots, S_n, \ldots$$

is non-convergent, so that the given G.P. is non-convergent.

If $r=1$, we have, *ab initio*

$$S_n=a+a+\ldots\ldots\text{to } n \text{ terms},$$
$$=na.$$

In this case again, S_n does not tend to any limit, so that the given G.P. is not convergent.

Thus the infinite geometric series $a+ar+ar^2+\ldots\ldots$ is convergent if and only if $|r|<1$, and its sum to infinity is then $\frac{a}{1-r}$.

Example 26. *Sum the series* $1+\frac{1}{5}+\frac{1}{5^2}+\ldots\ldots$ *to infinity.*

Solution. Let S_n denote the sum of the given G.P. to n terms. Since $a=1$, $r=\frac{1}{5}$, we have

$$S_n=\frac{1-\left(\frac{1}{5}\right)^n}{1-\frac{1}{5}}=\frac{5}{4}\left\{1-\left(\frac{1}{5}\right)^n\right\}$$

Since $\left(\frac{1}{5}\right)^n\to 0$ as $n\to\infty$, therefore, $S_n\to\frac{5}{4}$ as $n\to\infty$.

Hence the required sum is $\frac{5}{4}\cdot$

Aliter. The given series is a G.P. having $r=\frac{1}{5}\cdot$

Since $r=\frac{1}{5}$, which is less than 1, therefore, the given series is convergent and

$$S=\frac{1}{1-\frac{1}{5}}=\frac{5}{4}$$

Example 27. *Sum the series* $\frac{3}{4}-\frac{3}{4^2}+\frac{3}{4^3}$*......to infinity.*

Solution. The given series is a G.P. having

$$a=\frac{3}{4},\ r=-\frac{1}{4}\cdot$$

Since $|r|=\left|-\frac{1}{4}\right|=\frac{1}{4}$, which is less than 1, therefore, the given series is convergent and the sum to infinity

$$=\frac{\frac{3}{4}}{1-\left(-\frac{1}{4}\right)}=\frac{3}{5}$$

Hence the required sum is $\frac{3}{5}$

Example 28. *Is it possible to sum the series* $2+4+8+16+\cdots$ *to infinity ?*

Solution. The given series is a G.P. such that $a=2$, $r=2$.

Since $|r|=2$, which is greater than 1, the given series is non-convergent. Therefore, it is not possible to talk of its sum to infinity.

Example 29. *The 2nd term of an infinite convergent G.P. is $\frac{2}{9}$ and the sum to infinity is unity. Find the series.*

Solution. Let a be the first term and r the common ratio of the given G.P.

$$\therefore \qquad ar=\frac{2}{9}, \qquad \ldots(i)$$

$$\text{and} \qquad \frac{a}{1-r}=1. \qquad \ldots(ii)$$

Eliminating a, we have

$$\frac{2}{9r(1-r)}=1,$$

$$\text{or} \qquad 9r^2-9r+2=0,$$

$$\text{or} \qquad (3r-1)(3r-2)=0.$$

$$\therefore \qquad r=\frac{1}{3} \text{ or } \frac{2}{3}.$$

When $r=\frac{1}{3}$, from (i) we find that $a=\frac{2}{3}$

Therefore, the series is

$$\frac{2}{3}+\frac{2}{3^2}+\frac{2}{3^3}+\ldots\ldots \qquad \ldots(iii)$$

When $r=\frac{2}{3}$, from (ii) we find that $a=\frac{1}{3}$,

so that the series is

$$\frac{1}{3}+\frac{2}{3^2}+\frac{2^2}{3^3}+\frac{2^3}{3^4}+\ldots\ldots$$

Hence the given series is

$$\text{either} \qquad \frac{2}{3}+\frac{2}{3^2}+\frac{2}{3^3}+\ldots\ldots$$

$$\text{or} \qquad \frac{1}{3}+\frac{2}{3^2}+\frac{2^2}{3^3}+\ldots\ldots$$

5·18. RECURRING DECIMALS

A recurring decimal can be expressed as an infinite geometric series, and thus be converted into a rational fraction. The following examples will illustrate the method.

Example 30. *Express $\cdot\bar{7}$ as an infinite geometric series and hence reduce it to a rational fraction.*

Solution. $\cdot\bar{7}=\cdot777\ldots\ldots,$

$$\text{or} \qquad \cdot\bar{7}=\cdot7+\cdot07+\cdot007+\cdot0007+\ldots\ldots \qquad \ldots(i)$$

Since the R.H.S. of (*i*) is an infinite geometric series having $\cdot 7$ as the first term and $\cdot 1$ as the common ratio, we have

$$\cdot\bar{7}=\frac{\cdot 7}{1-\cdot 1}=\frac{\cdot 7}{\cdot 9}=\frac{7}{9}.$$

EXERCISE 5 (*i*)

Sum to infinity the following series :

1. $4+1\cdot 6+\cdot 64+\ldots\ldots$
2. $\frac{1}{2}+\frac{1}{6}+\frac{1}{18}+\ldots\ldots$
3. $6+\sqrt{30}+5+\ldots\ldots$
4. $\frac{1}{3}-\frac{1}{9}+\frac{1}{27}-\frac{1}{81}+\ldots\ldots$
5. $2+4x+8x^2+\ldots\ldots$, where $|x|<\frac{1}{2}$.
6. Find an infinite geometric series whose second term is 3 and the sum is 16.
7. Find the common ratio of a G.P., whose first term is 1 and each term is twice the sum to infinity of all the terms which follow it.
8. Find by the method of summation of infinite geometric series the value of (*i*) $\cdot\bar{3}$, (*ii*) $\cdot 2\bar{7}$, (*iii*) $\cdot 7\overline{23}$.
9. The sum of an infinite geometric series is $\frac{4}{3}$ and the sum of the G.P. obtained by squaring every term is equal to $\frac{16}{27}$. Find the series.

5·19. TO FIND THE SUM OF THE FIRST n NATURAL NUMBERS

Let $\quad S_n=1+2+3+\ldots\ldots+n.$ ...(*i*)

The above series is an A.P. whose first term is 1 and common difference is also 1.

$$\therefore \quad S_n=\frac{n}{2}\{2\times 1+(n-1)\,1\}=\frac{n(n+1)}{2}. \qquad \ldots(ii)$$

Thus the sum of the first n natural numbers is $\frac{1}{2}n(n+1)$.

Denoting the sum of the first n natural numbers by Σn, (*ii*) may also be written as

$$\Sigma n=\tfrac{1}{2}n\,(n+1).$$

5·20. TO FIND THE SUM OF THE SQUARES OF THE FIRST n NATURAL NUMBERS.

Let $S_n = 1^2+2^2+3^2+\dots\dots+n^2$. ...(i)

Consider the identity

$$x^3-(x-1)^3=3x^2-3x+1.$$

Putting $x=1, 2, 3, \dots\dots, n-1$ and n in succession, we have

$$1^3-0^3=3.1^2-3.1+1,$$
$$2^3-1^3=3.2^2-3.2+1,$$
$$3^3-2^3=3.3^2-3.3+1,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$(n-1)^3-(n-2)^3=3.(n-1)^2-3.(n-1)+1,$$

and $$n^3-(n-1)^3=3n^2-3n+1.$$

Adding, we have

$$n^3=3(1^2+2^2+\dots\dots+n^2)-3(1+2+\dots\dots+n)+n,$$
$$=3S_n-3\,.\,\frac{n(n+1)}{2}+n.$$

or $$3S_n=\frac{n}{2}\,(2n^2+3n+1)=\frac{n}{2}\,(n+1)(2n+1).$$

$$\therefore\ S_n=\frac{n(n+1)(2n+1)}{6}. \qquad \dots(ii)$$

Thus the sum of the squares of the first n natural numbers is

$$\frac{1}{6}\,n(n+1)(2n+1).$$

Denoting the sum of the squares of the first n natural numbers by Σn^2,

(ii) may also be written as

$$\boxed{\Sigma n^2=\frac{1}{6}\,n(n+1)(2n+1)}$$

5·21. TO FIND THE SUM OF THE CUBES OF THE FIRST n NATURAL NUMBERS

Let $S_n=1^3+2^3+3^3+\dots\dots+n^3$. ...(i)

Consider the identity

$$x^4-(x-1)^4=4x^3-6x^2+4x-1.$$

Putting $x=1, 2, 3, \dots\dots, n-1$ and n in succession, we have

$$1^4-0^4=4.1^3-6.1^2+4.1-1,$$
$$2^4-1^4=4.2^3-6.2^2+4.2-1,$$

$$3^4-2^4=4.3^3-6.3^2+4.3-1,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$(n-1)^4-(n-2)^4=4(n-1)^3-6(n-1)^2+4(n-1)-1,$$

and $$n^4-(n-1)^4=4n^3-6n^2+4n-1.$$

Adding, we have

$$n^4=4[1^3+2^3+3^3+\dots\dots+n^3]-6\Sigma n^2+4\Sigma n-n,$$

or $$4S_n=n^4+\frac{6n(n+1)(2n+1)}{6}-\frac{4n(n+1)}{2}+n,$$

$$=(n^4+n)+n(n+1)(2n+1-2),$$

$$=n(n+1)(n^2-n+1+2n-1),$$

$$=n(n+1)(n^2+n),$$

$$=n^2(n+1)^2.$$

$\therefore$ $$S_n=\frac{n^2(n+1)^2}{4}. \quad \dots(ii)$$

Thus *the sum of the cubes of the first n natural numbers*

$$=\frac{n^2(n+1)^2}{4}.$$

Denoting the sum of the cubes of the first n natural numbers by Σn^3, (*ii*) may also be written as

$$\boxed{\Sigma n^3=\left\{\frac{n(n+1)}{2}\right\}^2.} \quad \dots(iii)$$

Corollary. Since the R.H.S. of $(ii)=(\Sigma n)^2$, we have

$$\Sigma n^3=(\Sigma n)^2.$$

5·22. TO FIND THE SUM TO n TERMS OF THE SERIES WHOSE nTH TERM IS an^3+bn^2+cn+d.

Here $$t_n=an^3+bn^2+cn+d.$$

Putting $n=1, 2, 3, \dots\dots, n$ in succession, we have

$$t_1=a.1^3+b.1^2+c.1+d,$$

$$t_2=a.2^3+b.2^2+c.2+d,$$

$$t_3=a.3^3+b.3^2+c.3+d,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$t_n=a.n^3+b.n^2+c\,n+d.$$

Adding, we have

$$S_n = a\Sigma n^3 + b\Sigma n^2 + c\Sigma n + dn$$
$$= \frac{a}{4} n^2(n+1)^2 + \frac{b}{6} n(n+1)(2n+1)$$
$$+ \frac{c}{2} n(n+1) + dn$$

Example 31. *Find the sum to n terms of the series*

$$2^2+4^2+6^2+\ldots\ldots+(2n)^2.$$

Solution. Let S_n denote the required sum.

Then $S_n = 2^2+4^2+6^2+\ldots\ldots+(2n)^2,$

$$= 4\{1^2+2^2+\ldots\ldots+n^2\},$$
$$= 4 \cdot \frac{n}{6}(n+1)(2n+1)$$
$$= \frac{2}{3} n(n+1)(2n+1).$$

Example 32. *Find the sum to n terms of the series whose nth term is $3n^2+n-2$.*

Solution. Let t_n be the nth term and S_n the sum to n terms. Then

$$t_n = 3n^2+n-2.$$

Putting $n=1, 2, 3, \ldots\ldots, n$ in succession, we have

$$t_1 = 3.1^2+1-2,$$
$$t_2 = 3.2^2+2-2,$$
$$t_3 = 3.3^2+3-2,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$t_n = 3n^2+n-2.$$

Adding, we have

$$S_n = 3\Sigma n^2 + \Sigma n - 2n,$$
$$= 3\,\frac{n}{6}(n+1)(2n+1) + \frac{n}{2}(n+1) - 2n,$$
$$= n(n^2+2n-1).$$

Example 33. *Find the sum to n terms of the series*

$$3.5+4.7+5.9+\ldots\ldots$$

Solution. The nth term of the given series

$= n$th term of $3, 4, 5, \ldots\ldots \times n$th term of $5, 7, 9, \ldots,$

$$= \{3+(n-1).1\} \times \{5+(n-1).2\},$$
$$= (n+2)(2n+3),$$

or $t_n = 2n^2+7n+6.$

$$\therefore \qquad S_n=\Sigma(2n^2+7n+6)=2\Sigma n^2+7\Sigma n+6n,$$
$$=\frac{2\,n(n+1)(2n+1)}{6}+\frac{7n(n+1)}{2}+6n,$$
$$=\frac{4n^3+27n^2+59n}{6}.$$

Example 34. *Sum to n terms the series*

$$1^2+(1^2+2^2)+(1^2+2^2+3^2)+\ldots\ldots$$

Solution. If the nth term of the series be denoted by t_n, we have

$$t_n=1^2+2^2+3^2+\ldots\ldots+n^2,$$
$$=\frac{n(n+1)(2n+1)}{6},$$
$$=\frac{2n^3+3n^2+n}{6},$$
$$=\frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n.$$
$$S_n=\frac{1}{3}\Sigma n^3+\frac{1}{2}\Sigma n^2+\frac{1}{6}\Sigma n,$$
$$=\frac{1}{3}\,\frac{n^2(n+1)^2}{4}+\frac{1}{2}\,\frac{n(n+1)(2n+1)}{6}$$
$$+\frac{1}{6}\,\frac{n(n+1)}{2},$$
$$=\frac{n(n+1)}{12}\,\{n(n+1)+2n+1+1\},$$
$$=\frac{n(n+1)}{12}\,(n^2+3n+2),$$
$$=\frac{n(n+1)^2(n+2)}{12}.$$

Hence the sum to n terms is $\frac{n}{12}(n+1)^2(n+2)$.

EXERCISE 5 (j)

Sum the following series to n terms :

1. $1^2+3^2+5^2+\ldots\ldots$
2. $1^3+3^3+5^3+\ldots\ldots$
3. $1.3+3.5+5.7+\ldots\ldots$
4. $4.7+7.10+10.13+\ldots\ldots$

Sum the series :

5. $6^2+7^2+8^2+\ldots\ldots+20^2$.
6. $2^2+5^2+8^2+\ldots\ldots$ to 15 terms.
7. $5.6+6.7+7.8+\ldots\ldots$ to 25 terms.
8. Find the sum to n terms of the series whose nth term is $2n^2-3n+7$.

9. Find the sum to n terms of the series whose nth term is

$$n(n+1)(n+2).$$

10. Find the sum of the series $1+(1+2)+(1+2+3)+\ldots\ldots$ to n terms.

5·23. ARITHMETICO-GEOMETRIC SERIES

A series, each term of which is obtained by multiplying the corresponding terms of an A.P. and a G.P., is called an **Arithmetico-geometric series**. As an example, let us consider the series

$$1+4x+7x^2+\ldots\ldots \qquad \ldots(i)$$

The terms $1, 4x, 7x^2, \ldots\ldots$ of (i) are formed by multiplying corresponding terms of the A.P.

$$1+4+7+\ldots\ldots,$$

and the G.P. $\quad 1+x+x^2+\ldots\ldots$

The series (i) is, therefore, an arithmetic-geometric series. The standard form of such a series is

$$a+(a+d)x+(a+2d)x^2+\ldots\ldots+(a+\overline{n-1}\,d)\,x^{n-1}.$$

5·24. TO FIND THE SUM TO n TERMS OF THE ARITHMETICO-GEOMETRIC SERIES WHOSE nTH TERM IS $(a+\overline{n-1}\,d)\,x^{n-1}$

Let S_n denote the sum to n terms of the given series.

Then $S_n=a+(a+d)\,x+(a+2d)\,x^2+\ldots\ldots+(a+\overline{n-1}d)\,x^{n-1}$,

$xS_n= \quad ax \quad +(a+d)\,x^2+\ldots\ldots+ \qquad (a+\overline{n-1}d)\,x^n.$

By substraction, we have

$$(1-x)S_n=a+dx+dx^2+\ldots\ldots+dx^{n-1}-(a+\overline{n-1}\,d)\,x^n,$$

$$=a+\frac{dx(1-x^{n-1})}{1-x}-(a+\overline{n-1}\,d)x^n.$$

$$\therefore \qquad S_n=\frac{a}{1-x}+\frac{dx(1-x^{n-1})}{(1-x)^2}-\frac{(a+\overline{n-1}\,d)x^n}{1-x}. \qquad \ldots(i)$$

Corollary. If $|x|<1$, then x^{n-1}, x^n and nx^n all $\to 0$ as $n\to\infty$.

$$\therefore \qquad \lim_{n\to\infty} S_n=\frac{a}{1-x}+\frac{dx}{(1-x)^2}.$$

Hence the given series is convergent when $|x|<1$, and the sum to infinity is given by

$$S=\frac{a}{1-x}+\frac{dx}{(1-x)^2}.$$

Example 35. *Sum to n terms the series* $1+3x+5x^2+7x^3+\ldots$ *Find also the sum to infinity when* $|x|<1$.

Solution. Let t_n and S_n denote the nth term and the sum to n terms respectively.

Then $t_n=$(nth term of 1, 3, 5, ...)$\times$(nth term of 1, x, x^2,...),

$$=(2n-1)\,x^{n-1}$$

$$S_n=1+3x+5x^2+\ldots\ldots+(2n-1)\,x^{n-1},$$

$$xS_n=\quad x+3x^2+\ldots\ldots\qquad+(2n-1)\,x^n.$$

By subtraction, we have

$$(1-x)S_n=1+2x+2x^2+\ldots\ldots+2x^{n-1}-(2n-1)\,x^n,$$

$$=1+\frac{2x(1-x^{n-1})}{1-x}-(2n-1)x^n.$$

$$\therefore \qquad S_n=\frac{1}{1-x}+\frac{2x(1-x^{n-1})}{(1-x)^2}-\frac{(2n-1)x^n}{1-x}.$$

When $|x|<1$, then x^{n-1}, x^n and $nx^n\to 0$ as $n\to\infty$.

$$\therefore \lim_{n\to\infty} S_n=\frac{1}{1-x}+\frac{2x}{(1-x)^2}=\frac{1+x}{(1-x)^2}.$$

Hence the sum to infinity$=\dfrac{1+x}{(1-x)^2}$.

Example 36. *Sum to n terms and to infinity the series*

$$1-\frac{2}{3}+\frac{3}{9}-\frac{4}{27}+\ldots\ldots$$

Solution. The given series may be written as

$1+2(-\frac{1}{3})+3(-\frac{1}{3})^2+4(-\frac{1}{3})^3+\ldots\ldots+n(-\frac{1}{3})^{n-1}+\ldots\ldots$

Let S_n denote the sum to terms. Then,

$$S_n=1+2(-\tfrac{1}{3})+3(-\tfrac{1}{3})^2+\ldots\ldots+n(-\tfrac{1}{3})^{n-1}$$

$$(-\tfrac{1}{3})\,S_n=\quad(-\tfrac{1}{3})+2(-\tfrac{1}{3})^3+\ldots\ldots+(n-1)(-\tfrac{1}{3})^{n-1}+n(-\tfrac{1}{3})^n$$

By subtraction, we have

$$\tfrac{4}{3}\,S_n=1+(-\tfrac{1}{3})+(-\tfrac{1}{3})^2+\ldots\ldots-n(-\tfrac{1}{3})^n,$$

$$=\frac{1-(-\frac{1}{3})^n}{1-(-\frac{1}{3})}-n(-\tfrac{1}{3})^n.$$

$$\therefore \qquad S_n=\tfrac{9}{16}\{1-(-\tfrac{1}{3})^n\}-\tfrac{3}{4}n(-\tfrac{1}{3})^n.$$

Since $(-\frac{1}{3})^n$ and $n\,(-\frac{1}{3})^n\to 0$, as $n\to\infty$,

$$\therefore \qquad \lim_{n\to\infty} S_n=\frac{9}{16}.$$

Hence the sum to infinity$=\dfrac{9}{16}$.

EXERCISE 5 (*k*)

Sum the following series to n terms :

1. $1+2x+3x^2+4x^3+\ldots\ldots$

2. $2+5x+8x^2+11x^3+\dots\dots$
3. $3-7x+11x^2-15x^3\dots\dots$

Sum the following series to n terms. Also find the sum to infinity when $|x| < 1$:

4. $1+4x+7x^2+\dots\dots$
5. $1-x+2x^2-3x^3+\dots\dots$

Sum the following to series infinity :

6. $1+3x+6x^2+10x^3+\dots\dots$, where $|x| < 1$.
7. $1-6x+15x^2-28x^3+\dots\dots$, where x is numerically less than unity.
8. $1+\frac{2}{5}+\frac{3}{25}+\frac{4}{125}+\dots\dots$
9. $1-\frac{3}{2}+\frac{5}{4}-\frac{7}{8}+\dots\dots$
10. Prove that when $x=2$, the sum of the series
$2x+5x^2+8x^3+11x^4+\dots\dots$ to n terms is $8+(3n-4)2^{n+1}$.

TEST YOUR UNDERSTANDING V

In each of the following problems one of the four alternatives is correct. Write down the letter corresponding to the correct alternative.

1. The sum of the series $1+5+9+13+\dots\dots$ to 10 terms is :
(*a*) 170 (*b*) 380
(*c*) 190 (*d*) 320.
2. The sum of the first n terms of the series
$$\frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+\dots\dots$$
is equal to
(*a*) 2^n-n-1 (*b*) $1-2^{-n}$
(*c*) $n+2^{-n}-1$ (*d*) 2^n-1. (*I.I.T. J.E.E., 1988*)
3. The nth term of the sequence 6, 3, 0, −3, −6, ...is
(*a*) $6n$ (*b*) $3n+3$
(*c*) $8-3n$ (*d*) $9-3n$.
4. The sum of all the natural numbers with two digits is
(*a*) 5050 (*b*) 4905
(*c*) 4950 (*d*) 5000.
5. 3 is the geometric mean of a and b. Possible values of a and b are
(*a*) 1, 5 (*b*) 9, 0
(*c*) 4, 2 (*d*) 9, 1.
6. The first term of a G.P. is 2 and the sum to infinity is 6. The common ratio is

(a) $\frac{2}{3}$ (b) $\frac{1}{3}$

(c) 3 (d) $\frac{1}{2}$.

7. The rational number having the decimal expansion $\cdot 4\bar{9}$ is

(a) $\frac{49}{99}$ (b) $\frac{1}{2}$

(c) $\frac{49}{90}$ (d) $\frac{5}{11}$.

8. The sum to infinity of the series
$1+2x+4x^2+8x^3+\ldots\ldots$, for $-\frac{1}{2}<x<\frac{1}{2}$ is

(a) $\frac{1}{1+2x}$ (b) $\frac{1}{1-2x}$

(c) $\frac{2}{1-x}$ (d) $\frac{2x}{1-2x}$.

9. The sum of n terms of an A.P. is $n(3-2n)$. The common difference is

(a) -4 (b) 4

(c) -3 (d) 3.

10. The ninth term of a G.P. is 16. If the common ratio is 2, the first term is

(a) $\frac{1}{256}$ (b) $\frac{1}{64}$

(c) $\frac{1}{32}$ (d) $\frac{1}{16}$.

REVIEW EXERCISE V

1. Show that the sum of the first n even integers is equal to $\left(1+\frac{1}{n}\right)$ times the sum of the first n odd integers.

2. If S_1, S_2, S_3 be the sums of n, $2n$ and $3n$ terms of an A.P., show that $S_3=3(S_2-S_1)$.

3. If in an A.P. the sum of p terms be the same as the sum of q terms, show that the sum of $(p+q)$ terms is zero.

4. The sums of the first p, q, r terms of an A.P. are a, b, c respectively. Show that

$$\frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q)=0.$$

5. Find the sum of all the even numbers from 100 to 200 both inclusive.

6. The sum of a certain number of terms of an A.P. is 36 and the first and the last terms are 1 and 11 respectively. Find the number of terms and the common difference.
7. If the sum of n terms of an A.P. is $3n+4n^2$, find the rth term.
8. The sum of n terms of each of two given A.P.'s are in the ratio $5n+7 : 7n+9$. Find the ratio of their 15th terms.
9. Prove that the sum of an even number of terms of an A.P. is equal to the sum of the two middle terms multiplied by half of the number of terms.
10. Prove that the sum of an odd number of terms of an A.P. is equal to the middle term multiplied by the number of terms.
11. Find the sum of the series $a+br+br^2+ar^3+$.........to $2n$ terms.
12. Find the sum to infinity of the series
$$2+\frac{3}{2}+\frac{9}{8}+\frac{27}{32}+\ldots\ldots$$
13. Prove that $\cdot\overline{45}=\frac{5}{11}$.
14. Three numbers whose sum is 15 are in A.P. If 1, 4 and 19 be added to them respectively, the results are in G.P. Determine the numbers.
15. Sum $2.5+5.8+8.11+\ldots\ldots$ to n terms.
16. Sum $1.2^2+2.3^2+3.4^2+\ldots\ldots$to n terms.
17. Find the sum to n terms of the series whose nth term is n^2-n+1.
18. Sum the series $1-2x+3x^2-4x^3+\ldots\ldots$to n terms.
19. Sum the series $1+5x+9x^2+\ldots\ldots$to n terms. Find also the sum to infinity when $|x|<1$.
20. Sum the series to infinity :
$$\frac{1^2}{2}+\frac{3^2}{2^2}+\frac{5^2}{2^3}+\frac{7^2}{2^4}+\ldots\ldots.$$

SUMMARY

1. An A.P. with n terms can be written as
$$a,\ a+d,\ a+2d,\ \ldots\ldots,\ a+(n-1)d,$$
where a is the first term and d is the common difference.
2. The sum of the first n terms, S_n, of an A. P. whose first term is a and common difference is d, is given by
$$S_n=\frac{n}{2}\,[2a+(n-1)d],$$
If l is the last term, then
$$S_n=\frac{n}{2}\,(a+l).$$
3. A G.P. with n terms can be wrtiten as
$$a,\ ar,\ ar^2\ldots\ldots,\ ar^{n-1}$$

4. The sum to n terms S_n, and the sum to infinity of a G.P. whose first term is a and common ratio is r, are given by

$$S_n=\frac{a(1-r^n)}{1-r},\ S_\infty=\frac{a}{1-r}\ (\text{provided } |r|<1).$$

5. The arithmetic mean of two numbers x and y is $\frac{1}{2}(x+y)$.
6. The geometric mean of two numbers x and y is $\sqrt{xy}$.
7. The sum of the first n natural numbers is given by

$$\sum_{k=1}^{n} k=\frac{1}{2}n(n+1).$$

8. The sum of the squares of the first n natural numbers is given by

$$\sum_{k=1}^{n} k^2=\frac{1}{6}n(n+1)(2n+1).$$

9. The sum of the cubes of the first n natural numbers is given by

$$\sum_{k=1}^{n} k^3=\frac{1}{4}n^2(n+1)^2=\left(\sum_{k=1}^{n} k\right)^2$$

HISTORICAL NOTE

Familiarity with arithemetic sequences goes back to Egyptains. Problems involving arithmetic sequences are given in *Rhind Papyrus* (1550 B.C.). *Aryabhata's* work, *Aryabhatiyam,* includes a formula for the sum of n terms of an A.P. after the pth term.

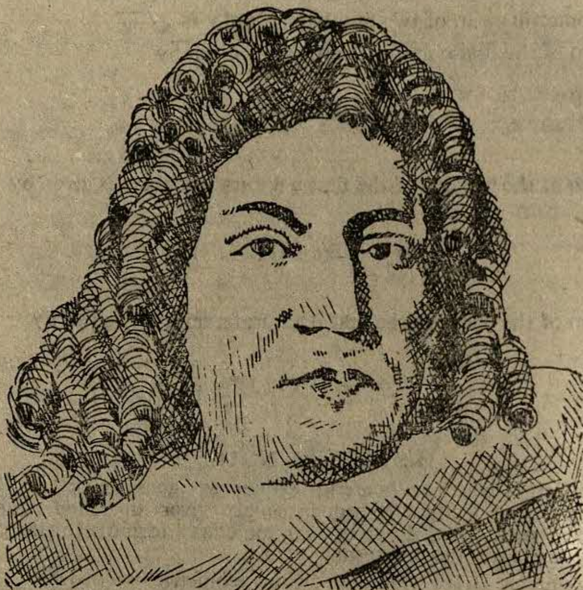
Babylonians were familiar with geometric sequences. Problems involving such sequences are also given in the *Rhind Papyrus.* Euclid gave a rule for summing a geometric series which is equivalent to the modern formula. *Bhaskara* gives several problems involving geometric progressions in his *Lilavati.* The first modern treatment of a G.P. is found in the *Algorithmus de Integris* of *Prosdocimo de Beldamandi* written around 1410 A.D. The first infinite geometric series known to have been summed is

$$1+\frac{1}{4}+\frac{1}{4^2}+\ldots\ldots+\left(\frac{1}{4}\right)^n+\ldots\ldots$$

by Archimedes. The general formula for summing an infinite G.P. was given by *Vieta* (1590 A.D.).

Rules for finding the sum of the squares and cubes of the first n natural numbers are given in the works of *Brahmagupta, Mahavira* and *Bhaskara.* Among the medieval western writers, Fibonacci gave rules for finding sum of squares of first n even numbers in his *Liber Qudratorum* (1225 A.D.). Rule for finding the sum of the first n natural numbers is given in *Pacioli's Suma* (1494 A.D.).

□ □



JACOB BERNOULLI (1654-1705)

Also known as James, Jacques, Jakob and Johann, the Swiss mathematician Jacob Bernoulli was born at Basel.

Bernoulli's *Ars Conjectandi*, published posthumously in 1713, is the first treatise on the theory of probability. It contains the first adequate proof of the binomial theorem for positive integral powers. It also contains a good account of permutations and combinations.

Bernoulli was, however, fascinated the most by curves. He found the equations of the catenary, tractrix and isochrone, the lemniscate of Bernoulli being too well-known to mention. Bernoulli was, however most partial to the logarithmic spiral. He discovered many of its interesting properties and left instructions to the effect that this curve be inscribed on his tombstone. He died at Basel in 1705.

Permutations and Combinations

6·1. INTRODUCTION

We often come across questions such as the following :

1. In how many ways can three pictures be arranged in a row ?
2. In how many ways can six people be seated at a round table ?
3. In how many ways can a team of five players be selected out of a group of eight players ?
4. In how many ways can 8 greeting cards be selected out of 10 and displayed in a row ?

Answers to these questions and many other important and more difficult ones can often be given without actually writing down all the different possibilities. In the present chapter we shall study some basic principles of the art of counting without counting which will enable us to answer questions such as the ones listed above in an elegant manner.

6·2. THE FUNDAMENTAL PRINCIPLE OF COUNTING

(*a*) How many numbers of two digits can be formed out of the digits 1, 2, 3, 4, no digit being repeated ?

The first digit can be any one of the four digits 1, 2, 3, 4, *i e.*, the first digit can be chosen in four ways. Having chosen the first digit, we are left with three digits from which the second digit can be chosen. Therefore, the possible ways of choosing the two digits are :

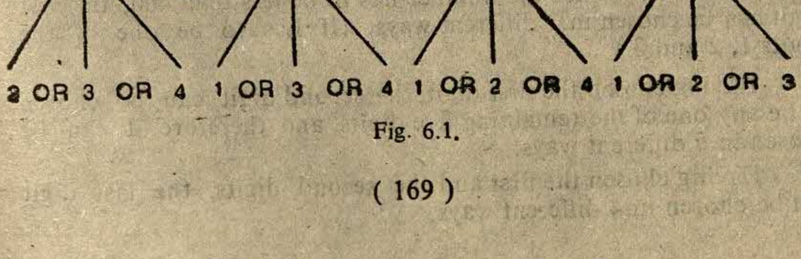


Fig. 6.1.

Since the first digit can be chosen in four ways and for each choice of the first digit there are three ways of choosing the second digit, therefore, there are 4×3 ways, *i.e.*, 12 ways of choosing both the digits. Thus 12 numbers can be formed.

(*b*) Anupama wishes to buy a birthday card for her brother Saurabh and send it by post. Five different types of cards are available at the card-shop, and four different types of postage stamps are available at the post-office. In how many ways can she choose the card and the stamp ? She can choose the card in five ways. For each choice of the card she has four choices for the stamp. Therefore, there are 5×4 ways, *i.e.*, 20 ways of choosing the card and the stamp.

(*c*) Ashu wishes to go from Delhi to Bombay by train and return from Bombay to Delhi by air. There are six different trains from Delhi to Bombay and five different flights from Bombay to Delhi. In how many ways can he perform the journey ?

Since he can choose any one of the six trains for going to Bombay, and for each such choice he has five choices for returning to Delhi, he can perform the journey in 6×5 ways, *i.e.*, 30 ways.

The above illustrations suggest that if one operation can be performed independently in m different ways and another operation can be performed independently in n different ways, then the number of ways in which both the operations can be performed in succession is mn.

The above principle can be generalized to the case of three or more operations. We thus have the following **Fundamental Principle of Counting.**

If one operation can be performed independently in m_1 different ways, and if a second operation can be performed in m_2 different ways, and a third operation can be performed in m_3 different ways and so on for any finite number of operations, then the total number of ways in which all the operations can be performed in the stated order is $m_1 m_2 m_3 \cdots$

Example 1. *How many three digit numbers less than 400 can be formed from the digits 1, 2, 3, 4, 5, 6 ? How many of these numbers are divisible by 5 ?*

Solution. Since the number has to be less than 400, the first digit can be chosen in 3 different ways. (It has to be one of the digits 1, 2 and 3.)

Having chosen the first digit, the second digit can be chosen to be any one of the remaining five digits, and therefore, it can be chosen in 5 different ways.

Having chosen the first and the second digits, the last digit can be chosen in 4 different ways.

By the fundamental principle of counting, all the digits can be chosen in $3 \times 5 \times 4$ ways, *i.e.*, 60 ways.

Thus 60 numbers can be formed. To find as to how many of these numbers are divisible by 5, we proceed as follows :

The first digit can be chosen in 3 different ways, the last digit can be chosen in 1 way (it must be 5), and the middle digit can be chosen in 4 different ways (it can be any of the remaining four digits). Therefore, by the fundamental principle of counting, there are $3 \times 1 \times 4$, *i.e.*, 12 numbers which are divisible by 5.

EXERCISE 6 (*a*)

1. Six roads lead to the top of a mountain. In how many ways can a tourist go up and down the mountain ?
2. There are 7 kinds of envelopes and 4 kinds of stamps of the same denomination. In how many ways can an envelope and a stamp be chosen for sending a letter ?
3. In how many ways can a consonant and a vowel be chosen out of the letters of the word GREAT ?
4. There are 25 boys and 15 girls in a class. In how many ways can a boy and a girl be selected to represent the class in a debate ?
5. There are 10 varieties of pens and 8 varieties of pencils available at a shop. In how many ways can a pen and pencil be chosen ?
6. There are four roads between the cities A and B, and three between the cities B and C. In how many ways can a person drive from A to C and return back going through B each time, without driving on the same road twice ?
7. How many three digit numbers can be formed by using the digits 0, 1, 2, 3 ?
8. How many three-digit numbers can be formed by using the digits 0, 6, 7, 8, no digit being repeated ?
9. How many numbers between 100 and 1000 can be formed if all the digits are different, and the digits 0, 1, 3, 5 are not to be used ?
10. In how many ways can a vowel, a consonant and a digit be chosen out of the 26 letters of the English alphabet and the 10 digits ?

6·3. FACTORIAL NOTATION

We often come across products of the form 1.2, 1.2.3, 1.2.3.4,...

Instead of writing all the factors of such a product in full, it is convenient to use a special notation. We write

$$1\,!=1,$$
$$2\,!=1.2,$$
$$3\,!=1.2.3,$$
$$\cdots\cdots\cdots\cdots$$
$$n\,!=1.2.3...n.$$

'n !' denotes the product of the first n natural numbers. We read 'n!' as 'n factorial'. n ! is also written as '$\underline{n|}$' and read as 'factorial n'. It is easy to see that

$1\,!=1$, $2\,!=1.2=2$, $3\,!=1.2.3=6$, $4\,!=1.2.3.4.=24$, and so on.

Observe that for $n > 1$, $n\,!=1.2.3\cdots(n-1).n$,

$$=((n-1)\,!).n,$$
$$=n((n-1)\,!). \qquad \cdots(1)$$

So far, 'n factorial' has been defined for a positive integer n. It is convenient to give a meaning to 0 ! as well. By convention, $0\,!=1$. Now (1) has a meaning even for $n=1$.

Remarks :

1. Factorials of fractions and negative integers are not defined.
2. We can use a recursive definition to describe factorial n. The relations

$$(n+1)\,!=(n+1)\,(n\,!), \text{ for } n \geqslant 1,$$

and $$1\,!=1$$

describe 'factorial n' for all $n \geqslant 1$.

Example 1. *Find the value of*$=\dfrac{11\,!}{6\,!\,5\,!}$.

Solution. $$\frac{11\,!}{6\,!\,5\,!}=\frac{11.10.9.8.7\,(6\,!)}{(6\,!).1.2.3.4.5},$$
$$=\frac{11.10.9.8.7}{1.2.3.4.5}=462.$$

Example 2. *Show that*

$$\frac{(2n)\,!}{n\,!}=(n+1)\,(n+2)\cdots\cdots\text{to } n \text{ factors},$$
$$=2.6.10.14\cdots\cdots\text{to } n \text{ factors}.$$

Solution. $$(2n)\,!=1.2.3\cdots n\,(n+1)\,(n+2)\cdots(2n)$$
$$=(n\,!)\,[(n+1)\,(n+2)\cdots\text{to } n \text{ factors}].$$

Therefore, $$\frac{(2n)\,!}{n\,!}=(n+1)\,(n+2)\cdots\text{to } n \text{ factors}.$$

Again, $(2n)\,!=1.2.3\cdots n\,(n+1)\,(n+2)\cdots(2n),$
$=(1.3.5\cdots\text{to } n \text{ factors})\,(2.4.6\cdots\text{to } n \text{ factors}),$
$=(1.3.5.\text{ to } n \text{ factors}).\,2^n\,(n\,!),$
$=(2.6.10\cdots\text{to } n \text{ factors})\,n\,!,$

so that $$\frac{(2n)\,!}{n\,!}=2.6.10\cdots\text{to } n \text{ factors}.$$

Example 3. *Prove by mathematical induction that $2^n < n\,!$ for all $n \geqslant 4$.*

Solution. Let $T(n)$ be the statement $2^n < n\,!$

Step 1. Since $2^4=16$, $4\,!=24$, therefore, $2^4 < 4\,!$ *i.e.*, $T(4)$ is true.

Step 2. Let $T(k)$ be true for some $k \geqslant 4$, so that $2^k < k\,!$ Then $2^{k+1}=2.2^k<2\,(k\,!)<(k+1)k\,!$. Thus $2^{k+1}<(k+1)\,!$, so that $T(k+1)$ is true. Since $T(4)$ is true, and the truth of $T(k)$ implies that of $T(k+1)$, therefore, by the principle of finite induction, $T(n)$ is true for all $n \geqslant 4$.

EXERCISE 6 (*b*)

Find thc value of :

1. $4\,!$ 2. $5\,!$ 3. $6\,!$ 4. $\dfrac{10\,!}{7\,!}$

5. $\dfrac{12\,!}{10\,!}$ 6. $\dfrac{10\,!}{6\,!\,4\,!}$ 7. $\dfrac{12\,!}{9\,!\,3\,!}$ 8. $\dfrac{10\,!}{(5\,!)^2}$

Write in factorial form :

9. $6\times5\times4$ 10. 12×11

11. $\dfrac{15\times14}{1\times2}$ 12. $\dfrac{14\times13\times12}{1\times2\times3}$

Factorize :

13. $n\,!-(n-1)\,!$ 14. $(n+1)\,!+(n-1)\,!$

15. $(n+1)\,!+n\,!+(n-1)\,!$

16. Solve for x :
$$\frac{x}{5\,!}+\frac{2x}{6\,!}=1.$$

17. If $n\,!=2[(n-2)\,!]$, find n.

6·4. PERMUTATIONS

Any arrangement of a finite set of objects is called a **permutation** of the set. For example, consider the set {Tea, Coffee, Cocoa}. "Coffee, Tea, Cocoa" is a permutation of this set. "Tea, Cocoa, Coffee" is another permutation of this set. Since a permutation is

an arrangement, therefore, in a permutation the order in which the elements are arranged is important. By changing the order we get a new permutation. As an illustration let us write down all the permutations of the numbers 1, 2, 3. There are in all six permutations, namely

1 2 3	2 1 3	3 1 2
1 3 2	2 3 1	3 2 1

Let us now consider all the permutations of the four numbers 1, 2, 3, 4. There are in all 24 permutations, namely

1 2 3 4	2 1 3 4	3 1 2 4	4 1 2 3
1 2 4 3	2 1 4 3	3 1 4 2	4 1 3 2
1 3 2 4	2 3 1 4	3 2 1 4	4 2 1 3
1 3 4 2	2 3 4 1	3 2 4 1	4 2 3 1
1 4 2 3	2 4 1 3	3 4 1 2	4 3 1 2
1 4 3 2	2 4 3 1	3 4 2 1	4 3 2 1

Observe that out of these 24 permutations, six have 1 in the first place, six have 2 in the first place, and so on. The six permutations having 1 in the first place can be easily written down if we observe that to write a permutation beginning with one we have only

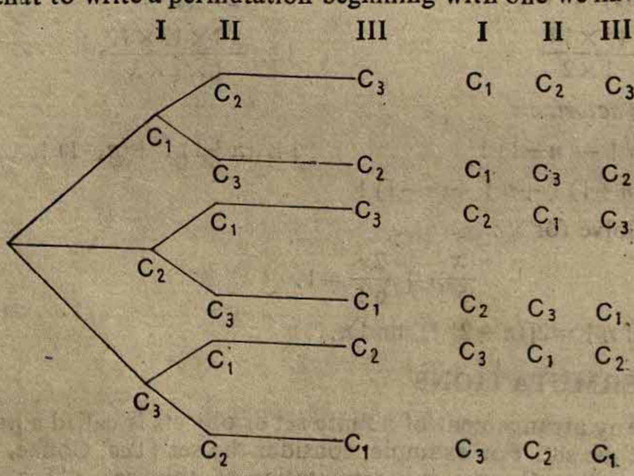


Fig. 6·2.

to follow it by some permutation of 2, 3, 4. Conversely, every permutation of 2, 3, 4 gives rise to a permutation of 1, 2, 3, 4 beginning with 1. Similar is the case with permutations beginning with 2, 3 or 4.

Remark. An easy way to list all possible permutations of a given set of objects is to make use of tree diagrams. Suppose we have three greeting cards (C_1, C_2, C_3) which we wish to send to three friends (I, II, III). We know by the fundamental principle of counting that the cards can be sent to the friends in $3\times2\times1=6$ different ways. Fig. 6·2, called a *tree diagram*, shows these six ways.

The tree diagram of Fig. 6·3 shows all the permutations of four cells A, B, C and D.

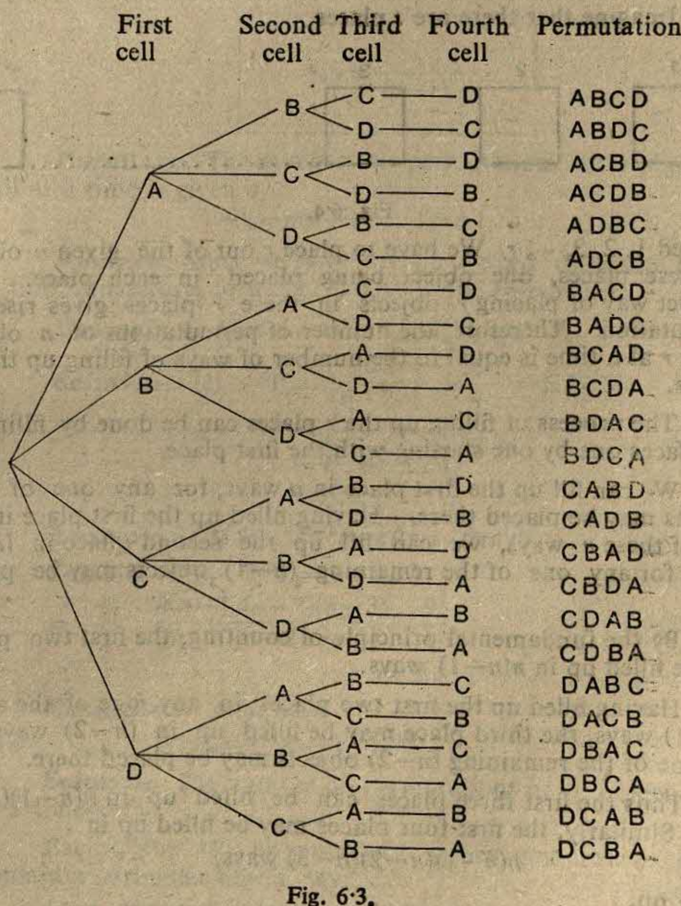


Fig. 6·3.

6·4·1. The Symbol ^nP_r and its Meaning

Sometime we wish to consider arrangements of the objects of a finite set but in every arrangement, we want only a specified number of objects. For example we may be interested in finding all 2-digits numbers where the digits are chosen from the set $\{1, 2, 3, 4\}$. Thus 12, 21, 13, 31 are some of the required arrangements (there being 12 such arrangements in all). Note that every time we took only two objects from the given objects 1, 2, 3, 4. These arrangements are called *permutations of the four digits 1, 2, 3, 4 taken two at a time* and their number is denoted by $^4P_2$. Thus $^4P_2 = 12$. More generally, the *number of permutations of n objects taken r at a time ($n \geqslant r$) is denoted by ^nP_r.*

6·4·2. To find the Number of Permutations of n Different Objects Taken r at a Time

Imagine that there are r places

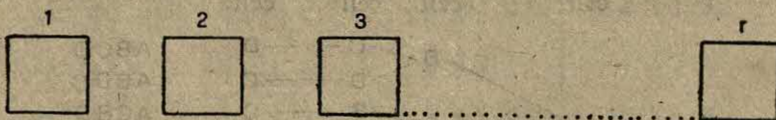


Fig. 6·4.

marked 1, 2, 3, ..., r. We have to place r out of the given n objects in these places, one object being placed in each place. Each distinct way of placing r objects in these r places gives rise to a permutation. Therefore, the number of permutations of n objects taken r at a time is equal to the number of ways of filling up these r places.

The process of filling up the r places can be done by filling up the places one by one starting with the first place.

We can fill up the first place in n ways, for any one of the n objects may be placed there. Having filled up the first place in any one of these n ways, we can fill up the second place in $(n-1)$ ways, for any one of the remaining $(n-1)$ objects may be placed there.

By the fundamental principle of counting, the first two places can be filled up in $n(n-1)$ ways.

Having filled up the first two places in any one of the above $n(n-1)$ ways, the third place may be filled up in $(n-2)$ ways, for any one of the remaining $(n-2)$ objects may be placed there.

Thus the first three places can be filled up in $n(n-1)(n-2)$ ways. Similarly, the first four places may be filled up in

$$n(n-1)(n-2)(n-3) \text{ ways,}$$

and so on.

Proceeding in this manner, we find that all the r places can be filled up in

$$n(n-1)(n-2) \cdots (n-r+1) \text{ ways.}$$

Therefore

$$\boxed{{}^n\mathrm{P}_r = n(n-1) \ldots (n-r+1).} \qquad \ldots(1)$$

We can easily express ${}^n\mathrm{P}_r$ in factorial notation. Multiplying and dividing the right-hand side of (1) throughout by $(n-r)$!, we have

$${}^n\mathrm{P}_r = \frac{[n(n-1) \cdots (n-r+1)]\,(n-r)\,!}{(n-r)\,!},$$

i.e.,

$$\boxed{{}^n\mathrm{P}_r = \frac{n\,!}{(n-r)\,!}.} \qquad \ldots(2)$$

Corollary. The number of permutations of n objects taken all at a time is given by

$${}^n\mathrm{P}_n = n(n-1) \ \ldots \ 1 = n\,!$$

Remark. Since we have already agreed to assign the value 1 to 0 !, therefore, formula (2) is valid for $r=n$ as well.

Example 4. *Find the value of* (*i*) ${}^6P_2$ (*ii*) ${}^7P_4$.

Solution. (*i*) ${}^6\mathrm{P}_2 = \dfrac{6\,!}{(6-2)\,!} = \dfrac{6\,!}{4\,!} = 6.5 = 30.$

(*ii*) ${}^7\mathrm{P}_4 = 7.6 \cdots (7-4+1)$, [Using (1)]

$= 7 \,.\, 6 \,.\, 5 \,.\, 4,$

$= 840.$

Example 5. *If* ${}^nP_4 : {}^{n-1}P_3 = 9 : 1$, *find* n.

Solution. ${}^n\mathrm{P}_1 : {}^{n-1}\mathrm{P}_3 = 9 : 1$

$$\Rightarrow \quad \frac{n(n-1)(n-2)(n-3)}{(n-1)(n-2)(n-3)} = \frac{9}{1},$$

$$\Rightarrow \quad n = 9.$$

Example 6. *Show that*

$${}^nP_r = {}^{n-1}P_r + r \,.\, {}^{n-1}P_{r-1}.$$

Solution. The number of permutations of n objects taken r at a time $= {}^n\mathrm{P}_r$. ...(*i*)

Each of the ${}^n\mathrm{P}_r$ permutations either contains or does not contain a particular object, say 'x'.

If we exclude 'x', we are left with $n-1$ objects.

The number of permutations of $n-1$ objects taken r at a time $= {}^{n-1}P_r$. ...(*ii*)

If we include 'x', we first place it in any one of the r places. This can be done in r ways. The remaining $(r-1)$ places can be filled up from the $(n-1)$ objects that are left, and this can be done in ${}^{n-1}P_{r-1}$ ways.

$\therefore$ The number of permutations containing x

$$= r \cdot {}^{n-1}P_{r-1} \qquad ...(iii)$$

From (*i*), (*ii*) and (*iii*), we have

$${}^{n}P_r = {}^{n-1}P_r + r \cdot {}^{n-1}P_{r-1}$$

EXERCISE 6 (*c*)

1. Find the value of

   (*i*) ${}^{7}P_4$, (*ii*) ${}^{8}P_2$, (*iii*) ${}^{10}P_3$.

2. If ${}^{n}P_5 : {}^{n-1}P_4 = 8 : 1$, find n.
3. If ${}^{15}P_{r-1} : {}^{16}P_{r-2} = 3 : 4$, find r.
4. If ${}^{56}P_{r+6} : {}^{54}P_{r+3} = 30800 ; 1$, find r. (*Roorkee Entrance, 1983*)
5. If ${}^{22}P_{r+1} : {}^{20}P_{r+2} = 11 : 52$, find r.
6. Show that

   $${}^{n+1}P_r = {}^{n}P_r + r\,{}^{n}P_{r-1}.$$

7. Find the number of permutations of 6 objects taken 3 at a time.
8. Three students enter a bus in which four seats are vacant. In how many ways can they occupy the seats ?
9. How many words can be formed from the letters of the word DELHI using all of them at a time ?
10. How many numbers of three digits can be formed with the digits 2, 4, 5, 6, 7, no digit being repeated ?

6·5. SOME SIMPLE APPLICATIONS OF PERMUTATIONS

Example 7. *How many numbers greater than 10 can be formed out of the digits 4, 5, 6, 7, no digit being repeated* ?

Solution. Since the number has to be greater than 10, it may contain two, three or four digits. Thus there are three possibilities which are mutually exclusive (*i.e.*, no two of them can hold simultaneously), and exhaustive (*i.e.*, they cover all the possible cases).

The number of ways of forming a two-digit number $= {}^{4}P_2 = 4.3 = 12$.

The number of ways of forming a three-digit number $= {}^{4}P_3 = 4.3.2 = 24$.

The number of ways of forming a four-digit number$={}^4P_4$ $=4.3.2.1=24$.

Since the above three possibilities cover all the possible cases, therefore, in all $12+24+24=60$ numbers greater than 10 can be formed out of the digits 4, 5, 6 and 7.

Example 8. *In how many ways can 10 books be arranged on a shelf so that a particular pair of books shall be (i) always together, (ii) never together ?*

Solution. (*i*) Since we want the books to be always together, let us tie them together and consider the pair as one book. Then we have 9 books to be arranged on the shelf. The number of ways in which this can be done$={}^9P_9=9\,!$.

In each of these 9 ! ways, the two books which have to be always together can be arranged among themselves in 2 ! ways without disturbing the others.

Therefore, the number of ways in which the two books are together$=2\,!\times 9\,!=2\,(9\,!)$.

(*ii*) Since 10 books can be arranged in 10 ! ways, therefore, the number of arrangements in which the two books are never together $=10\,!-2\,(9\,!)=8\,(9\,!)$.

Example 9. *In how many ways can 4 ladies and 6 gentlemen be seated in a line, so that no two ladies may come together ?*

Solution. We may perform the seating arrangement in two operations :

(*i*) Seating the 6 gentlemen in a row :

$$\times G_1 \times G_2 \times G_3 \times G_4 \times G_5 \times G_6 \times$$

Evidently this can be done in 6 ! ways.

(*ii*) In between the six gentlemen, there are five places. Also there are two places at the two ends. There are thus 7 places in all. If the ladies sit anywhere out of these seven places, no two of them are ever together. The number of ways in which the ladies can sit thus is ${}^7P_4$.

Hence the required number of ways of seating 4 ladies and 6 gentlemen

$$={}^7P_4\times 6\,!,$$

$$=\frac{7\,!\times 6\,!}{3\,!},$$

$$=120\,(7\,!).$$

EXERCISE 6 (*d*)

1. In how many of the permutations of 10 things taken 4 at a time, will a particular thing (*i*) never occur, (*ii*) always occur ?

2. In how many of the permutations of n things taken r at a time, will three particular things
(*i*) not occur, (*ii*) always occur ?

3. In how many of the arrangements of 12 books taken seven at a time will 4 specified books be
(*i*) always included, (*ii*) always excluded ?

4. In how many arrangements of the letters in the word GOLDEN will the vowels never occur together ?

5. In how many arrangements of the letters in the word 'ENGLISH' will the vowels occur together ?

6. Seven papers are to be held for an examination, two of them in Mathematics. In how many ways can the papers be held so that Mathematics papers are never together ?

7. How many arrangements of the letters of the word 'DAUGHTER' can be formed in which the vowels occur together ?

8. In how many ways can six books on English, five on Hindi and three on Science can be arranged on a shelf so that the books on each subject are always together ?

9. In how many arrangements of the letters of the word 'ARTICLE' will,
(*i*) the vowels occupy even places ;
(*ii*) the relative position of the vowels and the consonants remain unchanged ?

10. Find the number of words which can be formed with two different consonants and one vowel out of 7 different consonants and 3 different vowels, so that the vowel may always be between the consonants.

11. Find the number of ways in which 5 ladies and 5 gentlemen may be seated in a row, so that no two ladies are together.

12. Find the number of ways in which m boys and n girls may be arranged in a row so that no two of the girls are together, it being assumed that $m>n$.

13. The digits 1, 2, 4, 6 are written in every possible order. Find the sum of all the numbers so formed.
[**Hint** : The digit 1 will occur in unit's place, ten's place, hundred's place or thousand's place each in one-fourth of all the numbers formed. etc.]

14. The digits 1, 5, 7, 9 are written in every possible way. Find the sum of all the numbers so formed.

6·6. PERMUTATIONS INVOLVING IDENTICAL OBJECTS

Example 10. *In how many ways can the letters of the word FIFTY be arranged?*

Solution. The word FIFTY contains two F's which are identical. Let us distinguish them from each other by adding suffixes 1 and 2, *i.e.*, let us consider $F_1 I F_2 T Y$. We can arrange, F_1, I, F_2, T, Y in 5 ! ways. Here the two permutations $F_1 I F_2 T Y$ and $F_2 I F_1 T Y$ have been counted as two separate permutations. This means that the arrangement FIFTY is counted twice in the 5 ! permutations of 2 F's, I, T, and Y. Since we can arrange F_1 and F_2 in 2 ! ways, every distinct arrangement is counted 2 ! times in the permutations of $F_1 I F_2 T Y$.

Therefore, the number of arrangements of the letters of FIFTY is $\dfrac{5\,!}{2\,!}=60$.

Example 11. *In how many ways can the letters of the word INNINGS be arranged?*

Solution. There are seven letters in all, three N's, two I's, one G, and one S.

The number of permutations of $I_1N_1N_2I_2N_3GS$ is 7 !. Since N_1, N_2, N_3 can be arranged in 3 ! ways, and I_1, I_2 can be arranged in 2 ! ways, therefore, the number of permutations of $I_1N_1N_2I_2N_3GS$ is 3 ! × 2 ! times the number of permutations of INNINGS.

Therefore, the number of permutations of the letters of the word INNINGS $=\dfrac{7\,!}{3\,!\,2\,!}=420$.

Remark. From the above two examples we find that the *number of permutations of n objects when p of them are alike of one kind, q of them are alike of a second kind, r of them are alike of a third kind, and the rest are all different is* $\dfrac{n\,!}{p\,!\,q\,!\,r\,!}$.

Example 12. *Find the number of arrangements which can be made out of the letters of the word 'CALCUTTA'.*

Solution. There are 8 letters in all, of which there are two C's, two A's, two T's, and two others (L, U) are different.

$$\therefore \text{ the number of arrangements}=\frac{8\,!}{2\,!\,2\,!\,2\,!},$$

$$=\frac{8\times7\times6\times5\times4\times3\times2\times1}{2\times1\times2\times1\times2\times1}$$

$$=5040.$$

Example 13. *Find the number of arrangements of the letters of the word 'ALGEBRA' without altering the relative position of the vowels and the consonants.*

Solution. There are 7 letters, of which two are A's and the rest are all different.

The vowels A, A, E occupy 1st, 4th and 7th places. The number of ways in which they can be arranged in these places is

$$\frac{3\,!}{2\,!}=3.$$

The consonants L, G, B, R are all different. The number of ways in which they can arranged in the remaining places is 4 !.

Since each way of arranging the vowels can be *associated* with each way of arranging the consonants, we find that the total number of arrangements

$$=3\times 4\,!,$$
$$=3\times 24=72.$$

Example 14. *A boy has 6 pockets. In how many ways can he put 4 marbles in his pockets ?*

Solution. There is no restriction as to the number of marbles that can be put in any pocket. For each marble there are 6 possible choices.

Therefore, the required number of ways $=6^4=1296$.

EXERCISE 6 (*e*)

1. In how many ways can 15 marbles be arranged in a row if 7 of them are red, 5 are blue and 3 are yellow ?
2. How many different permutations can be made from the letters of the word :

   (*i*) MEERUT, (*ii*) AMBALA,
   (*iii*) COLLEGE, (*iv*) UNDERSTAND.
3. How many arrangements can be made with the letters of the word 'ALLAHABAD' and in how many of them the vowels occur together ?
4. In how many ways can the letters of the word 'ARRANGE' be arranged if the two R's are not allowed to come together ?
5. A library has on one shelf, twenty books, in which there are 5 copies of one book, 4 copies of another book, and the rest of the books are all different. In how many ways can the books be arranged on the shelf ?
6. How many different arrangements of the letters of the expression $a^3b^2cd^4$ can be made when it is written out at full length ?

7. How many numbers greater than a million can be formed with the digits 2, 3, 0, 3, 4, 2, 3 ?

8. In how many ways can the letters of the word 'CORRECTION' be arranged so that the vowels may occupy the even places ?

9. In how many ways can 5 different things be divided between four persons ? (Each person may receive one, more or none of the things.)

10. There are three different routes joining two places. In how many ways can a person perform the double journey ?

11. There are three candidates for a post, and one is to be elected by the votes of 5 men. In how many ways can the votes be given ?

12. In how many ways can n prizes be given away to n students, so that

    (*i*) each boy may receive a prize,

    (*ii*) a boy may receive any number of prizes,

    (*iii*) no boy should receive all the prizes, but may receive more than one prize ?

13. The first and second prizes in each of the four subjects— English, Mathematics, Physics and Chemistry, are to be awarded to a class of 15 students. Find the number of ways in which this can be done.

14. A letter-lock consists of three rings, each marked with 15 different letters ; find in how many ways it is possible to make an unsuccessful attempt to open the lock.

    [**Hint** : All possible attempts except one are unsuccessful.]

15. How many numbers between 20,000 and 36,000 can be formed with the digits 2, 3, 5, 6, 9, if each digit may be repeated any number of times ?

16. A purse contains a rupee, a fifty paise piece, a twenty paise piece, a ten paise piece and one five paise piece. In how many ways is it possible to draw a sum of money from the bag ?

17. A code consists of two different symbols only. How many different messages of 5 symbols each can be formed, the symbols being allowed to be repeated ?

6·7. CIRCULAR ARRANGEMENTS

When we arrange objects in a row, there is always a first place to consider In the case of arrangements in a circle, there is no first, second, third,..., last place. What really matters is the position of an object relative to the others.

Example 15. *Six persons are invited for a dinner. In how many ways can they be seated at a round table ?*

Solution. Since the persons are to be seated at a round table. there is no first or last place to be considered. No seat being special, we have only to consider the relative positions of the guests,

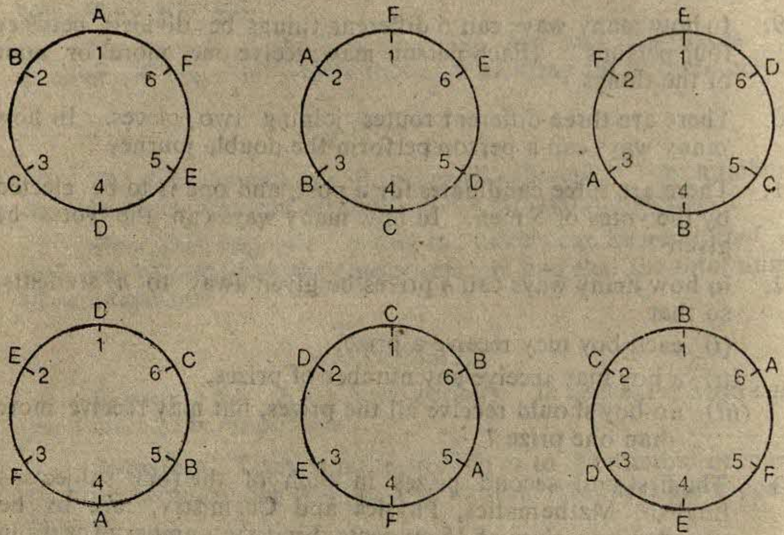


Fig. 6·5.

Let us number the chairs from 1 to 6. There are 6 ways of selecting a person to occupy first chair, 5 ways of selecting a person to occupy the second chair,··· and so on. Thus there are 6 ! ways of arranging the guests in the six seats.

From our point of view, this number is not the correct number for (as we shall see just now), it includes arrangements which are not essentially different from each other.

Consider the six arrangements shown in Fig. 6·5 (we have labelled the six guests as A, B,..., F).

The second arrangement is obtained from the first by moving the guests in the anti-clockwise direction along the table by one seat each. The third arrangement is obtained by repeating this process with the second arrangement, and so on. In other words, all the arrangements shown above can be obtained from the first one (or from any other !) by moving all the guests by a certain number of positions in the anti-clockwise direction along the table. Observe that in all the six arrangements shown above every guest has the same neighbour to the left, and the same neighbour to the right. (For example, B is always to the right of A, and F is always to the left of A.)

Therefore, the 6 ! ways of arranging the six guests in the seats numbered 1 to 6 is *six times* the number of ways in which they can be arranged round a circular table, so that there are in all $\frac{1}{6}$ (6!) = 5! ways of seating the guests at the round table.

Hence the required number of ways = 5 ! = 120.

Remarks. 1. Generalizing the above problem, we can easily see that *the number of ways of arranging n persons in a circle is $(n-1)$!.*

2. In the above example we have assumed that all the chairs are identical and there is nothing special about any position.

Example 16. *In how many ways can six persons be seated at a round table so that all shall not have the same neighbours in any two arrangements ?*

Solution. This example is similar to the previous one but with one difference. Consider the following two arrangements :

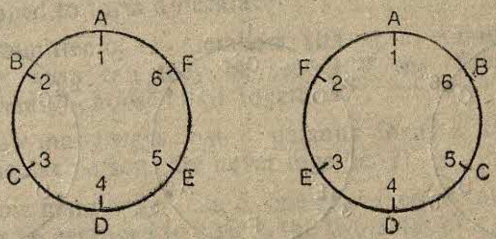


Fig. 6·6

In both the above arrangements every guest has the same neighbours. For example, in the first arrangement A has B to the right and F to the left ; in the second arrangement A has B to the left and F to the right. In the previous example both these arrangements were counted as two different arrangements. In the present example, since we are concerned only with the neighbours being the same and not with their being to the left or right, therefore we shall not consider these arrangements as different. The total number of arrangements is, therefore, half the number of arrangements in the preceding example.

$$\text{The required number of arrangements} = \frac{1}{2} \times (5\,!),$$
$$= 60.$$

Example 17. *In how many ways can six beads be chosen from nine different beads and threaded so as to form a ring ?*

Solution. The number of ways of arranging six beads, chosen from nine different beads in six places numbered 1 to 6 is ${}^9P_6$. Therefore, the number of ways of arranging six beads (in a circle) from nine different beads is $\frac{1}{6} \cdot {}^9P_6$. For, if A, B, C, D, E, F be the six beads selected out of the nine given beads, the following six arrangement are to be regarded as same (and of coure, there would be other such sets of six arrangements which are to be regarded as same).

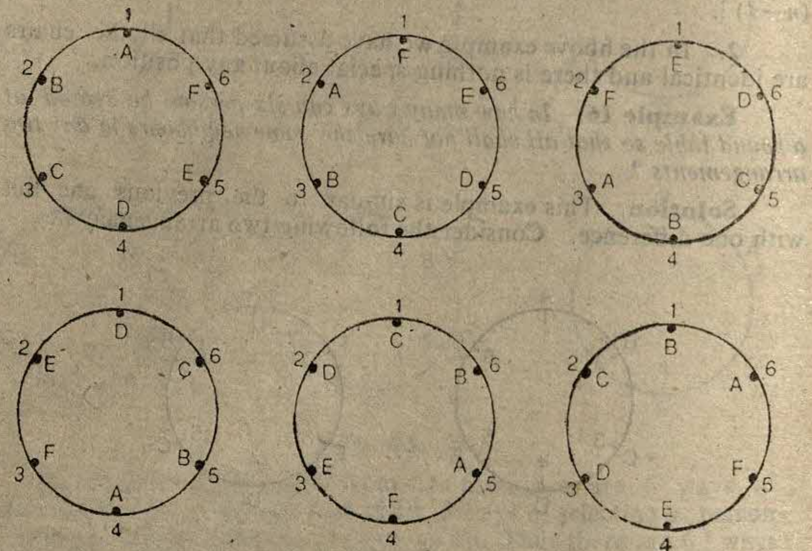


Fig. 6·7.

Since a ring can be turned upside down, the two arrangements such as the following are to be regarded as the same :

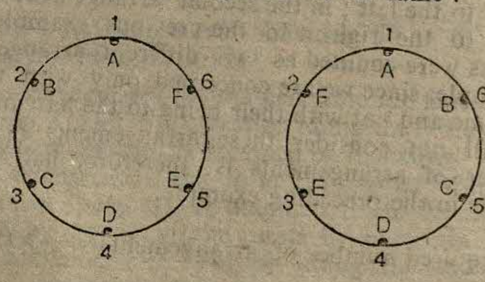


Fig. 6·8.

However, while counting the number of circular arrangements, these two arrangements have been counted as separate arrangements.

Therefore, the number of arrangements in a ring must be half the number of circular arrangements. Therefore, the number of arrangements in a ring$=\frac{1}{2}\left(\frac{1}{6}.{}^9P_6\right)=\frac{1}{2}.\frac{1}{6}.9.8.7.6.5.4=5040.$

EXERCISE 6 (*f*)

1. In how many ways can eight people be seated at a round table ?
2. In how many ways can seven persons be seated at a round table so that all shall not have the same neighbours in any two arrangements ?
3. In how many ways can ten beads be threaded so as to form a necklace ?
4. In how many ways can a necklace of seven beads be formed out of nine different beads ?
5. In how many ways can 12 persons be seated at a round table ?
6. Find the number of ways in which n different beads can be arranged to form a necklace.
7. A committee of 11 members sits at a round table. In how many ways can they be seated if the 'President' and the 'Secretary' choose to sit together ?
8. In how many ways can 8 persons form a ring so that two particular persons are never together ?
9. In how many ways can 5 one-rupee coins and six ten-paise coins be arranged in a ring, the two sides of a coin being taken as different from each other ?
10. In how many ways can 6 gentlemen and 4 ladies be arranged in a ring, so that no two ladies occupy consecutive places ?
11. In how many ways can 21 beads of which 7 are alike of one kind and 14 are alike of another kind, be formed into a ring ?

6·8. COMBINATIONS

Any selection which can be made by taking some (or all) of a number of objects is called a **combination**. For example, a team of two players selected out of five players is a combination of 5 players taken 2 at a time. In the case of a combination, the order of the objects does not matter. The objects in a combination when arranged give rise to a permutation. The number of combinations of n objects taken r at a time is usually denoted by the symbol ${}^nC_r$.

As an illustration of the above concepts, consider the four letters A, B, C, D. It is possible to select three letters out of these in 4 ways, namely, ABC, ABD, ACD and BCD. We say that there

are four combinations of 4 letters taken 3 at time. In symbols, we write ${}^4C_3=4$.

Each of the four combinations written above gives rise to 3 ! permutations. For example, A, B, C can be arranged as

ABC
ACB
BCA
BAC
CAB
CBA.

ABC, ACB, BCA, BAC, CAB, CBA represent different permutations but all of them represent the same combination. From the above discussion we find that ${}^4P_3={}^4C_3\times(3\,!)$.

We shall see that a similar relation holds between ${}^nP_r$ and ${}^nC_r$. In fact, we shall prove that

$$ {}^nP_r={}^nC_r\times(r\,!). $$

6·8·1. To Find the Number of Combinations of n Dissimilar Objects Taken r($\geqslant$1) at a Time

Let the required number of combinations be denoted by ${}^nC_r$.

Each of these combinations is a collection of r dissimilar objects which can be arranged among themselves in r ! ways. Hence each combination gives rise to r ! permutations. Hence ${}^nC_r$ combinations will give rise to ${}^nC_r\times r$! permutations. But the number of permutations of n things taken r at a time is ${}^nP_r$.

$$ \therefore \quad {}^nC_r\times r\,!={}^nP_r, $$

or

$$ {}^nC_r=\frac{{}^nP_r}{r\,!}=\frac{n(n-1)\ldots(n-r+1)}{r\,!}, \qquad \ldots(i) $$

or

$$ {}^nC_r=\frac{{}^nP_r}{r\,!}=\frac{n\,!}{(n-r)\,!\,r\,!}. \qquad \ldots(ii) $$

Hence

$$ {}^nC_r=\frac{n(n-1)\cdots(n-r+1)}{r\,!}=\frac{n\,!}{(n-r)\,!\,r\,!}. $$

Note. It is convenient to use (i) when a numerical result is required and (ii) when it is sufficient to leave the result in the factorial notation.

Example 18. *Find the number of diagonals that can be drawn by joining the angular points of a heptagon.*

Solution. A heptagon has seven angular points and seven sides.

The join of two angular points is either a side or a diagonal.

The number of lines joining the angular points

$$={}^7C_2=\frac{7\times 6}{1\times 2}=21.$$

But the number of sides $=7$.

$\therefore$ number of diagonals $=21-7=14$.

6·8·2. Complementary Combinations

Theorem 6·1. *The number of combinations of n different objects taken r at a time is equal to the number of combinations of n different objects taken n−r at a time.*

Proof. Every time we take a collection of r objects out of n given objects, there is left a collection of $n-r$ objects Thus the number of combinations of n objects taken r at a time is the same as the number of combinations of n objects taken $(n-r)$ at a time.

Hence $\boxed{{}^nC_r={}^nC_{n-r}.}$ ...(A)

The above proposition may also be proved as follows :

$$\begin{aligned}{}^nC_{n-r}&=\frac{n\,!}{(n-r)\,!\,\{n-(n-r)\}\,!},\\&=\frac{n\,!}{(n-r)\,!\,r\,!},\\&={}^nC_r.\end{aligned}$$

Note. So far, the relation (A) holds good for $r<n$, for the right hand side is without any meaning when $r=n$. If we set ${}^nC_0=1$, (A) becomes valid for $r=n$ also. We shall in future *take* ${}^nC_0=1$, and (A) will therefore, be true for $r\leqslant n$.

Theorem 6·2. ${}^nC_r={}^{n-1}C_{r-1}+{}^{n-1}C_r$.

Proof. ${}^nC_r$ denotes that number of combinations of n different objects taken r at a time. If we mark one of the objects say 'x', some combinations will include 'x', others will not include it.

The number of combinations which include 'x' is ${}^{n-1}C_{r-1}$, for we have to select $(r-1)$ objects out of the remaining $(n-1)$ objects .

The number of combinations which do not include 'x' is ${}^{n-1}C_r$, for leaving aside 'x' we have $(n-1)$ objects left and we have to select r objects out of them.

Hence $\boxed{{}^nC_r={}^{n-1}C_{r-1}+{}^{n-1}C_r}$

Aliter :

$${}^{n-1}C_{r-1}+{}^{n-1}C_r=\frac{(n-1)\,!}{(r-1)\,!\,\{(n-1)-(r-1)\}\,!}+\frac{(n-1)\,!}{r\,!\,(n-1-r)\,!},$$

$$=\frac{(n-1)!}{(r-1)!\,(n-1)!}+\frac{(n-1)!}{r!\,(n-r-1)!},$$

$$=\frac{r(n-1)!}{r(r-1)!\,(n-r)!}+\frac{(n-r)(n-1)!}{r!\,(n-r-1)!\,(n-r)!},$$

$$=\frac{r(n-1)!}{r!\,(n-r)!}+\frac{(n-r)(n-1)!}{r!\,(n-r)!},$$

$$=\frac{(n-1)!}{r!\,(n-r)!}\{r+n-r\},$$

$$=\frac{n(n-1)!}{r!(n-r)!}=\frac{n!}{r!\,(n-r)!}.$$

Example 19. *If* ${}^{18}C_r={}^{18}C_{r+2}$, *find* ${}^{r}C_5$.

Solution. $\quad {}^{18}C_r={}^{18}C_{r+2}$ ...(*i*)

Also, $\quad {}^{18}C_r={}^{18}C_{18-r}$ ...(*ii*)

From (*i*) and (*ii*), we have

$$18-r=r+2,$$

or

$$r=8.$$

$${}^{r}C_5={}^{8}C_5={}^{8}C_3=\frac{8.7.6}{1.2.3}=56.$$

Example 20. *If* ${}^{28}C_{2r} : {}^{24}C_{2r-4}=225 : 11$, *find* r.

Solution. We have

$${}^{28}C_{2r}=\frac{28!}{(2r)!\,(28-2r)!}.$$

$${}^{24}C_{2r-4}=\frac{24!}{(2r-4)!\,(24-2r+4)!}.$$

$$\therefore \quad \frac{{}^{28}C_{2r}}{{}^{24}C_{2r-4}}=\frac{28!}{(2r)!\,(28-2r)!}\times\frac{(2r-4)!\,(28-2r)!}{24!},$$

$$=\frac{28\times 27\times 26\times 25}{2r(2r-1)(2r-2)(2r-3)}$$

Also, $\quad \dfrac{{}^{28}C_{2r}}{{}^{24}C_{2r-4}}=\dfrac{225}{11}.$

so that $\quad \dfrac{28\times 27\times 26\times 25}{2r(2r-1)(2r-2)(2r-3)}=\dfrac{225}{11}.$

$$\therefore \quad 2r(2r-1)(2r-2)(2r-3)=\frac{28\times 27\times 26\times 25\times 11}{225},$$

or

$$(4r^2-6r)(4r^2-6r+2)=28\times 26\times 11\times 3.$$

Putting $4r^2-6r=x$, we have

$$x(x+2)=24024,$$

or

$$x^2+2x-24024=0,$$

or

$$(x+1)^2=24025$$

$$\therefore \quad x+1=\pm 155,$$

i.e., $x=154$ or -156.

When $x=154$,

$$4r^2-6r-154=0,$$

or $2r^2-3r-77=0,$

or $$r=\frac{3\pm\sqrt{9+616}}{4},$$

$$=\frac{3\pm25}{4}=7 \text{ or } -\frac{22}{4}$$

When $x=-156$,

$$4r^2-6r+156=0,$$

which has imaginary roots.

Hence $r=7$.

Example 21. *In how many ways can 12 distinct objects be divided into two groups containing 4 and 8 objects respectively ?*

Solution. Every time we take a set of 4 objects, a second set of 8 objects is left behind.

Hence the required number of ways=number of combinations of 12 objects taken 4 at a time

$$={}^{12}C_4=\frac{12\,!}{4\,!\,8\,!}=\frac{12\times11\times10\times9}{1\times2\times3\times4}=495.$$

Example 22. *In how many ways can a party of eight children be divided into two groups of four children ?*

Solution. The number of combinations of 4 children selected from 8 children

$$={}^{8}C_4=\frac{8\,.\,7\,.\,6\,.\,5}{1\,.\,2\,.\,3\,.\,4}=70.$$

Every time we choose one group of four children, the remaining four children form the other group.

Let us use the letter A, B,......, H to denote the children. Two possible choices for the first group are ABCD and EFGH. The children in the second group in these two cases are EFGH and ABCD respectively. But the two groups 'ABCD, EFGH' and 'EFGH, ABCD' are the same. Therefore, the number 70 of combinations of four children out of eight is twice the number of possible divisions into two equal groups of four children.

Therefore there are $\frac{1}{2}\times70$ ways, *i.e.,* 35 ways of dividing the children into two equal groups.

Example 23. *In how many ways can a committee of 5 be formed from 6 men and 4 women so as to include at least 2 women ?*

Solution. There are three mutually exclusive cases. The committee may consist of

(*i*) 3 men and 2 women,

or (*ii*) 2 men and 3 women,

or (*iii*) 1 man and 4 women.

We can select 3 men and 2 women in

$${}^6C_3 \times {}^4C_2 = 20 \times 6 = 120 \text{ ways.}$$

We can select 2 men and 3 women in

$${}^6C_2 \times {}^4C_3 = 15 \times 4 = 60 \text{ ways.}$$

We can select 1 man and 4 women in

$${}^6C_1 \times {}^4C_4 = 6 \times 1 = 6 \text{ ways.}$$

$\therefore$ the required number of ways $=120+60+6=186$.

Hence the number of ways in which the committee can be formed $=186$.

Example 24. *In how many ways can we choose three balls out of eight balls, four of which are black, two are white, one is red nd one is green ?*

Solution. There are three mutually exclusive cases :

(1) We may choose *all the three balls of the same colour.* This can be done in 1 way. (We may choose three black balls.)

(2) We may choose *two balls of one colour and one ball of a diffrent colour.*

Now two balls of the same colour can be chosen in 2 ways (both the balls may be black or white), and the third ball of a different colour can be chosen in 3 ways, therefore, there are 2×3, *i.e.*, 6 ways of choosing the three balls when two balls are of the same colour and the third ball is of a different colour.

(3) We may choose *three balls of three different colours.* Since there are balls of four different colours from which a choice of three colours has to be made, therefore, the three balls can be chosen in ${}^4C_3$ ways, *i.e.*, 4 ways. Hence there are $1+6+4$ ways, *i.e.*, 11 ways of choosing the balls.

Example 25. *Out of 5 consonants and 3 vowels, how many words can be formed, each containing two consonants and one vowel ?*

Solution. A word is an arrangement of letters. To each selection of 3 letters there will correspond 3 ! words. We shall first find the number of ways in which we can select 2 consonants and 1 vowel.

There are 3 vowels and we have to take 1 vowel ; this can be done in ${}^3C_1$ or 3 ways.

There are 5 consonants and we have to take 2 consonants ; this can be done in ${}^5C_2$ or 10 ways.

The choice of vowels does not affect the choice of consonants in any way, *i.e.*, these two combinations are independent of each other. For, each of the 3 ways of chosing the vowels there are 10 ways of choosing the consonants.

$\therefore$ We can make a selection of 2 consonants and one vowel in $3 \times 10 = 30$ ways.

But each selection gives 3 ! words.

$\therefore$ The number of words that can be formed $= 30 \times 3\,!$

$$= 180.$$

Hence 180 words can be formed.

Example 26. *Find the number of (i) combinations, (ii) permutations of the letters of the word 'INDEPENDENCE' taken 4 at a time.*

Solution. There are 12 letters of which 6 are distinct. In fact, we have 4E's, 3N's, 2D's, I, P, C.

The combinations may be made as follows :

(*i*) all 4 alike ;

(*ii*) 3 alike, 1 different ;

(*iii*) 2 alike, 2 alike but different from the first ;

(*iv*) 2 alike, 2 different ;

(*v*) all 4 different.

Now

(*i*) can be done in 1 way only, for there is a single set of 4 E's.

(*ii*) can be done in $2 \times 5 = 10$ways, for we may take one of the two sets EEE or NNN and one of the remaining 5 letters ;

(*iii*) can be done in ${}^3C_2 = 3$ ways, for we may select two sets out of the three pairs EE, NN, DD.

(*iv*) can be done in ${}^3C_1 \times {}^5C_2 = 30$ ways, for with one of the sets EE, NN, DD, we may take 2 different letters out of the 5 remaining letters.

(*v*) can be done in ${}^6C_4 = 15$ ways.

Hence the total number of combinations

$$= 1 + 10 + 3 + 30 + 15 = 59.$$

To find the number of permutations, we have to permute the letters in each case

Thus the number of permutations is

in (*i*) $1 \times \dfrac{4\,!}{4\,!} = 1$,

in (*ii*) $10 \times \dfrac{4\,!}{3\,!} = 40$,

in (*iii*) $3 \times \dfrac{4\,!}{2\,!\,2\,!} = 18$,

in (*iv*) $30 \times \dfrac{4\,!}{2\,!} = 360$,

and in (*v*) $15 \times 4\,! = 360$.

Hence the total number of permutations

$$=1+40+18+360+360,$$
$$=779.$$

EXERCISE 6 (*g*)

1. Find the value of (*i*) ${}^{24}C_4$, (*ii*) ${}^{19}C_{15}$.
2. Find the number of combinations of 50 things taken 46 at a time.
3. In how many ways can a party of 4 be selected from 10 persons ?
4. In how many ways can a party of 4 be selected out of 9, if one man is always to be (*i*) included, (*ii*) excluded ?
5. How many diagonals does a decagon have ?
6. A person has 10 books including one dictionary. In how many ways can a selection of three books be made, when
   (*i*) the dictionary is always included,
   (*ii*) the dictionary is always excluded ?
7. From a company of 20 soldiers, 4 are placed on guard every two hours. For what length of time can different sets be selected ?
8. How many triangles can be formed by joining 12 points, 7 of which are in the same straight line ?
9. A teacher wishes to take eight students to the Zoological gardens three at a time as often as he can without taking just the same students together more than once. How often will he go and how often will each student go ?
10. There are n points in a plane of which no three are in a straight line except p which are all in a straight line. Find
    (*i*) the number of straight lines formed by joining them,
    (*ii*) the number of triangles formed by them.
11. For an examination a student has to take up English, Hindi and three of the following subjects :
    (*i*) Mathematics, (*ii*) Economics, (*iii*) History, (*iv*) Political Science, (*v*) Sanskrit. How many different combinations are possible ?
12. If ${}^{2n}C_3 : {}^{n}C_2 = 44 : 3$, find n.
13. If ${}^{n+1}C_{r+1} : {}^{n}C_r : {}^{n-1}C_{r-1} = 11 : 6 : 3$, find n and r.
14. If ${}^{n}C_{12} = {}^{n}C_8$, find the value of ${}^{22}C_n$.
15. If ${}^{18}C_r = {}^{18}C_{r+2}$, find ${}^{r}C_5$.
16. In how many ways can a class of 105 students be divided into three sections A, B and C consisting of 40, 35 and 30 students respectively ?

17. In how many ways can 52 playing cards be placed in 4 heaps of 13 cards each ? In how many ways can they be dealt out to four players giving 13 cards each ?
18. Find the number of different ways of dividing 15 books into three sets of 5 each, to be given away as prizes to three different students.
19. Find the number of combinations of the letters of the word 'commitment' taken four at a time.
20. Find the number of permutations of the letters of the word 'series' taken three at a time.
21. Find the number of combinations and of arrangements which can be made by taking 4 letters at a time from the word 'examination'.
22. Find the number of combinations and of permutations of the letters of the word 'MATHEMATICS' taken four at a time.

TEST YOUR UNDERSTANDING VI

In eace of the following problems, four alternatives are given out of which only one is correct. Put a tick mark (✓) against the correct alternative.

1. The number of three-digit numbers that can be formed by using the digits 0, 3, 4, 5 is
   (*a*) 27 (*b*) 64 (*c*) 48 (*d*) 12.
2. If $\frac{x}{6\,!}-\frac{6x}{7\,!}=\frac{1}{5\,!}$, the value of x is
   (*a*) 21 (*b*) 84 (*c*) 20 (*d*) 42.
3. If ${}^{15}P_{r-1} : {}^{14}P_r = 3 : 4$, the value of r is
   (*a*) 19 (*b*) 14 (*c*) 12 (*d*) 15.
4. The number of permutations of 6 things taken 4 at a time in which 2 particular things will always occur is
   (*a*) 6 (*b*) 144 (*c*) 24 (*d*) 72.
5. The number of ways in which 3 ladies and 5 gentlemen can be seated in a row, so that no two ladies may come together is
   (*a*) 14400 (*b*) 7200 (*c*) 10800 (*d*) 9600.
6. The number of ways in which 4 red balls and 3 black balls can be arranged in a row is
   (*a*) 144 (*b*) 35 (*c*) 210 (*d*) 840.
7. The number of ways in which 4 different things can be divided among 3 persons so that each person may receive one, more or none of the things is
   (*a*) 255 (*b*) 256 (*c*) 80 (*d*) 81.
8. The number of ways in which five persons can be seated at a round table is
   (*a*) 120 (*b*) 24 (*c*) 12 (*d*) 60.

9. The number of ways in which five beads can be threaded so as to form a ring is
(*a*) 12 (*b*) 24 (*c*) 60 (*d*) 120.

10. The number of ways in which a party of 9 children can be divided into three groups of three children each is
(*a*) 280 (*b*) 336 (*c*) 216 (*d*) 504.

REVIEW EXERCISE VI

1. In how many ways can the letters of the word 'HONESTY' be arranged so that (*i*) S and T may always be together, (*ii*) S and T may never be together ?
2. In how many ways can the letters of the word 'SECOND' be arranged so that the vowels may occupy the even places ?
3. How many different numbers can be formed from the digits 2, 3, 5, 7, 9 ; how many of them are odd ?
4. Find how many words can be formed out of the letters of the word 'DAUGHTER', the vowels always coming together.
5. In how many ways can 10 examination papers be arranged so that the best and the worst never come together ?
6. How many different words can be formed with the letters of the word 'BHARAT' ? In how many of these B and H are never together, and how many of these begin with B and end with T ?
7. How many signals can be made by hoisting 4 flags of different colours one above the other, when any number of them may be used at a time ?
8. How many signals can be made by hoisting 5 flags of different colours, when
   (*i*) all the flags are hoisted together,
   (*ii*) all the flags may or may not be used ?
9. In a crossword puzzle 20 words are to be guessed of which 8 words have an alternative solution each also. Find the total number of solutions.
10. Find the number of ways in which *n* different beads can be arranged to form a necklace.
11. In how many ways can four persons be seated at a round table so that all shall not have the same neighbours in any two arrangements ?
12. In how many ways can 11 members of a committee sit at a round table so that the Secretary and the Joint Secretary are always the neignbours of the President.
13. Find the number of permutations of the letters of the word 'INDIA'.

14. Find the number of different words into which the word 'INTERFERE' can be converted by changing the order of the letters, it being given that no two consonants should come together.
15. In how many ways can 5 mangoes be distributed among 3 boys when there is no restriction as to the number of mangoes each boy may receive ?
16. A man has 6 friends. In how many ways can he invite one or more of them to a party ?
17. Four persons are to be chosen out of ten. In how many ways can it be done and how often would any person be chosen ?
18. Find the number of diagonals which can be drawn in a plane figure of 16 sides.
19. Find the smallest natural number n such that for all $m \geqslant n$, m ! has a zero in the unit's place.
20. Verify that $1! + 4! + 5! = 145$ and that $4! + 0! + 5! + 8! + 5! = 40585$. Can you find more such numbers ?

SUMMARY

1. $n! = n(n-1)(n-2)......3.2.1.$
2. $0! = 1.$
3. The number of permutations of n different objects taken r at a time$= {}^nP_r = n(n-1)(n-2)......$to r factors,

$$= n(n-1)(n-2)......(n-r+1),$$

$$= \frac{n!}{(n-r)!}.$$

4. The number of combinations of n different objects taken r at a time $= {}^nC_r = \frac{n!}{(n-r)!\,r!}.$
5. ${}^nC_r = {}^nC_{n-r}.$
6. ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r.$

HISTORICAL NOTE

The use of permutations and combinations in India goes back to the Vedic period. In Vedic literature we find computations of the number of ways in which poetic rhythms of a verse can be altered. In the 6th century B.C., *Sushruta*, the author of the famous treatise on medicine *Sushruta Samhita* wrote that it is possible to make out 63 combinations out of 6 different *rasas* (रस) taken one or more at a time. In the 3rd century B.C. *Pingala* in his well-known work *Chhanda-Sutra* (छन्द सूत्र) describes the method of finding the number of combinations of a given number of letters taken one or more at a time.

The credit for making a systematic study of permutations and combinations goes, however, to Jain mathematicians who studied the topic under the name *Vikalpa*. *Mahavira*, the most well-known among Jain mathematicians was the first mathematician in the world to give general formulae for ${}^nP_r$ and ${}^nC_r$ in his famous work *Ganita Sara Sangraha*.

Bhaskara, who lived in the 12th century A.D., dealth with permutations and combinations under the name *Anka Pasha* in his well-known work *Lilavati*.

Ars Conjectandi written by the noted Swiss mathematician *Jacob Bernoulli* (1654- 1705 A.D.) and published in 1713 A.D. after his death was the first book to deal with the subject matter of permutations and combinations in a systematic and reasonably complete manner.

□□

SIR ISAAC NEWTON (1642-1727)

Isaac Newton was born in Woolsthorpe on Christmas Day, 1642*. When he was still a child, he showed great skill and delight in devising clever mechanical models and in conducting experiments. At the age of 18, he entered Trinity College, Cambridge. At the age of 23, he proved the generalized binomial theorem and created his method of fluxions which is known today as the Differential Calculus. During the next one year, he performed his first experiments in optics, and formulated the basic principles of his theory of gravitation. In 1685, Newton completed the first book of his Principia, his greatest work which proved to be the most influential and the most admired work in the history of science. In 1689, Newton represented the University in parliament. In 1703, he was elected President of the Royal Society, a position to which he was annually reelected until his death in 1727 after a lingering and painful illness and was buried in Westminster Abbey. As a mathematician he is ranked almost universally as the greatest, the world has ever produced. Newton was the greatest genius that ever lived. His accomplishments were poetically expressed by Pope in the lines :

> Nature and Nature's laws lay hid in night.
> God said, Let Newton be, and all was light.

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*According to old calendar.

CHAPTER 7

Binomial Theorem

7·1. BINOMIAL EXPRESSIONS

An expression consisting of two terms only is called a **binomial expression**. Thus

$x+a$, $3x^2-y$, $5x^3-4y^7$ and $2x^{1/3}+i\sqrt{3}y^{-2}$ are all binomial expressions.

7·2. THE BINOMIAL THEOREM FOR A POSITIVE INTEGRAL INDEX

Statement.

When n is a positive integer, $(x+a)^n$ is identically equal to

$$x^n+{}^nC_1x^{n-1}a+\ldots+{}^nC_rx^{n-r}a^r+\ldots+a^n.$$

Proof. Consider the product

$$(x+a)(x+a)\cdots(x+a)$$

of $x+a$ by itself n times.

It is the sum of all the products that can be obtained by multiplying together one term from each bracket. Further,

(*i*) We may take x from each of the brackets ; the product is then x^n.

(*ii*) We may take a from one bracket and x from each of the remaining brackets, and this may be done in n ways ; there are, thus, n products $x^{n-1}a$.

(*iii*) When r is an integer less than n, we can select r out of the n brackets in ${}^nC_r$ ways, and if, having chosen r brackets, a be taken out of each of them and x out of the $(n-r)$ brackets that have not been chosen, the product is $x^{n-r}a^r$; there are ${}^nC_r$ products $x^{n-r}a^r$.

(*iv*) We may take a from each of the brackets ; the product is then a^n.

Thus the sum of all the terms obtained on multiplying the n brackets is

$$x^n+{}^nC_1x^{n-1}a+{}^nC_2x^{n-2}a^2+\cdots+{}^nC_rx^{n-r}a^r+\cdots+a^n, \qquad \ldots(A)$$

and this proves the theorem.

Another Proof. The binomial theorem can be proved by using the principle of mathematical induction.

Let T_n be the statement :

Given any two numbers x and a and any positive integer n, then

$$(x+a)^n = {}^nC_0x^n + {}^nC_1x^{n-1}a + \cdots + {}^nC_rx^{n-r}a^r + \cdots + a^n.$$

We shall show that T_n is true for all positive integers n by using PFI.

Step 1. T(1) asserts that

$$(x+a)^1 = {}^1C_0x^1 + {}^1C_1x^{1-1}a = x+a,$$

which is true.

Step 2. Suppose T(k) is true, *i.e.*, suppose

$$(x+a)^k = {}^kC_0x^k + {}^kC_1x^{k-1}a + \ldots + {}^kC_rx^{k-r}a^r + \ldots + {}^kC_ka^k. \quad \cdots(1)$$

Multiplying both sides of (1) by x and a, in turn, we have

$$x(x+a)^k = {}^kC_0x^{k+1} + {}^kC_1x^ka + \cdots + {}^kC_rx^{k+1-r}a^r + \cdots + {}^kC_kxa^k,$$

$$a(x+a)^k = {}^kC_0x^ka + \cdots + {}^kC_{r-1}x^{k+1-r}a^r + \ldots + {}^kC_ka^{k+1}.$$

Adding the corresponding sides of the above identities] we have,

$$(x+a)^{k+1} = {}^kC_0x^{k+1} + ({}^kC_1 + {}^kC_0)x^ka + \cdots$$
$$+ ({}^kC_r + {}^kC_{r-1})x^{k+1-r}a^r + \cdots + {}^kC_ka^{k+1}$$

Since ${}^kC_0 = 1 = {}^{k+1}C_0$, ${}^kC_r + {}^kC_{r-1} = {}^{k+1}C_r$, ${}^kC_k = 1 = {}^{k+1}C_{k+1}$, $\cdots(2)$

therefore, we can write (2) as

$$(x+a)^{k+1} = {}^{k+1}C_0x^{k+1} + {}^{k+1}C_1x^ka + \ldots + {}^{k+1}C_rx^{k+1-r}a^r + \cdots$$
$$+ {}^{k+1}C_{k+1}a^{k+1},$$

which is of the same form as (1) and can be obtained from it by replacing k throughout by $k+1$, and consequently T($k+1$) is true. By PFI, T(n) is true for all positive integers n.

Notes. 1. The above theorem was first discovered by Sir Isaac Newton, the English genius, who ranks second to none in the world of mathematics.

2. Interchanging x and a in (A), we have

$$(a+x)^n = a^n + {}^nC_1a^{n-1}x + \cdots + {}^nC_ra^{n-r}x^r + \ldots + x^n. \quad \cdots(B)$$

But $(a+x)^n = (x+a)^n$, and so (A) and (B) are identically equal. (A) is called the expansion of $(x+a)^n$ in descending powers of x ; (B) is called the expansion of $(x+a)^n$ in ascending powers of x.

3. The coefficient of $x^r\,a^{n-r}$ is ${}^nC_{n-r}$ in (A) and ${}^nC_r$ in (B). We thus have another proof of the equality of ${}^nC_r$ and ${}^nC_{n-r}$.

4. The coefficients ${}^nC_1, {}^nC_2, \cdots, {}^nC_r, \cdots$ are called the **binomial coefficients.**

5. Writing the values of binomial coefficients in (A) and (B) we have

$$(x+a)^n=x^n+nx^{n-1}a+\frac{n(n-1)}{2\,!}x^{n-2}a^2+\cdots$$
$$+\frac{n(n-1)\cdots(n-r+1)}{r\,!}x^{n-r}a^r+\cdots+a^n,$$
$$=a^n+na^{n-1}x+\frac{n(n-1)}{2\,!}a^{n-2}x^2+\cdots$$
$$+\frac{n(n-1)\ldots(n-r+1)}{r\,!}a^{n-r}x^r+\ldots+x^n.$$

6. The number of terms in the binomial expansion of $(x+a)^n$ is $n+1$.

7. Putting $n=3$, 4 in (A), we have as particular cases :

$$(x+a)^3=x^3+3x^2a+3xa^2+a^3,$$
$$(x+a)^4=x^4+4x^3a+6x^2a^2+4xa^3+a^4.$$

8. The statement of the binomial theorem as given above is valid only when the index n is a positive integer. The more general form of the binomial theorem which we will discuss later on will enable us to obtain the expansion of $(x+a)^n$ not only when n is a positive integer, but also when n is a negative integer or a positive or negative fraction, but in these cases we shall notice that

(*i*) the number of terms shall be infinite ;

(*ii*) the expansion will not be valid for all values of x and a.

9. The 'binomial' theorem can also be used to raise a 'trinomial' expression like $x+a+b$ to any power n by first treating $a+b$ as one term and later applying binomial theorem to obtain the expansion of the powers of $(a+b)$. In the same way, the binomial theorem can be used to find the expansion of the nth power of any multinomial expression $x+a+b+c+d+\cdots$ (see solved Example 2 on page 202).

7·3. THE EXPANSION OF $(x-a)^n$

$$(x-a)^n=\{x+(-a)\}^n$$
$$=x^n+{}^nC_1x^{n-1}(-a)+{}^nC_2x^{n-2}(-a)^2+\ldots$$
$$+{}^nC_rx^{n-r}(-a)^r+\cdots+(-a)^n.$$
$$\therefore\quad (x-a)^n=x^n-{}^nC_1x^{n-1}a+{}^nC_2x^{n-2}a^2$$
$$+\cdots+(-1)^r\,{}^nC_rx^{n-r}a^r+\cdots+(-1)^na^n.$$

7·4. THE GENERAL TERM

The $(r+1)$th term in the expansion of $(x+a)^n$ by the binomial theorem is called the general term and is denoted by T_{r+1}. From (A), we find that

$$T_1=x^n,$$

$$T_2 = {}^nC_1 x^{n-1} a,$$
$$T_3 = {}^nC_2 x^{n-2} a^2,$$
$$\dots\dots\dots\dots\dots\dots$$
$$T_{r+1} = {}^nC_r x^{n-r} a^r.$$

By putting $r=0, 1, 2, \dots, n$ in the general term we can get all the terms of the expansion.

Hence $(x+a)^n = \sum\limits_{r=0}^{n} {}^nC_r\, x^{n-r} a^r.$

Example 1. *Expand* $(2b-3y)^4$.

Solution. Replacing x by $2b$, a by $(-3y)$ and n by 4 in the expansion of $(x+a)^n$, we have

$$\begin{aligned}(2b-3y)^4 &= (2b)^4 + {}^4C_1(2b)^3(-3y) + {}^4C_2(2b)^2(-3y)^2 \\ &\qquad + {}^4C_3(2b)(-3y)^3 + {}^4C_4(-3y)^4, \\ &= 16b^4 - 96b^3y + 216b^2y^2 - 216y^3 + 81y^4.\end{aligned}$$

Example 2. *Expand* $(x-a+b)^3$.

Solution. $$\begin{aligned}(x-a+b)^3 &= \{(x-a)+b\}^3, \\ &= (x-a)^3 + {}^3C_1(x-a)^2b + {}^3C_2(x-a)b^2 + {}^3C_3b^3 \\ &= x^3 + {}^3C_1x^2(-a) + {}^3C_2x(-a)^2 + {}^3C_3(-a)^3 \\ &\qquad + 3(x^2-2xa+a^2)b + 3(x-a)b^2 + b^3, \\ &= x^3 - 3x^2(a-b) + 3x(a^2-2ab+b^2) \\ &\qquad - (a^3-3a^2b+3ab^2-b^3).\end{aligned}$$

Aliter. $$\begin{aligned}&(x-a+b)^3, \\ &= \{x-(a-b)\}^3 \\ &= x^3 + {}^3C_1x^2\{-(a-b)\} + {}^3C_2x\{-(a-b)\}^2 \\ &\qquad + {}^3C_3\{-(a-b)\}^3 \\ &= x^3 - 3x^2(a-b) + 3x(a-b)^2 - (a-b)^3, \\ &= x^3 - 3x^2(a-b) + 3x(a^2-2ab+b^2) \\ &\qquad - (a^3-3a^2b+3ab^2-b^3).\end{aligned}$$

Example 3. *Evaluate* $(a+\sqrt{5})^4 - (a-\sqrt{5})^4$.

Solution.

$$\begin{aligned}(a+\sqrt{5})^4 &= a^4 + {}^4C_1a^3(\sqrt{5}) + {}^4C_2a^2(5\surd)^2 + {}^4C_3a(\sqrt{5})^3 \\ &\qquad + {}^4C_4(\sqrt{5})^4, \\ &= a^4 + 4a^3(\sqrt{5}) - 6a^2(5) + 4a(5\sqrt{5}) + 25. \qquad \dots(i)\end{aligned}$$

$$\begin{aligned}(a-\sqrt{5})^4 &= a^4 + {}^4C_1a^3(-\sqrt{5}) + {}^4C_2a^2(-\sqrt{5})^2 \\ &\qquad + {}^4C_3\, a(-\sqrt{5})^3 + {}^4C_4(-\sqrt{5})^4, \\ &= a^4 - 4a^3(\sqrt{5}) + 6a^2(5) - 4a(5\sqrt{5}) + 25. \qquad \dots(ii)\end{aligned}$$

From (*i*) and (*ii*), we get

$$(a+\sqrt{5})^4-(a-\sqrt{5})^4=2(4a^3(\sqrt{5})+4a(5\sqrt{5})),$$
$$=8\sqrt{5}a^3+40\sqrt{5}a.$$

Example 4. *Write down the 10th term of* $\left(\frac{a}{b}-\frac{2b}{a^2}\right)^{13}$.

Solution. Let the tenth term be denoted by T_{10}.

$$T_{10}=T_{9+1},$$
$$={}^{13}C_9\left(\frac{a}{b}\right)^{13-9}\left(-\frac{2b}{a^2}\right)^9.$$
$$={}^{13}C_4\frac{a^4}{b^4}\left(-512\frac{b^9}{a^{18}}\right).$$
$$=-715\times512\frac{a^4}{b^4}\cdot\frac{b^9}{a^{18}}.$$
$$=-366080\,\frac{b^5}{a^{14}}.$$

Example 5. *Find the coefficient of* x^{-17} *in the expansion of*

$$\left(x^4-\frac{1}{x^3}\right)^{15}.$$

Solution. Let x^{-17} occur in the $(r+1)$th term.

Now $\quad T_{r+1}={}^{15}C_r(x^4)^{15-r}(-x^{-3})^r,$

$$={}^{15}C_r(-1)^r x^{60-7r}.$$

It contains x^{-17} if $60-7r=-17$,

or $\quad 7r=77,$

or $\quad r=11.$

Hence x^{-17} occurs in the 12th term and its coefficient is

$${}^{15}C_r(-1)^r={}^{15}C_{11}(-1)^{11}$$
$$=-{}^{15}C_4.$$
$$=-\frac{15\times14\times13\times12}{4\times3\times2}=-1365.$$

Example 6. *Find the term independent of x in the expansion of*

$$\left(2x^2+\frac{1}{x}\right)^{12}.$$

Solution. Let the $(r+1)$th term be independent of x.

Now $\quad T_{r+1}={}^{12}C_r(2x^2)^{12-r}\left(\frac{1}{x}\right)^r.$

$$={}^{12}C_r\,2^{12-r}\,x^{24-3r}.$$

T_{r+1} will be independent of x, if $24-3r=0$, *i.e.*, $r=8$. Hence 9th term is independent of x and is equal to

$$\begin{aligned} &{}^{12}C_8 \cdot 2^4 \cdot x^0, \\ &={}^{12}C_4 \cdot 16, \\ &=\frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 16}{4 \cdot 3 \cdot 2}=7920. \end{aligned}$$

Example 7. *Find the middle term in the expansion of* $(1+x)^{2n}$, *where n is a positive integer.*

Solution. The number of terms in the expansion will be $2n+1$. Since the number of terms is odd, therefore, the middle term is

$$\frac{(2n+1)+1}{2} \text{ i.e., } (n+1)\text{th},$$

Middle term

$$\begin{aligned} &=T_{n+1}={}^{2n}C_n\, x^n=\frac{2n\,!}{n\,!\,n\,!}x^n, \\ &=\frac{2n\,(2n-1)(2n-2)\ldots4 \cdot 3 \cdot 2 \cdot 1}{(n\,!)^2}x^n, \\ &=\frac{2n\,(2n-2)\cdots4 \cdot 2 \cdot (2n-1)(2n-3)\cdots3 \cdot 1}{(n\,!)^2}x^n, \\ &=\frac{2^n n(n-1)\cdots2 \cdot 1\,(2n-1)(2n-3)\cdots3 \cdot 1}{(n\,!)^2}x^n, \\ &=\frac{2^n\,(2n-1)(2n-3)\ldots3 \cdot 1}{n\,!}x^n. \end{aligned}$$

Example 8. *Find the middle terms in the expansion of*

$$\left(\frac{x}{2}-y\right)^9.$$

Solution. The number of terms in the expansion is 10. The 5th and the 6th terms are the two middle terms.

$$T_5=T_{4+1}={}^9C_4\left(\frac{x}{2}\right)^5(-y)^4=\frac{63}{16}x^5y^4.$$

$$T_4=T_{5+1}={}^9C_5\left(\frac{x}{2}\right)^4(-y)^5=-\frac{63}{8}x^4y^5.$$

Note. It may be noted that there are two middle terms when n is odd ; for, then the number of terms is even and $\frac{n+1}{2}$th and $\frac{n+3}{2}$th terms are the two middle terms. When n is even, there is

an odd number of terms in the binomial expansion. The $\left(\frac{n}{2}+1\right)$th term is the middle term in this case.

Example 9. *Use binomial theorem to find the value of* $(101)^4$.

Solution.
$$\begin{aligned}(101)^4 &= (100+1)^4,\\ &= (100)^4 + {}^4C_1\,(100)^3\,(1) + {}^4C_2\,(100)^2\,(1)\\ &\qquad + {}^4C_1\,(100)(1)^3 + {}^4C_4\,(1)^4,\\ &= 100000000+4000000+60000+400+1,\\ &= 104060401.\end{aligned}$$

EXERCISE 7 (*a*)

1. Expand :
   (*i*) $(2x+3y)^5$
   (*ii*) $\left(xy^{1/2}+\frac{2}{3}x^{1/2}\,y\right)^6$
   (*iii*) $\left(\frac{2}{3}x-\frac{3}{2x}\right)^6$
   (*iv*) $\left(3xy-\frac{2x}{y}\right)^5$
   (*v*) $(1+2x-3x^2)^5$.
2. Find the value of :
   (*i*) $(\sqrt{2}+1)^6-(\sqrt{2}-1)^6$
   (*ii*) $[\sqrt{(x^2-a^2)}+x]^5-[\sqrt{(x^2-a^2)}-x]^5$.
3. Find the 7th term of $\left(\frac{4x}{5}-\frac{5}{2x}\right)^9$.
4. Find the 6th term of $\left(2x-\frac{1}{2}\right)^9$.
5. Find the rth term of $(1+x)^{2n}$.
6. Find the coefficient of x^{18} in $(a^4-bx^3)^{10}$.
7. Find the coefficient of x^{15} in $(x-x^2)^{10}$.
8. Show that the coefficients of x^m and x^n in $(1+x)^{m+n}$ are equal.
9. Find the term independent of x in the expansion of $\left(x^3-\frac{1}{x^2}\right)^{10}$.
10. Find the value of the constant term in the expansion of $\left(\sqrt{x}-\frac{2}{x^2}\right)^{10}$.

11. Show that there will be no term containing x^{2r} in the expansion of $(x+x^{-2})^{n-3}$, unless $\frac{1}{3}(n-2r)$ is a positive integer.

12. Find the middle term of $\left(\frac{2}{x^2}+\frac{x^2}{2}\right)^9$.

13. Show that the middle term of $\left(x-\frac{1}{x}\right)^{2n}$ is

$$(-2)^n \frac{1\,.\,3\,.\,5\cdots(2n-1)}{n\,!}.$$

14. Find the two middle terms of $\left(3a-\frac{a^3}{6}\right)^9$.

15. Show that, if n be even, the coefficient of the middle term of $(1+x)^n$ is $\frac{1\,.\,3\,.\,5\ldots(n-1)}{1\,.\,2\,.\,3\cdots(\frac{1}{2}n)}\cdot 2^{n/2}$ and that, if n be odd, the coefficient of each of the two middle terms is

$$\frac{1\,.\,3\,.\,5\cdots n}{1\,.\,2\,.\,3\cdots\frac{1}{2}(n+1)}\;2^{\frac{1}{2}(n-1)}.$$

16. If x^p occurs in the expansion of $\left(x^2+\frac{1}{x}\right)^{2n}$, prove that its coefficient is $\frac{(2n\,!)}{[\frac{1}{3}(4n-p)]\,!\,[\frac{1}{3}(2n+p)]\,!}$

17. Use binomial theorem to find
(*i*) 99^4, (*ii*) 999^3, (*iii*) 51^4.

18. Find the binomial expansion whose first three terms are 1, 35, 525.

19. Show that the coefficient of the middle term of $(1+x)^{2n}$ is equal to the sum of the coefficients of the two middle terms of $(1+x)^{2n-1}$.

20. Find the middle term, pth term from the beginning, and pth term from the end in the expansion of $\left(2x+\frac{1}{x}\right)^{2n}$.

7·5. TO FIND THE GREATEST TERM IN A BINOMIAL EXPANSION

Example 10. *Find the greatest term in $(5+2y)^{17}$ if $y=\frac{1}{2}$.*

Solution. Now,

$$\frac{T_{r+1}}{T_r}=\frac{17-r+1}{r}\cdot\frac{2y}{5}=\frac{18-r}{5r}. \qquad \left(\because\; y=\frac{1}{2}\right)$$

$\therefore \quad T_{r+1} >, =, \text{or} < T_r$

according as $\dfrac{18-r}{5r} >, =, \text{or} < 1,$

i.e., according as $18-r >, =, \text{or} < 5r,$

i.e., according as $r <, =, \text{or} > 3.$

Since r is a positive integer,

so long as $r < 3$, $T_{r+1} > T_r$, *i.e.*, the terms go on increasing.

For $r=3$, we have $T_r = T_{r+1}$, *i.e.*, $T_3 = T_4$.

For $r > 3$, $T_{r+1} < T_r$, *i.e.*, the terms go on decreasing.

Hence the 3rd and the 4th terms are equal when $y=\frac{1}{2}$ and they are greater than any other term.

Example 11. *Find the numerically greatest term in the expansion of $(3-2x)^8$ when $x=1$.*

Solution. $$\frac{T_{r+1}}{T_r} = \frac{8-r+1}{r} \cdot \left(\frac{-2x}{3}\right) = \frac{9-r}{r} \cdot \left(-\frac{2}{3}\right),$$

or $$\left|\frac{T_{r+1}}{T_r}\right| = \frac{9-r}{r} \cdot \frac{2}{3}.$$

Hence $\quad |T_{r+1}| >, =, \text{or} < |T_r|$

according as $\dfrac{9-r}{r} \cdot \dfrac{2}{3} >, =, \text{or} < 1,$

or as $\quad 18 >, =, \text{or} < 5r,$

or as $\quad r <, =, \text{or} > \dfrac{18}{5} = 3\dfrac{3}{5}.$

Hence so long as $r < 3\frac{3}{5}$, $|T_{r+1}| > |T_r|$, *i.e.*, the terms increase in absolute value.

$\therefore$ T_{3+1}, *i.e.*, T_4 is the numerically greatest term and its numerical value is ${}^8C_3 (3)^5 (2)^3$

$$= 56.243.8 = 108864.$$

7·5·1. To Find the Binomial Expansion When Three Consecutive Terms Are Given.

Example 12. *If three consecutive coefficients of $(1+x)^n$ are 6, 15 and 20, find n.*

Solution. Let the three consecutive terms whose coefficients are given be the rth, $(r+1)$th and $(r+2)$th terms.

$\therefore \quad {}^nC_{r-1} = 6, \quad ...(i)$

${}^nC_r = 15, \quad ...(ii)$

${}^nC_{r+1} = 20. \quad ...(iii)$

Dividing (*ii*) by (*i*), we have

$$\frac{n-r+1}{r}=\frac{15}{6}=\frac{5}{2},$$

or $$\frac{n+1}{r}-1=\frac{5}{2},$$

or $$\frac{n+1}{r}=\frac{7}{2}. \quad ...(iv)$$

Again, dividing (*iii*) by (*ii*), we have

$$\frac{n-(r+1)+1}{r+1}=\frac{20}{15},$$

or $$\frac{n+1}{r+1}-1=\frac{4}{3},$$

or $$\frac{n+1}{r+1}=\frac{7}{3}. \quad ...(v)$$

Dividing (*iv*) by (*v*), we have

$$\frac{r+1}{r}=\frac{3}{2}, \text{ or } r=2. \quad ...(vi)$$

Substituting the value of r in (*iv*), we have

$$n=6.$$

Example 13. *Find the binomial expansion whose first, second and third terms are 243, 810 and 1080 respectively.*

Solution. Let the required binomial expansion be $(a+x)^n$.

Since $(a+x)^n=a^n+n\,a^{n-1}x+\dfrac{n(n-1)}{2\,!}\,a^{n-2}\,x^2+...+x^n$,

$$\therefore \quad a^n=243, \quad ...(i)$$

$$na^{n-1}\,x=810, \quad ...(ii)$$

and $$\frac{n(n-1)}{2}\,a^{n-2}\,x^2=1080. \quad ...(iii)$$

We have three equations in three unknown a, x and n.

Multiplying (*i*) and (*iii*) and dividing by the square of (*ii*), we have

$$\frac{n(n-1)}{2}\cdot\frac{a^{2n-2}\cdot x^2}{n^2\,.\,a^{2n-2}\,x^2}=\frac{243\times1080}{810\times810}.$$

or $$\frac{n-1}{2n}=\frac{2}{5}, \text{ or, } 5n-5=4n, \text{ or, } n=5.$$

Putting $n=5$ in (*i*), we have

$$a^5=243=3^5, \text{ i.e., } a=3. \quad ...(iv)$$

Substituting the values of n and a in (*ii*), we have

$$5.3^4\,.\,x=810,$$

i.e., $x=\frac{810}{5\times 81}=2.$

Hence the required binomial expansion is $(3+2)^8$.

EXERCISE 7 (*b*)

1. Find the numerically greatest term in the expansion of :
   (*i*) $\left(1+\frac{2}{3}x\right)^8$ when $x=\frac{3}{2}\cdot$
   (*ii*) $(2-3x)^9$ when $x=\frac{3}{2}\cdot$
   (*iii*) $\left(2+\frac{2}{7}x\right)^{11}$ when $x=14$.
   (*iv*) $(2x-12)^{28}$ when $x=9$.
   (*v*) $(2a+b)^{11}$ when $a=4$, $b=5$.
   (*vi*) $(3-2x)^{15}$ when $x=\frac{5}{2}\cdot$
2. Find the numerically greatest coefficient in each of the following :
   (*i*) $(1-x)^{11}$, (*ii*) $(x-y)^{12}$.
3. Find the term (or terms) with the numerically greatest coefficient in the expansion of $(x-2y)^{12}$.
4. The 2nd, 3rd, 4th terms in the expansion of $(x+y)^n$ are 240, 720, 1080 respectively ; find x, y, n.
5. If the 9th, 10th and 11th coefficients of $(1+x)^n$ are in A.P., find n.
6. If three consecutive coefficients of $(1+x)^n$ are 35, 21, 7, find n.
7. If the rth, $(r+1)$th and $(r+2)$th coefficients of $(1+x)^n$ are in A.P., show that $n^2-n(4r+1)+4r^2-2=0$.
8. If the 5th, 6th and 7th coefficients of $(1+x)^n$ are in A.P., find n.
9. Three consecutive coefficients in the expansion of $(1+x)^n$ are 462, 330, 165 ; find n.
10. Find the binomial expansion of which 4, 7, 7 and $\frac{35}{8}$ are four consecutive terms.
11. If the 2nd and the 3rd terms in $(a+b)^n$ are in the same ratio as the 3rd and 4th terms in $(a+b)^{n+3}$, find n.
12. If the first three terms of $(a+b)^n$ are in A.P., find n, it being given that $a=2b$.

13. Find the limits between which x must lie if the greatest term of $(1+x)^{72}$ has the greatest coefficient.

14. Find x if the greatest terms of $(1+x)^{51}$ are also the terms containing the greatest coefficients.

7·6. PROPERTIES OF BINOMIAL COEFFICIENTS

Here below we prove certain interesting theorems relating to binomial coefficients.

Theorem 7·1. *In a binomial expansion, the coefficients of the terms equidistant from the beginning and the end are equal.*

Propf. There are $(n+1)$ terms in the expansion of $(x+a)^n$.

The coefficient of the $(r+1)$th term from the beginning $={}^nC_r$. ...(i)

Since the $(r+1)$th term from the end is the same as the $[(n+1)-r]$th term from the beginning, its coefficient $={}^nC_{n-r}$. ...(ii)

Since ${}^nC_r={}^nC_{n-r}$, from (i) and (ii) we find that the coefficient of the $(r+1)$th term from beginning is the same as that of the $(r+1)$th term from the end.

Hence the theorem.

Throughout the remainder of this chapter we shall denote the binomial coefficients

$$1\ (={}^nC_0),\ {}^nC_1,\ {}^nC_2, \ldots\ldots, {}^nC_r \ldots\ldots, \text{ and } 1\ (={}^nC_n)$$

by $C_0, C_1, C_2, \ldots\ldots, C_r, \ldots\ldots$, and C_n respectively. In this notation, the binomial theorem may be written as

$$(x+a)^n=C_0x_n+C_1x^{n-1}a+\ldots+C_rx^{n-r}a^r+\ldots+C_na^n.$$

Theorem 7.2. *The sum of the binomial coefficients corresponding to the index n is 2^n.*

Proof. Putting $x=1$ in the expansion

$(1-x)^n=C_0+C_1x+C_2x^2+\ldots+C_nx^n$, we have

$$\boxed{2^n=C_0+C_1+\ldots\ldots+C_n} \quad \ldots(i)$$

Hence the required sum $=2^n$.

Cor. The relation (i) may be re-written as

$${}^nC_1+{}^nC_2+\ldots+{}^nC_n=2^n-1.$$

Since the L.H.S. is the sum of the number of combinations of n things taken one, two, three,......n, at a time, we have the following result :

The total number of combinations of n different things taken any number at a time is 2^n-1.

Theorem 7·3. *In the expansion of $(1+x)^n$, the sum of the coefficients of the odd terms is the same as the sum of the coefficients of even terms, each being equal to 2^{n-1}.*

Proof. Putting $x=1, -1$ successively in the expansion
$(1+x)^n=C_0+C_1x+C_2x^2+\cdots\cdots+C_nx^n$, we have

$$2^n=C_0+C_1+\cdots+C_n \qquad \ldots(i)$$

and $$0=C_0-C_1+\cdots+(-1)^nC_n. \qquad \ldots(ii)$$

Adding (i) and (ii), we have

$$C_0+C_2+C_4+\cdots=2^{n-1}. \qquad \ldots(iii)$$

Also, subtracting (iii) and (iv), we have

$$C_1+C_3+C_5+\cdots=2^{n-1}. \qquad \ldots(iv)$$

From (iii) and (iv), we have

$$C_0+C_2+C_4+\cdots=C_1+C_3+C_5+\cdots=2^{n-1}.$$

Note. *Since $C_r=\dfrac{^nP_r}{r\,!}$, the above result can also be put as*

$$1+\frac{^nP_2}{2\,!}+\frac{^nP_4}{4\,!}+\cdots\cdots=\frac{^nP_1}{1\,!}+\frac{^nP_3}{3\,!}+\frac{^nP_5}{5\,!}+\cdots\cdots=2^{n-1}.$$

Example 14 *If the expansion of $(1+x-2x^2)^6$ is*

$$1+a_1x+a_2x^2+\cdots\cdots a_{12}x^{12},$$

prove that $a_2+a_4+a_6+\cdots\cdots+a_{12}=31.$

Solution. $(1+x-2x^2)^6=1+a_1x+a_2x^2+\cdots+a_{12}x^{12}$, for all x.

Putting $x=1$ in both sides, we have

$$0=1+a_1+a_2+\cdots+a_{12}. \qquad \ldots(i)$$

Again, putting $x=-1$, we have

$$64=1-a_1+a_2-a_3+a_4-\cdots+a_{12}. \qquad \ldots(ii)$$

Adding (i) and (ii), we have

$$64=2[1+a_2+a_4+\cdots+a_{12}],$$

or $$1+a_2+a_4+\cdots\cdots+a_{12}=32.$$

Hence $$a_2+a_4+\cdots\cdots+a_{12}=31.$$

Example 15. *If $(1+x)^n=C_0+C_1x+C_2x^2+\ldots+C_nx^n$, prove that*

(i) $C_1+2C_3+\cdots+nC_n=n.2^{n-1}$

(ii) $C_0+2C_1+3C_2+\cdots+(n+1)C_n=(n+2)\,2^{n-1}.$

Solution.

(i) $$\text{L.H.S.}=n+2\,\frac{n\,(n-1)}{2}+3.\frac{n\,(n-1)\,(n-2)}{3\,!}+\cdots$$
$$+\cdots\text{to } n \text{ terms},$$

$$=n\left[1+(n-1)+\frac{(n-1)(n-2)}{2!}+\frac{(n-1)(n-2)(n-3)}{3!}+\ldots\ldots\text{to } n \text{ terms}\right],$$

$$=n(1+1)^{n-1}=n.2^{n-1}.$$

(*ii*) $$\text{L.H.S.}=[1+2n+3.\frac{n(n-1)}{2!}+4.\frac{n(n-1)(n-2)}{3}+\ldots\ldots\text{to } (n+1) \text{ terms}],$$

$$=[1+n+\frac{n(n-1)}{2!}+\frac{n(n-1)(n-2)}{3!}+\ldots\ldots\text{to } (n+1) \text{ terms}],$$

$$+[n+2.\frac{n(n-1)}{2!}+3.\frac{n(n-1)(n-2)}{3!}+\ldots\ldots\text{to } n \text{ terms}],$$

$$=(1+1)^n+n\,[1+(n-1)+\frac{(n-1)(n-2)}{2!}+\ldots\ldots\text{to } n \text{ terms}],$$

$$=(1+1)^n+n(1+1)^{n-1}$$

$$=2^n+n.2^{n-1}=2^{n-1}(n+2).$$

Example 16. *Show that*

(*a*) $C_0^2+C_1^2+\ldots+C_n^2=\dfrac{(2n)!}{(n!)^2}$.

(*b*) $C_0C_1+C_1C_2+\ldots+C_{n-1}C_n=\dfrac{(2n!)}{(n-1)!(n+1)!}$.

Solution. (*a*) $(1+x)^n=C_0+C_1x+C_2x^2+\ldots\ldots+C_nx^n$ ...(*i*)

Also, $(x+1)^n=C_0x^n+C_1x^{n-1}+\ldots\ldots+C_n$ ...(*ii*)

Multiplying corresponding sides of (*i*) and (*ii*), we have

$(1+x)^{2n}=(C_0+C_1x+\ldots+C_nx^n)(C_0x^n+C_1x^{n-1}+\ldots+C_n)$ ...(*iii*)

The coefficient of x^n on the right-hand side is $C_0^2+C_1^2+\ldots+C_n^2$.

Equating the coefficient of x^n on both sides of (*iii*), we have

$C_0^2+C_1^2+\ldots+C_n^2=$coefficient of x^n in $(1+x)^{2n}$,

$$={}^{2n}C_n=\frac{(2n)!}{n!\,n!}=\frac{(2n)!}{(n!)^2}.$$

(*b*) The coefficient of x^{n-1} on the right hand side of (*iii*) is

$$C_0C_1+C_1C_2+\ldots+C_{n-1}C_n.$$

Equating the coefficients of x^{n-1} on both sides of (*iii*), we have

$$C_0C_1+C_1C_2+\ldots+C_{n-1}C_n={}^{2n}C_{n-1}=\frac{(2n)!}{(n-1)!(n+1)!}.$$

Example 17. *With the usual notation, prove that*

(*i*) $^mC_r+{}^mC_{r-1}{}^nC_1+{}^mC_{r-2}{}^nC_2+\ldots+{}^nC_r={}^{m+n}C_r$.

(*ii*) $C_0+\frac{C_1}{2}+\frac{C_2}{3}+\frac{C_3}{4}+\ldots\ldots+\frac{C_n}{n+1}=\frac{2^{n+1}-1}{n+1}$.

Solution.

(*i*) $(1+x)^m=1+{}^mC_1x+{}^mC_2x^2+{}^mC_3x^3+\cdots+{}^mC_mx^m$ ...(*i*)

and $(1+x)^n=1+{}^nC_1x+{}^nC_2x^2+{}^nC_3x^3+\cdots+{}^nC_nx^n$ ...(*ii*)

Multiplying corresponding sides of the two sides and comparing the coefficients of x^r on both the sides, we have

$$^mC_r+{}^mC_{r-1}.\,{}^nC_1+{}^mC_{r-2}.\,{}^nC_2+\ldots+{}^nC_r$$

$$=\text{coefficient of } x^r \text{ in } (1+x)^{m+n}={}^{m+n}C_r=\frac{(m+n)\,!}{(m+n-r)\,!\,r\,!}.$$

This result is known as *Vandermonde's Theorem.*

(*ii*) $\text{L.H.S.}=1+\frac{n}{2}+\frac{1}{3}\cdot\frac{n\,(n-1)}{2\,!}+\frac{1}{4}\cdot\frac{n\,(n-1)\,(n-2)}{3\,!}+\ldots\text{to }(n+1)\text{ terms,}$

$$=1+\frac{n}{2\,!}+\frac{n\,(n-1)}{3\,!}+\frac{n\,(n-1)\,(n-2)}{4\,!}+\cdots\text{to }(n+1)\text{ terms,}$$

$$=\frac{1}{n+1}\left\{(n+1)+\frac{(n+1)\,n}{2\,!}+\frac{(n+1)\,n\,(n-1)}{3\,!}+\frac{(n+1)\,n\,(n-1)\,(n-2)}{4\,!}+\cdots\text{to }n+1\text{ terms}\right\},$$

$$=\frac{1}{n+1}\left\{1+(n+1)+\frac{(n+1)\,n}{2\,!}+\frac{(n+1)\,n\,(n-1)}{3\,!}+\cdots\text{to }n+2\text{ terms}-1\right\},$$

$$=\frac{1}{n+1}\{(1+1)^{n+1}-1\},$$

$$=\frac{1}{n+1}\,(2^{n+}-1).$$

EXERCISE 7 (*c*)

1. If $(1+x+x^2)^{3n}=C_0+C_1x+C_2x^2+\cdots$,
   prove that $C_0-C_1+C_2-C_3+C_4-\cdots=1$.
2. Show that

   (*i*) $1+(n-1)+\frac{(n-1)\,(n-2)}{2\,!}+\frac{(n-1)\,(n-2)\,(n-3)}{3\,!}+\ldots\ldots+1=2^{n-1}$.

   (*ii*) $1+\frac{(n-1)\,(n-2)}{2\,!}+\frac{(n-1)\,(n-2)\,(n-3)\,(n-4)}{4\,!}+\ldots\ldots=2^{n-2}$.

3. Show that
$1+{}^nC_1+{}^nC_2+{}^nC_3+\cdots+{}^nC_n=2+2+2^2+\cdots+2^{n-1}=2^n.$

4. In the expansion of $(x+y)^n$, which term from the beginning is
   (*i*) the third from the end ?
   (*ii*) the ninth from the end ?
   (*iii*) the *r*th from the end ?

Find the value of

5. ${}^{12}C_1+{}^{12}C_2+{}^{12}C_3+{}^{12}C_4+\cdots+{}^{12}C_{12}.$

6. ${}^{13}C_2+{}^{13}C_4+{}^{13}C_6+{}^{13}C_8+\cdots+{}^{13}C_{12}.$

7. ${}^{17}C_1+{}^{17}C_3+{}^{17}C_5+{}^{17}C_7+\cdots+{}^{17}C_{17}.$

If C_0, C_1, C_2, C_3 be the binomial coefficients corresponding to the index n, show that

8. $C_1-2.C_2+3.C_3-4.C_4+\cdots+(-1)^{n-1}.nC_n=0.$

9. $C_0+2.C_1+4.C_2+8.C_3+\cdots+2^nC_n=3^n.$

10. $C_2+2C_3+3C_4+\cdots+(n-1).C_n=1+(n-2)2^{n-1}.$

11. $C_0-2C_1+3C_2-\cdots+(-1)^n(n+1)C_n=0.$

12. $\dfrac{C_1}{C_0}+2.\dfrac{C_2}{C_1}+3.\dfrac{C_3}{C_2}+\ldots+n\dfrac{C_n}{C_{n-1}}=\dfrac{1}{2}n(n+1).$

13. $C_0-\dfrac{C_1}{2}+\dfrac{C_2}{3}+\cdots+(-1)^n\dfrac{C_n}{n+1}=\dfrac{1}{n+1}.$

7·7. BINOMIAL THEOREM FOR ANY RATIONAL INDEX

So far we have been concerned with the binomial theorem for a positive integral index. We shall now extend it to the case when n is any rational number. We shall however only state the result and apply it to approximations, evaluation of roots etc., and not go into the proof which is beyond the scope of this book.

7·7·1. Statement of the Binomial Theorem for a Rational Index

Let n be any rational number, positive or negative, integral or ractional, and x be any real number such that $|x|<1$. Then

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\cdots\cdots+\frac{n(n-1)\ldots(n-r+1)}{r!}x^r+\cdots\cdots \quad (A)$$

Notes. 1. When *n is a positive integer*, the series (A) terminates and its sum is $(1+x)^n$ for all values of x. In fact, (A) then reduces to the binomial theorem for a positive integral index. (A) thus includes the binomial theorem for a positive integral index as a particular case.

2. When n *is not a positive integer*, the series (A) is an infinite series ; for, then $n-r$ does not vanish for any positive integral value of r, whatever. When such is the case, the statement of the binomial theorem is equivalent to the following :

If $|x|<1$, the infinite series

$$1+nx+\frac{n(n-1)}{2!}x^2+\dots\dots+\frac{n(n-1)\dots(n-r+1)}{r!}x^r+\dots$$

is convergent and its sum to infinity is $(1+x)^n$.

3. When n *is a positive rational number*, the binomial theorem also holds for $|x|=1$. It is only for the sake of uniformity that we have stated the theorem in a restricted form.

4. That restrictions on the value of x are necessary can be easily seen by giving to n and x certain values which violate the conditions stated above, and then showing that they give rise to absurd results, *e.g.*, putting $n=-1$, $x=-2$, (A) gives $-1=1+2+2^2+2^3\dots$, which is obviously absurd.

5. When $n=p/q$, where p is an integer (positive or negative) and q is a positive integer, the series (A) is 'the positive qth root' of $(1+x)^p$.

For example, $1+\frac{2}{3}x+\frac{\frac{2}{3}\cdot\left(\frac{2}{3}-1\right)}{2!}x^2+\dots\dots$ is equal to $+\sqrt[3]{(1+x)^2}$.

6. Though the expansion (A) is not valid when $|x|>1$ and n is non-integer, it is then possible to expand $(1+x)^n$ in descending powers of x. For,

$$(x+1)^n=x^n(1+x^{-1})^n,$$

$$=x^n\left(1+nx^{-1}+\frac{n(n-1)}{2!}x^{-2}+\dots+\frac{n(n-1)\dots(n-r+1)}{r!}x^{-r}\right.$$

$$\left.+\dots\right), \text{ since } |x^{-1}|<1.$$

$$=x^n+nx^{n-1}+\frac{n(n-1)}{2!}x^{n-2}+\dots+\frac{n(n-1)\dots(n-r+1)}{r!}x^{n-r}+\dots$$

Thus

$$(x+1)^n=x^n+nx^{n-1}+\dots+\frac{n(n-1)\dots(n-r+1)}{r!}x^{n-r}+\dots\dots$$

provided $|x|>1$.

Thus

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\dots+\frac{n(n-1)\dots(n-r+1)}{r!}x^r+\dots,$$
provided $|x|<1$

and

$$(x+1)^n=x^n+nx^{n-1}+\frac{n(n-1)}{2!}x^{n-2}+\dots+\frac{n(n-1)\dots(n-r+1)}{r!}x^{n-r}$$
$+\dots\dots$, provided $|x|>1$.

When n is a positive integer, both the above results reduce to the following :

$$(x+1)^n=(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\dots+x^n,$$
$$=x^n+nx^{n-1}+\frac{n(n-1)}{2!}x^{n-2}+\dots\dots+1.$$

Example 18. *Use binomial theorem to show that*

$$(1-x)^{-1}=1+x+x^2+x^3+\dots \textit{ provided } |x|<1.$$

Solution. Since $|x|<1$, we have

$$(1-x)^{-1}=\{1+(-x)\}^{-1}=1+(-1)(-x)+\frac{(-1)(-1-1)}{2!}(-x)^2+\dots,$$
$$=1+x+x^2+\dots\dots$$

Note. In the above example, the series on the right hand side is an infinite G.P. having x ($|x|<1$) as the common ratio. Therefore, from what we have already learnt in Chapter 5, its sum to infinity$=\dfrac{1}{1-x}=(1-x)^{-1}$.

Thus an infinite G.P. is only a particular case ($n=-1$) of the binomial series (A).

EXERCISE 7 (*d*)

If $|x|<1$, *show that* :

1. $(1-x)^{-2}=1+2x+3x^2+\dots+(r+1)x^r+\dots$
2. $(1-x)^{-3}=1+3x+6x^2+\dots+\frac{1}{2}(r+1)(r+2)x^r+\dots$,
3. $(1+x)^{-2}=1-2x+3x^2-\dots+(-1)^r(r+1)x^r+\dots$
4. $(1+x)^{-3}=1-3x+6x^2-\dots+\frac{1}{2}(-1)^r(r+1)(r+2)x^r+\dots$
5. $(1+x)^{-n}=1-nx+\dfrac{n(n+1)}{2!}x^2+\dots$
$$+(-1)^r.\frac{n(n+1)\dots(n+r-1)}{r!}x^r+\dots$$

6. $$(1-x)^{-n}=1+nx+\frac{n(n+1)}{2!}x^2+\cdots+\frac{n(n+1)\cdots(n+r-1)}{r!}x^r+\cdots$$

7·8. THE EXPANSION OF $(x+a)^n$

When n is not a positive integer, we can use the binomial theorem to *expand* $(x+a)^n$ *in ascending powers of* x *if* $|x|<|a|$ *and in descending powers of* a *if* $|a|<|x|$. We may proceed as follows :

Case 1. When $|x|<|a|$.

$$(x+a)^n=a^n\left(1+\frac{x}{a}\right)^n.$$

Since $|x|<|a|$, $|x/a|<1$, so that by using the binomial theorem, we have

$$\left(1+\frac{x}{a}\right)^n=1+n\left(\frac{x}{a}\right)+\frac{n(n-1)}{2!}\left(\frac{x}{a}\right)^2+\cdots+\frac{n(n-1)\cdots(n-r+1)}{r!}\left(\frac{x}{a}\right)^r+\cdots,$$

or $$(x+a)^n=a^n\left\{1+n\left(\frac{x}{a}\right)+\frac{n(n-1)}{2!}\left(\frac{x}{a}\right)^2+\cdots\right\},$$

$$=a^n+nxa^{n-1}+\frac{n(n-1)}{2!}x^2a^{n-2}+\cdots \qquad \text{...(A)}$$

Case 2. When $|a|<|x|$,

$$(x+a)^n=x^n\left\{1+\frac{a}{x}\right\}^n.$$

Since $|a|<|x|$, $|a/x|<1$, so that by using the binomial theorem, we have

$$\left(1+\frac{a}{x}\right)^n=1+n\left(\frac{a}{x}\right)+\frac{n(n-1)}{2!}\left(\frac{a}{x}\right)^2+\cdots+\frac{n(n-1)\cdots(n-r+1)}{r!}\left(\frac{a}{x}\right)^r+\cdots,$$

or $$(x+a)^n=x^n\left\{1+n\left(\frac{a}{x}\right)+\frac{n(n-1)}{2!}\left(\frac{a}{x}\right)^2+\cdots\right\},$$

$$=x^n+nax^{n-1}+\frac{n(n-1)}{2!}a^2x^{n-2}+\cdots\cdots \qquad \text{...(B)}$$

From (A) and (B), we have

$$(x+a)^n = a^n + nxa^{n-1} + \frac{n(n-1)}{2!}x^2a^{n-2} + \ldots\ldots,$$

provided $|x| < |a|$,

and
$$(x+a)^n = x^n + nax^{n-1} + \frac{n(n-1)}{2!}a^2x^{n-2} + \ldots\ldots,$$

provided $|a| < |x|$.

Example 19. *Expand* $(2+3x)^{-4}$ *in (i) ascending powers of* x *(ii) descending powers of* x *stating in each case the range of values of* x *for which the expansion is valid.*

Solution. (*i*) *Expansion in ascending powers of* x :

$$(2+3x)^{-4} = \left\{2\left(1+\frac{3}{2}x\right)\right\}^{-4},$$

$$= \frac{1}{16}\left(1+\frac{3}{2}x\right)^{-4},$$

$$= \frac{1}{16}\left\{1+(-4)\left(\frac{3}{2}x\right) + \frac{(-4)(-5)}{2!}\left(\frac{3x}{2}\right)^2 + \frac{(-4)(-5)(-6)}{3!}\left(\frac{3x}{2}\right)^3 + \ldots\ldots\right\},$$

provided $\frac{3}{2}|x| < 1$, *i.e.*, provided $|x| < \frac{2}{3}$.

$$= \frac{1}{16}\left\{1 - 6x + \frac{45}{2}x^2 - \frac{135}{2}x^3 + \ldots\ldots\right\},$$

provided $|x| < \frac{2}{3}$.

(*ii*) *Expansion in descending powers of* x :

$$(2+3x)^{-4} = \left\{3x\left(1+\frac{2}{3x}\right)\right\}^{-4},$$

$$= \frac{1}{81x^4}\left\{1+(-4)\left(\frac{2}{3x}\right) + \frac{(-4)(-5)}{2!}\left(\frac{2}{3x}\right)^2 + \frac{(-4)(-5)(-6)}{3!}\left(\frac{2}{3x}\right)^3 + \ldots\ldots\right\},$$

provided $\left|\frac{2}{3x}\right| < 1$, *i.e.*, provided $|x| > \frac{2}{3}$,

$$= \frac{1}{81}\left\{\frac{1}{x^4} - \frac{8}{3}\cdot\frac{1}{x^5} + \frac{40}{9}\cdot\frac{1}{x^6} - \frac{160}{27}\cdot\frac{1}{x^7} + \ldots\ldots\right\},$$

provided $|x| > \frac{2}{3}$.

Example 20. *Write down the first four terms of the expansion of $(8+x)^{4/3}$, when $|x| < 8$.*

Solution.

$$(8+x)^{4/3}=\left\{8\left(1+\frac{x}{8}\right)\right\}^{4/3},$$

$$=8^{4/3}\left(1+\frac{x}{8}\right)^{4/3},$$

$$=16\left\{1+\frac{4}{3}\cdot\frac{x}{8}+\frac{\frac{4}{3}\left(\frac{4}{3}-1\right)}{2\,!}\left(\frac{x}{8}\right)^2\right.$$
$$\left.+\frac{\frac{4}{3}\left(\frac{4}{3}-1\right)\left(\frac{4}{3}-2\right)}{3\,!}\left(\frac{x}{8}\right)^3+\ldots\ldots\right\},$$

since $|x| < 8$,

$$=16+\frac{8}{3}x+\frac{x^2}{18}-\frac{x^3}{648}+\ldots\ldots$$

7.8 1. The general term

The $(r+1)$th term in the expansion of $(1+x)^n$ is called the general term.

When $|x| < 1$, T_{r+1} in the expansion of $(1+x)^n$ in ascending powers of x

$$=\frac{n(n-1)\ldots\ldots(n-r+1)}{r\,!}x^r.$$

When $|x| > 1$, T_{r+1} in the expansion of $(1+x)^n$ in descending powers of x

$$=\frac{n(n-1)\ldots\ldots(n-r+1)}{r\,!}x^{n-r}.$$

Example 21. *Write down the 5th term in the expansion of $(1-2x)^{5/2}$ in ascending powers of x, stating the range of values of x for which such an expansion is valid.*

Solution. The expansion in ascending powers of x is valid provided $|2x| < 1$, *i.e.*, provided $|x| < \frac{1}{2}$. The 5th term is then

$$T_5=\frac{\frac{5}{2}\left(\frac{5}{2}-1\right)\ldots\ldots\left(\frac{5}{2}-3\right)}{4\,!}(-2x)^4,$$

$$=\frac{\frac{5}{2}\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{4\,!}(-2x)^4=-\frac{5}{8}x^4.$$

Example 22. *Assuming that it is possible to expand* $(3-x)^{-25}$ *in a series in ascending powers of x, find the general term in the expansion.*

Solution. $(3-x)^{-25}=3^{-25}\left(1-\dfrac{x}{3}\right)^{-25}$.

General term$=T_{r+1}$,

$$=(3^{-25})\,\frac{(-25)(-25-1)\ldots\text{to } r \text{ factors}}{r\,!}(-x/3)^r$$

$$=3^{-25-r}\,\frac{25.26\ldots(r+24)}{r\,!}x^r,$$

$$=\frac{(r+24)\,!}{(24)\,!\,r\,!\,3^{r+25}}\cdot x^r.$$

7·8·2. The First Negative Term in the Expansion of $(1+x)^n$

To find the first negative term in the expansion of $(1+x)^n$ we find the smallest value of r for which $\dfrac{T_{r+1}}{T_r}<0$. T_{r+1}, when r takes this value is the first negative term. The method is illustrated by the following example.

Example 23. *Assuming that it is possible to expand* $(1+x)^{11/2}$ *in a series of ascending powers of x, find the first negative term in the expansion.*

Solution. Consider the expansion of $(1+x)^n$ in ascending powers of x, where $n=\frac{11}{2}$.

$$T_{r+1}=\frac{n\,(n-1)\ldots(n-r+1)}{r\,!}x^r,$$

$$T_r=\frac{n\,(n-1)\cdots(n-r+2)}{(r-1)\,!}x^{r-1}.$$

$$\frac{T_{r+1}}{T_r}=\frac{n-r+1}{r}\cdot x=\frac{\frac{11}{2}-r+1}{r}\cdot x,\ \text{since } n=\frac{11}{2}.$$

Since r, x are both positive, T_{r+1} and T_r will be of opposite signs if $\frac{13}{2}-r<0$, *i.e.*, if $r>\frac{13}{2}$. The least value of r for which this inequality holds is 7. Therefore, the least value of r for which T_r and T_{r+1} have opposite signs is 7.

$\therefore$ The first negative term is T_8.

Hence the first negative term is

$$T_8=\frac{\frac{11}{2}\cdot\frac{9}{2}\cdot\frac{7}{2}\cdots\cdots\left(\frac{11}{2}-7+1\right)}{7\,!}x^7,$$

$$=-\frac{11\,.\,9\,.\,7\,.\,5\,.\,3}{2^7\,7\,!}x^7,$$

$$=-\frac{33}{1024}x^7.$$

Aliter. Let T_{r+1} be the first negative term.

$$T_{r+1}=\frac{\frac{11}{2}\cdot\frac{9}{2}\cdots\cdots\left(\frac{11}{2}-r+1\right)}{r\,!}\,.\,x^r$$

Since T_{r+1} is to be first negative term, we have to find the least value of r for which $\frac{11}{2}-r+1<0$,

i.e., for which $r>\frac{13}{2}$.

The least value of r for which $\frac{11}{2}-r+1<0$ is 7.

Hence T_8 is the first negative term. As above

$$T_8=-\frac{33}{1024}x^7.$$

7·8·3. The Numerically Greatest Term in the Expansion of $(1+x)^n$.

To find the numerically greatest term we find the greatest value of r for which $\frac{T_{r+1}}{T_r}$ is greater than 1. T_{r+1}, where r takes this value, is the numerically greatest term. The method is clearly illustrated by the following example.

Example 24. *Find the numerically greatest term in the expansion of $(1+x)^n$ when $x=\frac{1}{5}$, $n=-60$.*

Solution. Let T_{r+1} be the numerically greatest term in the expansion of $(1+x)^n$ in ascending powers of x.

$$\frac{T_{r+1}}{T_r}=\frac{n-r+1}{r}x,$$

$$=\frac{-60-r+1}{r}\cdot\frac{1}{5}.$$

$$\left|\frac{T_{r+1}}{T_r}\right|=\left|-\frac{59+r}{5r}\right|=\frac{59+r}{5r}.$$

Now $\qquad \frac{59+r}{5r}>1,$

if $59+r > 5r,$

i e., if $r < 14\frac{3}{4}$.

T_{r+1} is numerically greater than T_r so long as $r \leqslant 14$ and T_{r+1} is numerically less than T_r for $r > 14$.

$\therefore$ T_{r+1} is the numerically greatest term when $r=14$.

$\therefore$ T_{15} is the numerically greatest term.

$$T_{15}=\frac{(-60)(-60-1)\ldots\ldots(-60-14+1)}{14\,!}\left(\frac{1}{5}\right)^{14},$$

$$=\frac{73\,!}{5^{14}\,14\,!\,59\,!}$$

EXERCISE 7 (*e*)

1. Write down the first four terms in the expansion of the following in ascending powers of x :

   (*i*) $(1-x)^{-2}$, when $|x|<1$.

   (*ii*) $(1-3x)^{-1/3}$, when $|x|<\frac{1}{3}$.

   (*iii*) $\frac{1}{\sqrt{(4-3x^2)}}$, when $|x|<\frac{2}{\sqrt{3}}$

   (*iv*) $(2-3x)^{3/2}$, when $|x|<\frac{2}{3}$.

2. Find the 7th term in the expansion of $(1-2x)^{9/2}$ in ascending powers of x, being given that $|x|<\frac{1}{2}$.

3. Find the fifth term in the expansion of $(1-2x^3)^{11/2}$ in ascending powers of x, stating the range of values of x for which the expansion is valid.

4. Write down the 5th term in the expansion of $(3a-2b)^{-1}$ distinguishing between the cases :

   (*i*) b/a numerically less than $\frac{3}{2}$.

   (*ii*) b/a numerically greater than $\frac{3}{2}$.

5. Find the general term in each of the following expansions in ascending powers of x :

   (*i*) $(1-x)^{-4}$, when $|x|<1$.

   (*ii*) $(4-3x^2)^{-1/2}$, when $|x|<\frac{2}{\sqrt{3}}$

(*iii*) $\left(1-\frac{2}{3}x\right)^{-1/2}$, when $|x|<\frac{3}{2}$.

(*iv*) $(1-nx)^{1/n}$, when $|x|<\frac{1}{n}$.

6. If x be a positive proper fraction, find the first negative term in the expansion of $(1+x^2)^{17/2}$ in ascending powers of x.

7. If x be a positive number less than $\frac{3}{2}$, find the first negative term in the expansion of $\left(1+\frac{4}{9}x^2\right)^{16/3}$ in ascendng powers of x.

7·8·4. To Find the Coefficient of a Given Power of x in the Expansion of $(1+x)^n$.

Example 25. *Find the coefficient of x^{10} in the expansion of $(1-2x)^{-1/2}$ in ascending powers of x, assuming that such an expansion is possible.*

Solution. Let x^{10} occur in the $(r+1)$th term.

The $(r+1)$th term $=T_{r+1}$

$$=\frac{-\frac{1}{2}(-\frac{1}{2}-1)\ldots(-\frac{1}{2}-r+1)}{r!}(-2x^2)^r.$$

Since T_{r+1} contains x^{10},

$\therefore \quad 2r=10$, *i.e.*, $r=5$.

Hence x^{10} occurs in the 6th term and its coefficient

$$=\frac{-\frac{1}{2}(-\frac{1}{2}-1)\ldots(-\frac{1}{2}-5+1)}{5!}(-2)^5,$$

$$=\frac{1.3.5.7.9}{2^5.1.2.3.4.5}(-1)^{5+5}.2^5,$$

$$=\frac{63}{8}.$$

Example 26. *Find the coefficient of x^{10} in the expansion of $\frac{1+x+2x^2}{(1+x)^4}$ in ascending powers of x, assuming that such an expansion is valid.*

Solution. $\frac{1+x+2x^2}{(1+x)^4}=(1+x+2x^2)(1+x)^{-4}$.

Coefficient of $x^{10}=1\times$ coefficient of x^{10} in $(1+x)^{-4}$

$+1\times$ coefficient of x^9 in $(1+x)^{-4}$

$+2\times$ coefficient of x^8 in $(1+x)^{-4}$.

Now the general term in the expansion of $(1+x)^{-4}$ in ascending powers of x is given by

$$T_{r+1}=\frac{-4(-4-1)(-4-2)...(-4-r+1)}{r\,!}x^r,$$

$$=(-1)^r\,\frac{4.5.6...(r+3)}{1.2.3......r}x^r,$$

$$=(-1)^r\,\frac{(r+1)(r+2)(r+3)}{3\,!}\,x^r.$$

The coefficient of x^r in the expansion of $(1+x)^{-4}$

$$=(-1)^r\,\frac{(r+1)(r+2)(r+3)}{6}.$$

Coefficient of x^{10}

$$=(-1)^{10}\,\frac{11.12.13}{6},$$

$$=\frac{11.12.13}{6}.$$

Coefficient of x^9

$$=(-1)^9\,.\,\frac{10.11.12}{6},$$

$$=-\frac{10.11.12}{6}.$$

Coefficient of x^8

$$=(-1)^8\,\frac{9.10.11}{6},$$

$$=\frac{9.10.11}{6}.$$

Hence the coefficient of x^{10} in the given expansion

$$=\frac{11.12.13}{6}-\frac{10.11.12}{6}+2\,.\,\frac{9.10.11}{6},$$

$$=\frac{11}{6}[156-120+180]=\frac{11.216}{6}=396.$$

EXERCISE 7 (*f*)

1. Find the coefficient of x^{14} in the expansion of $(1-4x^2)^{-1/2}$ in ascending powers of x, x being numerically less than $\frac{1}{2}$.

2. Find the coefficient of x^{12} in the expansion of $(a^5-b^3x^2)^{5/2}$ in ascending powers of x, assuming that the expansion is valid.

3. Find the coefficient of x^r in the expansion of $(2-3x)^{2/3}$ in ascending powers of x, being given that $|\,x\,|<\frac{2}{3}$

4. Find the coefficient of x^r in the expansion of each of the following in ascending powers of x (It may be assumed that the expansion is valid in each case) :

(i) $\dfrac{1+2x}{1-x}$ (ii) $\dfrac{1+2x+3x^2}{(1-x)^3}$

(iii) $\dfrac{(1+x)^r}{1-x}$.

5. If $|x|<1$, show that the coefficient of x^{11} in the expansion of $\sqrt{\left\{\dfrac{(1+x)^3}{(1-x)^5}\right\}}$ in ascending powers of x is

$$\frac{7 \cdot 9 \cdot 11 \cdot 13}{8\,(4\,!)}.$$

6. Find the coefficient of x^r in the expansion of $\sqrt{\dfrac{1+x}{1-x}}$ in ascending powers of x, $|x|$ being less than 1.

7. $|x|<1$, show that the coefficient of x^{2n} in the expansion of $\dfrac{1}{(1-x)^3(1+x)}$ in ascending powers of x is

$$\frac{1}{6}(n+1)(n+2)(n+3).$$

8. Find the coefficient of x^4 in the expansion of $(1+x+x^2)^{-5}$, where $|x|<1$.

9. Find the coefficient of x^2 in the expansion of $(1-x+x^2)^{-2}$, where $|x|<1$.

10. If $x>1$, show that

$$x^n=1+n\left(1-\frac{1}{x}\right)+\frac{n(n+1)}{2\,!}\left(1-\frac{1}{x}\right)^2+\cdots$$

11. If x is a positive proper fraction, show that

$$\frac{(1+x)^n}{(1-x)^n}=1+n\frac{2x}{1+x}+\frac{n(n+1)}{2\,!}\left(\frac{2x}{1+x}\right)^2+\cdots$$

[**Hint.** $\left(\dfrac{1+x}{1-x}\right)^n=\left(1-\dfrac{2x}{1+x}\right)^{-n}$. It is possible to expand the R H.S. in ascending powers of $\dfrac{2x}{1+x}$, since $\dfrac{2x}{1+x}$ is positive and less than 1 when x is positive and less than 1.]

7·9. APPROXIMATIONS

When $|x|<1$, the limit of the sum of r terms of the series

$$1+nx+\frac{n(n-1)}{2\,!}x^2+\cdots \qquad \cdots(A)$$

as r approaches infinity is $(1+x)^n$. This provides us with a method of finding approximations to $(1+x)^n$ correct to a given power of x.

To the first power of x, $(1+x)^n=1+nx$.

To the second power of x, $(1+x)^n=1+nx+\dfrac{n(n-1)}{2\,!}x^2$.

In general, the larger the number of terms retained, the closer the approximation obtained.

Note. In the following problems on approximations, we shall frequently use the phrase 'x is very small' to mean that 'x is numerically very small'.

Example 27. *If x be so small that its square and higher powers can be neglected, then show that*

$$\frac{(1-3x)^{1/2}+(1-x)^{5/3}}{(4-x)^{1/2}}=1-\frac{35}{24}x.$$

Solution. Neglecting square and higher powers of x, we have

$$\frac{(1-3x)^{1/2}+(1-x)^{5/3}}{(4-x)^{1/2}}=\frac{\left(1-\frac{3}{2}x\right)+\left(1-\frac{5}{3}x\right)}{4^{1/2}\left(1-\frac{x}{4}\right)^{1/2}}$$

$$=\frac{1}{2}\cdot\frac{2-\frac{19}{6}x}{1-\frac{x}{8}}$$

$$=\left(1-\frac{19}{12}x\right)\left(1-\frac{x}{8}\right)^{-1}$$

$$=\left(1-\frac{19}{12}x\right)\left(1+\frac{x}{8}\right)=1-\frac{35}{24}x.$$

Example 28. *If x be so small that its square and higher powers can be neglected, show that*

$$\frac{(16+5x)^{1/2}-(27-4x)^{1/3}}{5x+6}=\frac{1}{6}-\frac{13}{1296}x.$$

Solution. L.H.S.

$$=\frac{16^{1/2}\left(1+\frac{5}{16}x\right)^{1/2}-27^{1/3}\left(1-\frac{4}{27}x\right)^{1/3}}{6\left(1+\frac{5}{6}x\right)}.$$

$$=\frac{4\left(1+\frac{5}{32}x\right)-3\left(1-\frac{4}{81}x\right)}{6\left(1+\frac{5}{6}x\right)},$$

$$=\frac{\left(4+\frac{5}{8}x\right)-\left(3-\frac{4}{27}x\right)}{6\left(1+\frac{5}{6}x\right)},$$

$$=\frac{\left(1+\frac{167}{216}x\right)}{6\left(1+\frac{5}{6}x\right)},$$

$$=\frac{1}{6}\left(1+\frac{167}{216}x\right)\left(1+\frac{5}{6}x\right)^{-1},$$

$$=\frac{1}{6}\left(1+\frac{167}{216}x\right)\left(1-\frac{5}{6}x\right),$$

$$=\frac{1}{6}\left(1-\frac{13}{216}x\right)=\frac{1}{6}-\frac{13}{1296}x.$$

Example 29. *If x be nearly equal to 1, show that*

$$\frac{ax^b-bx^a}{x^b-x^a}=\frac{1}{1-x} \text{ nearly.}$$

Solution. Since x is nearly equal to 1, therefore, let $x=1+y$, where y is so small that its square and higher powers can be neglected.

$$\frac{ax^b-bx^a}{x^b-x^a}=\frac{a(1+y)^b-b(1+y)^a}{(1+y)^b-(1+y)^a},$$

$$=\frac{a(1+by)-b(1+ay)}{(1+by)-(1+ay)},$$

$$=\frac{a-b}{(b-a)y},$$

$$=-\frac{1}{y}=\frac{1}{1-x}.$$

7·9·1. Approximate Evaluation of Roots

Recall that it is possible to express every irrational number as an infinite non-recurring decimal. As an illustration of the same, we shall show how the nth root $(n=2, 3, 4, \ldots)$ of any rational

number can be evaluated correct to any number of decimals by applying the binomial theorem.

Let N be a positive rational number whose nth root is desired. We have first of all to find two consecutive positive integers $a-1$ and a such that $(a-1)^n < N < a^n$.

Let $\quad N = a^n - x.$

$\therefore \quad N^{1/n} = (a^n - x)^{1/n},$

$$= a\left(1 - \frac{x}{a^n}\right)^{1/n}. \qquad \ldots(i)$$

Since $x = a^n - N < a^n$, by applying the binomial theorem to (i), we have

$$N^{1/n} = a\left\{1 - \frac{1}{n}\cdot\frac{x}{a^n} + \frac{\frac{1}{n}\left(\frac{1}{n}-1\right)}{2\,!}\left(\frac{x}{a^n}\right)^2 + \ldots\right\}.$$

We have thus expressed $N^{1/n}$ as an infinite series. By converting each term on the R.H.S. into decimals, and retaining a suitable number of terms, we can evaluate $N^{1/n}$ to any number of decimal places.

Note. It is sometimes also convenient to write

$$N = (a-1)^n + y,$$

so that

$$N^{1/n} = (a-1)\left\{1 + \frac{y}{(a-1)^n}\right\}^{1/n},$$

$$= (a-1)\left[1 + \frac{1}{n}\cdot\frac{y}{(a-1)^n} + \frac{\frac{1}{n}\left(\frac{1}{n}-1\right)}{2\,!}\left\{\frac{y}{(a-1)^n}\right\}^2 + \ldots\right],$$

provided $\dfrac{y}{(a-1)^{1/n}} < 1.$

Example 30. *Find the value of* $\dfrac{1}{\sqrt{47}}$ *correct to four places of decimal.*

Solution. Since 47 lies between 6^2 and 7^2, therefore, we shall write 47 as $7^2 - 2$.

Now $\dfrac{1}{\sqrt{47}}=(47)^{-1/2}=(7^2-2)^{-1/2}$,

$$=\frac{1}{7}\left(1-\frac{2}{7^2}\right)^{-1/2},$$

$$=\frac{1}{7}\left(1+\frac{1}{7^2}+\frac{3}{2}\cdot\frac{1}{7^4}+\frac{5}{2}\cdot\frac{1}{7^6}+\cdots\right),$$

$$=\frac{1}{7}+\frac{1}{7^3}+\frac{3}{2}\cdot\frac{1}{7^5}+\frac{5}{2}\cdot\frac{1}{7^7}+\cdots,$$

$$=\cdot142857+\cdot002915+\cdot000088=\cdot14586.$$

Hence $\dfrac{1}{\sqrt{47}}=\cdot1459$ correct to four places of decimals.

Note. In the above example we have stopped at the fourth term, for the succeeding terms will not affect the result upto the fifth decimal place.

Example 31. *Find the square root of 999 correct to three decimal places.*

Solution. Since 999 lies between 30^2 and 31^2, we shall write

$$999=30^2+99.$$

$$999=(900+99)^{1/2}=(30^2+99)^{1/2},$$

$$=30\left(1+\frac{99}{900}\right)^{1/2},$$

$$=30(1+\cdot11)^{1/2},$$

$$=30\left[1+\frac{1}{2}(\cdot11)+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}(\cdot11)^2\right.$$

$$\left.+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(\cdot11)^3+\cdots\cdots\right],$$

$$=30\left[1+\frac{\cdot11}{2}-\frac{1}{8}(\cdot11)^2+\frac{1}{16}(\cdot11)^3\right.$$

$$\left.-\frac{5}{128}(\cdot11)^4+\cdots\right],$$

$$=30[1+\cdot055-\cdot001512+\cdot000083$$

$$-\cdot000006+\cdots],$$

$$=30(1{\cdot}053565)=31{\cdot}60695,$$

$$=31{\cdot}607 \text{ correct to three decimal places.}$$

Note. If the cube root of 999 be required, we write 999 as (10^3-1) and then proceed as above.

EXERCISE 7 (*g*)

If x be so small that square and higher powers of x may be neglected, show that

1. (*i*) $(1+2x)^{1/2}(1-4x)^{-5/2}=1+11x$ nearly.

(*ii*) $\dfrac{(9+2x)^{1/2}(3+4x)}{\sqrt[5]{1-x}}=9+\dfrac{74}{5}x$ nearly.

(*iii*) $\dfrac{\sqrt{(1+x)}+\sqrt[3]{1+2x}}{\sqrt[4]{1+3x}+\sqrt[5]{1+4x}}=1-\dfrac{23}{120}x$ nearly.

2. (*i*) $\dfrac{(1+x)^{1/2}+(1-x)^{2/3}}{(1+x)+(1+x)^{1/2}}=1-\dfrac{5}{6}x$ nearly.

(*ii*) $\dfrac{(9+7x)^{1/2}-(16+3x)^{1/4}}{4+5x}=\dfrac{1}{4}-\dfrac{17}{384}x$ nearly.

3. If x be so small that its cube and higher powers may be neglected, then show that

(*i*) $\dfrac{(1-x)^{-5/2}+(16+8x)^{1/2}}{(1+x)^{-1/2}+(2+x)^2}=1+\dfrac{23}{40}x^2$ nearly.

(*ii*) $\dfrac{(1+x)^{3/4}+(1+5x)^{1/2}}{(1-x)^2}=2+\dfrac{29}{4}x+\dfrac{297}{32}x^2$ nearly.

(*iii*) $\dfrac{3\left(x+\dfrac{4}{9}\right)^{1/2}\cdot\left(1-\dfrac{3}{4}x^2\right)^{1/3}}{2\left(1+\dfrac{9}{16}x^2\right)}=1-\dfrac{307}{256}x^2$ nearly.

4. If x be a quantity so small that x^3 may be neglected in comparison with l^3, show that

$$\sqrt{\left(\frac{l}{l+x}\right)}+\sqrt{\left(\frac{l}{l-x}\right)}=2+\frac{3x^2}{4l^2} \text{ very nearly.}$$

5. If the square and higher powers of the difference between x and unity can be neglected, then show that

$$\frac{mx^m-nx^n}{m-n}=x^{m+n} \text{ nearly.}$$

6. If x is very nearly equal to a, prove that

$$\frac{(5a^2x-x^3)^{1/2}-2a^{1/2}x}{(3ax-2x^2)^{1/4}-a^{1/2}}=6a \text{ very nearly.}$$

7. If N and n be nearly equal, then show that

$$\sqrt{\frac{N}{n}}=\frac{N}{N+n}+\frac{1}{4}\frac{N+n}{n} \text{ very nearly.}$$

[**Hint.** Let $N=n+x$, where x is so small that its square and higher powers may be neglected.

$$\text{L.H.S.}=\left(1+\frac{x}{n}\right)^{1/2}=1+\frac{x}{2n}.$$

Similarly simplify the R.H.S. also. Alternatively, we may put $N=m+x$ and $n=m-x$, where x is so small that its powers above one may be neglected.

$$\sqrt{\frac{N}{n}}=\frac{\sqrt{m^2-x^2}}{m-x}=\frac{m}{m-x}\left(1-\frac{x^2}{2m^2}\right) \text{ nearly,}$$

$$=\frac{m+x}{2m}+\frac{1}{4}\cdot\frac{2m}{m-x}.]$$

8. If $\sqrt{N}=a+x$, where x is small, then show that

$$\sqrt{N}=a\cdot\frac{3N+a^2}{N+3a^2} \text{ nearly.}$$

9. If $N^{1/3}=a-x$, where x is small, then show that

$$N^{1/3}=a\cdot\frac{2N+a^3}{N+2a^3} \text{ nearly.}$$

10. If m differs from n^2 by a small quantity, then show that the square root of m is approximately equal to

$$\frac{3}{2n}-\frac{1}{8n^3}(3n^2-m)^2.$$

11. Find the cube root of 126 correct to five decimal places.

12. Use binomial theorem to prove that $\left(\frac{9}{10}\right)^{4/5}=\cdot91917$, correct to five decimal places.

13. Calculate the value of $(998)^{1/3}$ correct to five decima

14. Evaluate $(\cdot998)^{-1/2}$ and $(\cdot99)^{1/3}$ correct to decimal.

15. Find the fifth root of :

(*i*) 244 correct to three decimal places.

(*ii*) 31 correct to three decimal places.

TEST YOUR UNDERSTANDING VII

In each of the following problems only one out of the four alternatives is correct. Write down the letter corresponding to the correct alternative :

1. The number of terms in the binomial expansion of $(2x+5)^{30}$ is
(*a*) 30 (*b*) 31 (*c*) 29 (*d*) 37.

2. The coefficients of x^7 in the expansion of $\left(x-\frac{1}{x}\right)^{10}$ is
(*a*) 45 (*b*) 120 (*c*) 0 (*d*) -1.

3. The coefficient of x^{-17} in the expansion of $\left(x-\frac{1}{x^2}\right)^{10}$ is
(*a*) -20 (*b*) 0 (*c*) 1 (*d*) -10.

4. The *k*th term in the expansion of $\left(3x^2+\frac{1}{x}\right)^{12}$ is independent of x. The value of k is
(*a*) 7 (*b*) 8 (*c*) 9 (*d*) 6.

5. The sum of the coefficients in the expansion of $(5p-4q)^{100}$ is
(*a*) 1 (*b*) -1 (*c*) 5^{100} (*d*) -20^{100}

6. The fourth term in the expansion of $(8+4x^2)^{4/3}$, when $|x|<\sqrt{2}$ is
(*a*) $\frac{8x^2}{9}$ (*b*) $-\frac{8}{81}x^6$ (*c*) $20x^3$ (*d*) $-x^{10}$.

7. The expansion of $(1+2x)^{1/2}$ by binomial theorem is valid when
(*a*) $x>\frac{1}{2}$ (*b*) $x<\frac{1}{2}$ (*c*) $-\frac{1}{2}<x<\frac{1}{2}$ (*d*) $-2<x<2$.

8. The number of terms in the expansion of $(1-3x)^{1/2}$ is
(*a*) 2 (*b*) 50 (*c*) 100 (*d*) infinitely many.

9. The *n*th term in the expansion of $(1-x)^{-2}$, when $|x|<1$, is
(*a*) nx^n (*b*) nx^{n-1} (*c*) x^{n-1} (*d*) $-x^n$.

10. If x is so small that its square and higher powers may be neglected, then $\left(\frac{1-x}{1+x}\right)^{1/2}$ is approximately equal to
(*a*) $1-x$ (*b*) $1+x$ (*c*) $2-x$ (*d*) $1-\frac{1}{2}x$.

REVIEW EXERCISE VII

1. Sum the series

$$x^n (x-1)^n + {}^nC_1\, x^{n-1} (x-1)^{n-1} (x+1)$$
$$+ {}^nC_2 x^{n-2} (x-1)^{n-2} (x+1)^2 + \dots + (x+1)^n.$$

2. Show that the middle term in the expansion of $(1+x)^{2n}$ is

$$\frac{1.3.5\dots(2n-1)}{n\,!}.2^n x^n. \qquad (AISSE\ 1986)$$

3. Which term in the expansion of $\left(2x^2-\frac{1}{x}\right)^{12}$ is independent of x? Find its value. *(AISSE 1987)*

4. Find the sum of all the terms in the expansion of $\left(x^3-\frac{1}{x^2}\right)^{10}$ when $x=1$.

5. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \dots + C_n x^n$, then show that the sum of the products of the C_i's taken two at a time, represented by $\underset{0<i<j<n}{\Sigma\ \Sigma} C_i C_j$, is equal to $2^{2n-1} - \frac{(2n)\,!}{2\,(n\,!)^2}$.

(I.I.T., J.E.E. 1983)

Prove that:

6. $\frac{C_0}{1} + \frac{C_2}{3} + \frac{C_4}{5} + \frac{C_6}{7} + \dots = \frac{2^n}{n+1}$.

7. $C_0C_2 + C_1C_3 + C_2C_4 + \dots + C_{n-2}C_n = \frac{(2n)\,!}{(n+2)\,!\,(n-2)\,!}$

8. $C_0C_r + C_1C_{r+1} + C_2C_{r+2} + \dots + C_{n-r}C_n = \frac{(2n)\,!}{(n+r)\,!\,(n-r)\,!}$

9. $C_0^2 - C_1^2 + C_2^2 - C_3^2 + \dots + (-1)^n C_n^2 = 0$, if n is odd, and

$= \frac{(-1)^{n/2}\, n\,!}{(\frac{1}{2}n\,!)^2}$, if n is even.

10. If n be a positive integer, prove that the coefficient of the middle term of $(1+x)^{2n}$ is equal to the sum of the squares of the coefficients of $(1+x)^n$.

11. If, in the expansion $(a+x)^n$, p be the sum of the odd terms and q the sum of the even terms, show that

(*i*) $p^2 - q^2 = (a^2-x^2)^n$,

(*ii*) $4pq = (a+x)^{2n} - (a-x)^{2n}$.

12. Write down the first four terms in the expansion of $(2-3x)^{3/2}$ when $|x|<\frac{2}{3}$.

13. Find the general term in the expansion of $3a/(a^3-x^2)$ in ascending powers of x, $|x|$ being less than $a^{3/2}$.

14. Find the numerically greatest term in the expansion of $(4+7x)^{3/2}$ when $x=\dfrac{3}{14}$.

15. Find the coefficient of x^r in the expansion of $\dfrac{(1-x)^2}{(1-2x)^3}$ in ascending powers of x, $|x|$ being less than $\frac{1}{2}$.

16. If x be so small that square and higher powers of x may be neglected, then show that

$$\frac{2+3x+(1-3x)^{1/2}}{1-\frac{1}{2}x+(4-x)^{1/2}}=1+\frac{3}{4}x, \text{ nearly.}$$

17. If p is less than unity, show that

$$\frac{1}{\sqrt[3]{(1-p)}}-\frac{1}{\sqrt[3]{(1+p)}}=\frac{2}{3}p,$$

neglecting powers of p higher than the first.

18. Evaluate $(623)^{1/4}$ to seven places of decimal.

19. Evaluate $(1{\cdot}004)^{1/10}$ to eight places of decimal.

20. If p is nearly equal to q and $n>1$, show that

$$\frac{(n+1)p+(n-1)q}{(n-1)p+(n+1)q}=\left(\frac{p}{q}\right)^n.$$

Apply this result to find the 7th root of $\dfrac{131}{132}$.

SUMMARY

1. When n is a positive integer, $(x+a)^n$ is identically equal to

$$x^n+{}^nC_1x^{n-1}a+{}^nC_2x^{n-2}a^2+\dots+{}^nC_rx^{n-r}a^r+\dots+a^n.$$

2. The $(r+1)$th term in the expansion of $(x+a)^n$ by binomial theorem is

$$T_{r+1}={}^nC_rx^{n-r}a^r.$$

3. If n be any rational number, positive or negative, integral or fractional, and x be any real number such that $|x|<1$, then

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\dots+\frac{n(n-1)\dots(n-r+1)}{r!}x^r+\dots$$

HISTORICAL NOTE

The binomial theorem for a positive integral index was given by Blaise Pascal (1623-1662 A.D.) in his book *Traite du Triangle Arithmetique*, published posthumously in 1665. The pascal triangle, an arrangement of numbe s giving the binomial coeffic ents for positive integral indices, is known after him. The same arrangement, in a slightly different form was given in India by *Pingala* in the 3rd century B.C. in the form of a diagram which he called the *Meru Prastar*.

The binomial theorem for fractional indices was discovered by Sir Isaac Newton (1642-1727) in 1665. It was communicated by him in two letters written in 1676 to Henry Oldenburg, the then Secretary of the Royal Society. It was published by Wallis in his *Algebra* (with due credit to Newton, of course) of 1685. Newton himself never published the binomial theorem, nor did he prove it. A proof was provided later on by C. Maclaurin (1698-1746).

Leibniz generalised the binomial theorem to the multinomial theorem to obtain the expansion of $(x+y+z)^n$. N.H. Abel extended the binomial theorem to the case of negative exponents.

□□

LEONHARD EULER (1707-1783)

Leonhard Euler, the most prolific mathematician of his time, was born in Basel, in Switzerland. He was educated at the University of Basel. He was invited to the Academy at Petersberg in 1727, and stayed there upto 1741 when accepting an invitation from Frederick the Great, he came to Berlin. In 1766 he went back to Petersberg.

Euler made signal contributions in every field of mathematics which existed in his day. He wrote on the most difficult topics with incredible ease, and his presentation came to be accepted as final. According to an estimate, during his lifetime he published 530 books and papers, and at his death, he left many manuscripts, which were published by the academy, during the next 47 years. The best known of his works, the *Introductio* was published in 1748, and caused a revolution in analytical mathematics.

CHAPTER 8

Exponential and Logarithmic Series

8·1. INTRODUCTION

In earlier chapters we have studied two infinite series, namely, the geometric series and the binomial series. In this chapter we shall study two more infinite series, the exponential and logarithmic series. Both these series are of central importance in mathematics.

8·2. THE NUMBER e

Consider the infinite series

$$1+\frac{1}{1\,!}+\frac{1}{2\,!}+\frac{1}{3\,!}+\ldots\ldots+\frac{1}{(n-1)\,!}+\ldots \qquad \ldots(i)$$

Let us denote the sum to n terms of the series (i) by S_n. Then

$$S_n=1+\frac{1}{1\,!}+\frac{1}{2\,!}+\ldots\ldots+\frac{1}{(n-1)\,!}. \qquad \ldots(ii)$$

We shall show that for all values of n, S_n lies between 2 and 3.

Since every term of S_n is positive and since the sum of the first two terms is 2, therefore, it follows that

$$S_n > 2, \quad \text{for all } n > 2. \qquad \ldots(iii)$$

To show that $S_n < 3$ for all n, observe that

$$\frac{1}{3\,!}=\frac{1}{1\,.\,2\,.\,3}<\frac{1}{2^2},$$

$$\frac{1}{4\,!}=\frac{1}{1\,.\,2\,.\,3\,.\,4}<\frac{1}{2^3},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{1}{(n-1)\,!}=\frac{1}{1\,.\,2\ldots(n-1)}<\frac{1}{2^{n-2}},$$

so that

$$\frac{1}{3\,!}+\frac{1}{4\,!}+\ldots\ldots+\frac{1}{(n-1)\,!}<\frac{1}{2^2}+\frac{1}{2^3}+\ldots\ldots\frac{1}{2^{n-2}} \qquad \ldots(iv)$$

The series $\frac{1}{2^2}+\frac{1}{2^3}+\ldots\ldots+\frac{1}{2^{n-2}}$ on the right-hand side of

(*iv*) is a geometric series whose first term is $\frac{1}{2^2}$, common ratio is $\frac{1}{2}$, and number of terms is $(n-3)$, and therefore, the sum is

$$\frac{\frac{1}{2^2}\left(1-\frac{1}{2^{n-3}}\right)}{1-\frac{1}{2}}, \text{ i.e., } \frac{1}{2}-\frac{1}{2^{n-2}}$$

Also the series on the left-hand side of (*iv*) equals

$S_n-1-\frac{1}{1!}-\frac{1}{2!}$.

Therefore, we can re-write (*iv*) as

$$S_n-1-\frac{1}{1!}-\frac{1}{2!}<\frac{1}{2}-\frac{1}{2^{n-2}}<\frac{1}{2},$$

i.e., $$S_n<1+1+\frac{1}{2}+\frac{1}{2},$$

i.e., $$S_n<3, \text{ for all } n, \qquad \ldots(v)$$

From (*iii*) and (*v*) we find that

$$2<S_n<3, \text{ for all } n>2. \qquad \ldots(vi)$$

It follows from (*vi*) that it is meaningful to talk of the sum of the infinite series (*i*), and that the sum lies between 2 and 3. The sum of the infinite series (*i*) is denoted by *e*, *i.e.*,

$$\boxed{e=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots}$$

In view of the above discussion we find that

$$\boxed{2<e<3.}$$

The number *e* is an irrational number. Its value is 2·718218... It is one of the two most important irrational numbers that occur in a variety of situations in mathematics (the other being π).

Example 1. (*a*) *Show that if we take only the first seven terms* $1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5}+\frac{1}{6!}$ *in the series expression for for e, then the error is less than ·00025.*

(*b*) *Show that e lies between 2·71803 and 2·71832.*

Solution. (*a*) The error E (say) in taking only the first seven terms of the series

$$1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\dots$$

is given by

$$E=\frac{1}{7!}+\frac{1}{8!}+\frac{1}{9!}+\dots,$$

$$=\frac{1}{7!}\left[1+\frac{1}{8}+\frac{1}{9.8}+\frac{1}{10.9.8}+\dots\right]$$

$$<\frac{1}{7!}\left[1+\frac{1}{8}+\frac{1}{8.8}+\frac{1}{8.8.8}+\dots\right]$$

$$=\frac{1}{7!}\cdot\frac{1}{1-\frac{1}{8}}$$

$$=\frac{1}{7!}\cdot\frac{8}{7},$$

$$=\frac{1}{4410}$$

$$<\frac{1}{4000}=\cdot 00025 \qquad \dots(i)$$

(b)

$1 = 1\cdot 00000$	...	$= 1\cdot 00000$
$\frac{1}{1!} = 1\cdot 00000$	...	$= 1\cdot 00000$
$\frac{1}{2!} = \cdot 50000$	...	$= \cdot 50000$
$\frac{1}{3!} = \cdot 16666$	...	$< \cdot 16667$
$\frac{1}{4!} = \cdot 04166$	...	$< \cdot 04167$
$\frac{1}{5!} = \cdot 00833$	...	$< \cdot 00834$
$\frac{1}{6!} = \cdot 00138$	...	$< \cdot 00139$
....		....
$\therefore\ 1+1+\frac{1}{2!}+\dots+\frac{1}{6!} = 2\cdot 71803\dots$		$< 2\cdot 71807$

$$\dots(ii)$$

From (*ii*) we find that

$$2\cdot 71803 < 1+\frac{1}{1!}+\frac{1}{2!}+\dots\dots+\frac{1}{6!} < 2\cdot 71807 \qquad \dots(iii)$$

Also,

$$1+\frac{1}{1!}+\frac{1}{2!}+\ldots\ldots+\frac{1}{6!}=e<1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{6!}+\mathrm{E}. \quad \ldots(iv)$$

From (i), (ii), (iii) and (iv) we find that

$$2{\cdot}71803<1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{6!}<e$$

$$=1+\frac{1}{1!}+\frac{1}{2!}+\ldots\ldots+\frac{1}{6!}+\mathrm{E},$$

$$<2{\cdot}71807+{\cdot}00025,$$

$$=2{\cdot}71832.$$

Hence e lies between 2·71803 and 2·71832.

Remark. From the above example we find that the value of e rounded off to three places of decimal is 2·718.

Example 2. *Sum the series* :

$$1+\frac{1+2}{2!}+\frac{1+2+3}{3!}+\frac{1+2+3+4}{4!}+\ldots$$

Solution. Let us denote the nth term of the given series by t_n and its sum by S.

$$t_n=\frac{1+2+3+\cdots+n}{n!},$$

$$=\frac{1}{2}\left[\frac{n(n-1)+2n}{n!}\right],$$

$$=\frac{1}{2}\cdot\frac{n(n-1)}{n!}+\frac{n}{n!},$$

$$=\frac{1}{2}\cdot\frac{1}{(n-2)!}+\frac{1}{(n-1)!},\ \text{for all } n\geqslant 2. \quad \ldots(i)$$

From (i), we find that

$$t_1=\quad 1,$$

$$t_2=\frac{1}{2}\cdot\frac{1}{0!}+\frac{1}{1!},$$

$$t_3=\frac{1}{2}\cdot\frac{1}{1!}+\frac{1}{2!},$$

$$t_4=\frac{1}{2}\cdot\frac{1}{2!}+\frac{1}{3!},$$

....

....

Add

$$S=\frac{1}{2}\left(\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\ldots\right)+\left(1+\frac{1}{1!}+\frac{1}{2!}+\ldots\right),$$
$$=\frac{1}{2}e+e,$$
$$=\frac{3}{2}e.$$

Example 3. *Sum the series*

$$\frac{1^2 \cdot 2}{1!}+\frac{2^2 \cdot 3}{2!}+\frac{3^2 \cdot 4}{3!}+\ldots$$

Solution. We shall split up the given series as a sum of several series each of which can be summed up as an exponential series. For this purpose we shall write the nth term say T_n, of the given series.

$$T_n=\frac{n^2(n+1)}{n!},$$
$$=\frac{n(n+1)}{(n-1)!},$$
$$=\frac{(n-1)(n-2)+4(n-1)+2}{(n-1)!},$$
$$=\frac{(n-1)(n-2)}{(n-1)!}+\frac{4(n-1)}{(n-1)!}+\frac{2}{(n-1)!},$$
$$=\frac{1}{(n-3)!}+\frac{4}{(n-2)!}+\frac{2}{(n-1)!}, \text{ provided } n\geqslant 3. \qquad \ldots(i)$$

Observe that the condition $n \geqslant 3$ is necessary because $(n-3)!$ has no meaning for $n=1, 2$.

Putting $n=3, 4, \ldots\ldots$ in (i), we have

$$T_3=\frac{1}{0!}+\frac{4}{1!}+\frac{2}{2!},$$
$$T_4=\frac{1}{1!}+\frac{4}{2!}+\frac{2}{3!},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

By adding columnwise, and using the series expansion for e,

$$\sum_{n=3}^{\infty} T_n=\left(1+\frac{1}{1!}+\frac{1}{2!}+\ldots\right)+4\left(\frac{1}{1!}+\frac{1}{2!}+\ldots\right)$$
$$+2\left(\frac{1}{2!}+\frac{1}{3!}+\ldots\right),$$
$$=e+4(e-1)+2(e-2),$$
$$=7e-8. \qquad \ldots(ii)$$

Also,

$$T_1+T_2=\frac{1^2 \cdot 2}{1\,!}+\frac{2^2 \cdot 3}{2\,!}=2+6=8. \qquad ...(iii)$$

From (*ii*) and (*iii*) we have

$$\sum_{n=1}^{\infty} T_n=(7e-8)+8=7e.$$

EXERCISE 8 (*a*)

1. Find the value of e rounded off to one decimal place.
2. Show that the error in taking only the first six terms $1+\frac{1}{1\,!}+\frac{1}{2\,!}+\frac{1}{3\,!}+\frac{1}{4\,!}+\frac{1}{5\,!}$ in the series expression for e is less than $\frac{1}{600}$.
3. Show that
$$1+\frac{3}{1\,!}+\frac{5}{2\,!}+\frac{7}{3\,!}+\frac{9}{4\,!}+......=3e.$$
4. Sum the series
$$\frac{1^2}{3\,!}+\frac{2^2}{4\,!}+\frac{3^2}{5\,!}+......$$
5. Show that
$$1\,.\,3+\frac{2\,.\,4}{1\,.\,2}+\frac{3\,.\,5}{1\,.\,2\,.\,3}+\frac{4\,.\,6}{1\,.\,2\,.\,3\,.\,4}+......=4e.$$
6. Sum the series
$$1^3+\frac{2^3}{2\,!}+\frac{3^3}{3\,!}+\frac{4^3}{4\,!}+...$$
7. Sum the series
$$1\,.\,4+\frac{2\,.\,5}{1\,!}+\frac{3\,.\,6}{2\,!}+\frac{4\,.\,7}{3\,!}+\frac{5\,.\,8}{4\,!}+......$$
8. Sum the series $\displaystyle\sum_{n=2}^{\infty}\frac{{}^nC_2}{(n+1)\,!}$.
9. Show that
$$1+\frac{1^2+2^2}{2\,!}+\frac{1^2+2^2+3^2}{3\,!}+......=\frac{17}{6}e.$$

8.3. THE EXPONENTIAL SERIES

If x be any real number (whether rational or irrational), then

$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots} \qquad \dots(A)$$

The series on the right-hand side of the above relation is called **the exponential series.** A word of caution is necessary here. You know the meaning of e^x when x is rational, but you have not yet studied irrational exponents. Therefore, with your present knowledge, statement (A) is meaningful only for rational values of x. It is a true statement but the proof is beyond the scope of the present book. In higher classes you will learn as to what e^x means when x is an irrational number. You will then also learn that (A) is a true statement for all real values of x. For the present we shall be content with applications of (A) for rational values of x. The following simple consequences of (A) are worth noting :

1. For $x=1$, we get

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots\dots \qquad \dots(i)$$

which is simply the series for e studied earlier in this chapter.

2. For $x=-1$, we get

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots\dots \qquad \dots(ii)$$

3. From (i) and (ii), we get

$$\frac{1}{2}(e - e^{-1}) = 1 + \frac{1}{3!} + \frac{1}{5!} + \dots\dots \qquad \dots(iii)$$

$$\frac{1}{2}(e + e^{-1}) = 1 + \frac{1}{2!} + \frac{1}{4!} + \dots\dots \qquad \dots(iv)$$

4. Replacing x throughout by $-x$ in (A), we have

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots\dots \qquad \dots(v)$$

5. From (A) and (v), we immediately have,

$$\frac{1}{2}(e^x + e^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\dots \qquad \dots(vi)$$

$$\frac{1}{2}(e^x - e^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\dots \qquad \dots(vii)$$

Remark :

The graph of the function $y=e^x$ is as shown in Fig. 8·1.

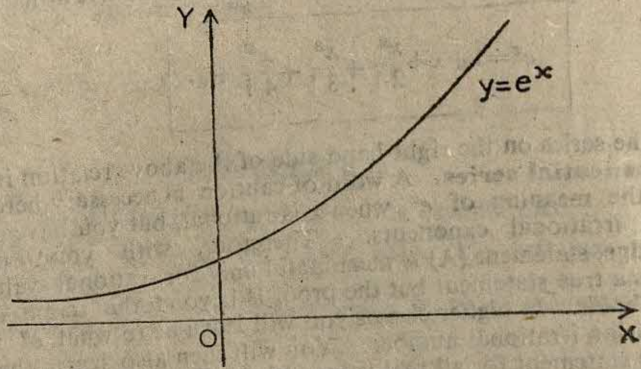


Fig. 8·1. Graph of $y=e^x$.

Example 4. *Find the coefficient of* x^n *in*

$$1+\frac{a+bx}{1!}+\frac{(a+bx)^2}{2!}+\ldots\ldots+\frac{(a+bx)^n}{n!}+\ldots\ldots$$

(*Roorkee Entrance 1985*)

Solution.

$$1+\frac{a+bx}{1!}+\frac{(a+bx)^2}{2!}+\ldots\ldots+\frac{(a+bx)^n}{n!}+\ldots\ldots$$

$$=e^{a+bx},$$

$$=e^a \,.\, e^{bx},$$

$$=e^a\left[1+bx+\frac{b^2x^2}{2!}+\frac{b^3x^3}{3!}+\ldots\ldots+\frac{b^nx^n}{n!}+\ldots\ldots\ldots\right].$$

$\therefore$ The required coefficient of $x^n=\dfrac{e^ab^n}{n!}$.

Example 5. *Find the coefficient of* x^n *in the expansion of*

$$\frac{(1+x+x^2)}{e^x}$$

Solution.

$$e^{-x}=1-x+\frac{x^2}{2!}+\ldots\ldots\ldots(-1)^n\frac{x^n}{n!}+\ldots\ldots \quad \ldots(i)$$

$$xe^{-x}=x-x^2+\ldots\ldots+(-1)^{n-1}\frac{x^n}{(n-1)!}+\ldots\ldots \quad \ldots(ii)$$

$$x^2e^{-x}=x^2+\ldots\ldots+(-1)^{n-2}\frac{x^n}{(n-2)!}+\ldots\ldots \quad \ldots(iii)$$

Adding (*i*), (*ii*) and (*iii*), we have

$$(1+x+x^2)e^{-x}=1+\frac{x^2}{2\,!}+\dots\dots$$
$$+\left[\frac{(-1)^n}{n\,!}+\frac{(-1)^{n-1}}{(n-1)\,!}+\frac{(-1)^{n-2}}{(n-2)\,!}\right].\,x^n+\dots\dots$$

The required coefficient

$$=\frac{(-1)^n}{n\,!}+\frac{(-1)^{n-1}}{(n-1)\,!}+\frac{(-1)^{n-2}}{(n-2)\,!},$$
$$=\frac{(-1)^n}{n\,!}\,[1-n+n(n-1)],$$
$$=\frac{(-1)^n(n-1)^2}{n\,!}.$$

EXERCISE 8 (*b*)

1. Write down the first five terms in the expansion of e^{2x^2}.
2. Find the coefficient of x^5 in the expansion of e^{4x}.
3. Expand $\dfrac{e^{5x}+e^{2x}}{e^{3x}}$ in ascending powers of x.
4. Show that
$$\frac{2}{3\,!}+\frac{4}{5\,!}+\frac{6}{7\,!}+\frac{8}{9\,!}+\dots\dots=\frac{1}{e}.$$
5. Show that
$$\frac{\dfrac{1}{2\,!}+\dfrac{1}{4\,!}+\dfrac{1}{6\,!}+\dots\dots}{\dfrac{1}{1\,!}+\dfrac{1}{3\,!}+\dfrac{1}{5\,!}+\dots\dots}=\frac{e-1}{e+1}.$$
(*Roorkee Entrance, 1981*)
6. Find the value of
$$x^2-y^2+\frac{1}{2\,!}(x^4-y^4)+\frac{1}{3\,!}(x^6-y^6)+\dots\dots$$
7. Show that
$$e^2-e=1+\frac{1+2}{2\,!}+\frac{1+2+2^2}{3\,!}+\frac{1+2+2^2+2^3}{4\,!}+\dots\dots$$
8. Show that
$$\frac{1+\dfrac{2^2}{2\,!}+\dfrac{2^4}{3\,!}+\dfrac{2^6}{4\,!}+\dots\dots}{1+\dfrac{1}{2\,!}+\dfrac{2}{3\,!}+\dfrac{2^2}{4\,!}+\dots\dots}=e^2-1.$$

9. Show that

$$\left(\frac{2}{1!}+\frac{4}{3!}+\frac{6}{5!}+\frac{8}{7!}+\dots\dots\right)\left(\frac{2}{3!}+\frac{4}{5!}+\frac{6}{7!}+\dots\right)=1$$

(*Roorkee Entrance, 1979*)

10. Find the value of e^2 rounded off to one decimal place.

11. Find the coefficient of x^n in the expansion of $(a-ax-x^2)/e^x$.

8·4. THE LOGARITHMIC SERIES

Recall that if a be any number greater than unity, then a number y is said to be the logarithm of a number x to the base a if $a^y=x$. In symbols,

$$\log_a x=y \Leftrightarrow a^y=x.$$

The numbers 10 and e are commonly used as bases for logarithms. The logarithm of a number to the base 10 is called its **common logarithm**. Logarithms to the base 10 are used for computational purposes. Logarithm tables give logarithms of numbers to base 10.

The logarithm of a number to the base e is called its **natural logarithm**. Natural logarithms occur very frequently (and in a natural manner !) in higher mathematics, as also in applications of mathematics to other sciences. In the remainder of this chapter we shall be mostly dealing with natural logarithms. We begin by obtaining an expression for $\log_e (1+x)$ as an infinite series.

Theorem 8·1. $\log_e (1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\dots\dots,$

provided $|x| < 1$.

Proof. We shall prove the result by expanding $(1+x)^y$ in two different ways, and comparing the coefficients of y in the two expansions.

(*i*) Let $|x| < 1$. By the binomial theorem, we have

$$(1+x)^y=1+yx+\frac{y(y-1)}{1.2}x^2+\frac{y(y-1)(y-2)}{1.2.3}x^3+\dots\dots \quad \dots(1)$$

The coefficient of y on the right-hand side of (1)

$$=x-\frac{x^2}{2}+\frac{x^3}{3}-\dots\dots \quad \dots(2)$$

(*ii*) Since $|x| < 1$, therefore, $(1+x)^y > 0$.

Since every positive real number m can be written as $e^{\log_e m}$, therefore, we can write $(1+x)^y=e^{\log_e (1+x)^y}=e^{y\log_e(1+x)}$. Expanding $e^{y\log_e(1+x)}$ by the exponential series, we have

$$(1+x)^y = e^{y \log_e (1+x)}$$
$$= 1 + y \log_e (1+x)$$
$$+ \frac{y^2}{2!} [\log_e (1+x)]^2 + \ldots\ldots \qquad \ldots(3)$$

The coefficient of y on the right-hand side of (2) is

$$\log_e (1+x) \qquad \ldots(4)$$

Equating the coefficients of y (as given in (2) and (4) in the two different expansions of $(1+x)^y$ as obtained in (*i*) and (*ii*), we have

$$\boxed{\log_e (1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\ldots} \qquad \ldots(A)$$

The expansion (A) is valid for $|x| < 1$. The series on the right-hand side of (A) is called the **logarithmic series.**

Remarks. 1. It can be shown that (A) is valid even when $x=1$ (but not for $x=-1$).

2. $\log_e x$ is also written as $\ln x$.

3. The graph of the function $y = \log_e (1+x)$ is as shown in Fig. 8·2.

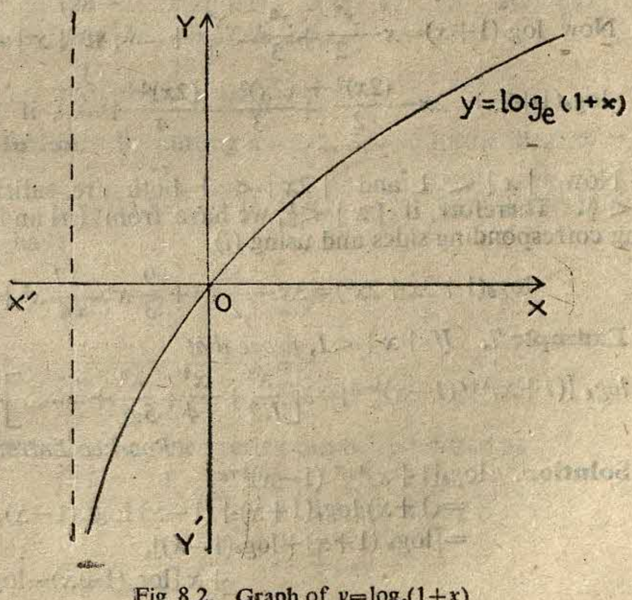


Fig. 8.2. Graph of $y = \log_e(1+x)$

Corollaries. 1. Replacing x by $-x$ throughout in (A), we have

$$\log_e(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\ldots\ldots \qquad \ldots(B)$$

2. From (A) and (B), we have

$$\log_e(1+x)-\log_e(1-x)=\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots\ldots\right)$$
$$-\left(-x-\frac{x^2}{2}-\frac{x^3}{3}-\ldots\ldots\right),$$

or $$\log_e\frac{1+x}{1-x}=2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots\ldots\right).$$

3. By putting $x=1$ throughout in (A), we have

$$\log_e 2=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\ldots$$

The above series is due to the German mathematician Leibniz and is called *Leibniz series* for $\log_e 2$.

Example 6. *If* $|x|<\frac{1}{2}$, *prove that*

$$\log_e(1+3x+2x^2)=3x-\frac{5}{2}x^2+\frac{9}{3}x^3-\frac{17}{4}x^4+\ldots\ldots$$

Solution. $\log_e(1+3x+2x^2)=\log_e[(1+x)(1+2x)]$,

$=\log_e(1+x)+\log_e(1+2x)$. $\qquad \ldots(i)$

Now $\log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\ldots$, if $|x|<1$. $\quad \ldots(ii)$

$\log_e(1+2x)=2x-\frac{(2x)^2}{2}+\frac{(2x)^3}{3}-\frac{(2x)^4}{4}+\ldots\ldots$, if $|2x|<1$. $\quad \ldots(iii)$

Now $|x|<1$ and $|2x|<1$ both are satisfied when $|x|<\frac{1}{2}$. Therefore, if $|x|<\frac{1}{2}$, we have from (*ii*) and (*iii*), by adding corresponding sides and using (*i*),

$$\log_e(1+3x+2x^2)=3x-\frac{5}{2}x^2+\frac{9}{3}x^3-\frac{17}{4}x^4+\ldots\ldots$$

Example 7. *If* $|x|<1$, *prove that*

$$\log_e[(1+x)^{1+x}(1-x)^{1-x}]=2\left[\frac{x^2}{1.2}+\frac{x^4}{3.4}+\frac{x^6}{5.6}+\ldots\ldots\right].$$

(*Roorkee Entrance, 1983*)

Solution. $\log_e[1+x)^{1+x}(1-x)^{1-x}]$

$=(1+x)\log_e(1+x)+(1-x)\log_e(1-x)$,

$=[\log_e(1+x)+\log_e(1-x)]$,

$+x[\log_e(1-x)-\log_e(1-x)]$.

Now $\log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\frac{x^6}{6}+\ldots\ldots,$ ...(A)

$$\log_e(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\frac{x^5}{5}-\frac{x^6}{6}\ldots\ldots,$$

whenever $|x|<1$.

Therefore,

$$\log_e(1+x)+\log_e(1-x)=2\left(-\frac{x^2}{2}-\frac{x^4}{4}-\frac{x^6}{6}\right)-\ldots\ldots,$$

$$\log_e(1+x)-\log_e(1-x)=2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots\ldots\right).$$

Multiplying the second of the above relations throughout by x and adding each side to the corresponding side of the first relation, we have on using (A),

$$\log_e[(1+x)^{1+x}(1-x)^{1-x}]$$
$$=2\left[x^2\left(1-\frac{1}{2}\right)+x^4\left(\frac{1}{3}-\frac{1}{4}\right)+x^6\left(\frac{1}{5}-\frac{1}{6}\right)+\ldots\ldots\right]$$
$$=2\left[\frac{x^2}{1.2}+\frac{x^4}{3.4}+\frac{x^6}{5.6}+\ldots\ldots\right],$$

if $|x|<1$.

Example 8. *By using the identity*

$$\frac{1}{(2n-1)\,2n(2n+1)}=\frac{1}{2(2n-1)}-\frac{1}{2n}+\frac{1}{2(2n+1)},$$

or otherwise, find the sum of the series

$$\frac{1}{1.2.3}+\frac{1}{3.4.5}+\frac{1}{5.6.7}+\ldots\ldots$$

Solution. By putting $n=1, 2, 3, \ldots\ldots$ in the identity

$$\frac{1}{(2n-1)\,2n(2n+1)}=\frac{1}{2(2n-1)}-\frac{1}{2n}+\frac{1}{2(2n+1)},$$

we have

$$\frac{1}{1.2.3}=\frac{1}{2.1}-\frac{1}{2}+\frac{1}{2.3},$$

$$\frac{1}{3.4.5}=\frac{1}{2.3}-\frac{1}{4}+\frac{1}{2.5}.$$

....

Therefore, the given series can be re-written as

$$\left(\frac{1}{2.1}-\frac{1}{2}+\frac{1}{2.3}\right)+\left(\frac{1}{2.3}-\frac{1}{4}+\frac{1}{2.5}\right)+\ldots\ldots$$
$$=\frac{1}{2.1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\ldots$$

$$=-\frac{1}{2}+\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\ldots\right),$$

$$=-\frac{1}{2}+\log_e 2.$$

Hence the desired sum $=\log_e 2-\frac{1}{2}$.

8·5. CALCULATION OF THE LOGARITHM OF A NUMBER USING SUITABLE LOGARITHMIC SERIES

Consider the series for $\log_e(1+x)$,

$$\log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots \qquad \ldots(A)$$

As we have already pointed out, for $x=1$, (A) yields the Leibniz series

$$\log_e 2=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}-\ldots \qquad \ldots(B)$$

By taking a sufficiently large number of terms of the series on the right-hand side of (B), we can calculate the value of $\log_e 2$ to any number of decimal places.

By putting $x=\frac{1}{n}$, where n is any natural number, (A) yields

$$\log_e\left(1+\frac{1}{n}\right)=\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\frac{1}{4n^4}+\ldots\ldots,$$

or

$$\log_e(n+1)-\log_e n=\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^4}-\frac{1}{4n^3}+\ldots\ldots \qquad \ldots(C)$$

where n is any natural number.

Putting $n=2, 3, 4\ldots\ldots$ in (C), we have

$$\log_e 3-\log_e 2=\frac{1}{2}-\frac{1}{2.2^2}+\frac{1}{3.2^3}\ldots\ldots,$$

$$\log_e 4-\log_e 3=\frac{1}{3}-\frac{1}{2.3^3}+\frac{1}{3.3^3}-\ldots$$

Since $\log_e 2$ can be found from (B), therefore, the above relations can be used to calculate $\log_e 3$, $\log_e 4$, $\log_e 5$ etc.

However, since the terms of the above series, specially that of (B) decrease very slowly, the series is not as useful as some other series which we shall describe below.

Let us consider the series :

$$\log_e\frac{1+x}{1-x}=2\left[x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots\ldots\right]. \qquad \ldots(D)$$

Statement (D) is valid for $|x|<1$. By substituting

$$x=\frac{1}{2n+1}$$

(where n is any natural number) in both sides of (D), we have

$$\log_e(n+1)-\log_e n=2\left[\frac{1}{2n+1}+\frac{1}{3(2n+1)^3}+\frac{1}{5(2n+1)^5}+\cdots\cdots\right]. \quad \text{...(E)}$$

Putting $n=1$ in (E), we have

$$\log_e 2=2\left(\frac{1}{3}+\frac{1}{3.3^3}+\frac{1}{5.3^5}+\cdots\cdots\right) \quad \text{...(F)}$$

The fourth term on the right-hand side of (F) is

$$\frac{2}{7.3^7}=\frac{2}{15309}<{\cdot}0002,$$

and the subsequent terms are much smaller. Therefore, by taking the first three terms on the right-hand side of (F), we can calculate the value of $\log_e 2$ correct to two places of decimal. Formula (F) gives better approximations than (B) because the terms in (F) decrease much faster than those in (B).

The series in (E) can be used to calculate $\log_e 2$, $\log_e 3$, $\log_e 5$, $\log_e 7$, $\log_e 10$ etc., in the same way (but more efficiently) as the series in (C).

By using the relation

$$\log_{10} n=(\log_e n)\,.\,(\log_{10} e),$$
$$=\frac{(\log_e n)}{(\log_e 10)},$$

it is possible to calculate the common logarithm of a number n provided its natural logarithm is given, and $\log_e 10$ is known. The number $(\log_e 10)^{-1}=\log_{10} e={\cdot}43429\cdots$

Example 9. *Show that*

$$\log_e(n+1)-\log_e(n-1)=2\left[\frac{1}{n}+\frac{1}{3n^3}+\frac{1}{5n^5}+\cdots\right],$$

and use it to calculate the value of $\log_e 3$ correct to three places of decimals. Given that $\log_{10} e={\cdot}43429$, also find $\log_{10} 3$.

Solution.

$$\log\,(n+1)-\log_e(n-1)=\log_e\frac{n+1}{n-1},$$

$$=\log_e\frac{1+\frac{1}{n}}{1-\frac{1}{n}},$$

$$=\log_e\left(1+\frac{1}{n}\right)-\log_e\left(1-\frac{1}{n}\right). \quad \text{...(A)}$$

By substituting $x=\frac{1}{n}$, $-\frac{1}{n}$ successively in the series expression for $\log_e(1+x)$, we have

$$\log_e\left(1+\frac{1}{n}\right)=\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\ldots,$$

$$\log_e\left(1-\frac{1}{n}\right)=-\frac{1}{n}-\frac{1}{2n^2}-\frac{1}{3n^3}+\ldots,$$

so that

$$\log_e\left(1+\frac{1}{n}\right)-\log_e\left(1-\frac{1}{n}\right)=2\left[\frac{1}{n}+\frac{1}{3n^3}+\ldots\right],$$

or $$\log_e(n+1)-\log_e(n-1)=2\left[\frac{1}{n}+\frac{1}{3n^3}+\frac{1}{5n^5}+\ldots\right]. \quad \text{...(B)}$$

Substituting $n=2$ in (B), and observing that $\log_e 1=0$,

$$\log_e 3=2\left[\frac{1}{2}+\frac{1}{3.2^3}+\frac{1}{5.2^5}+\ldots\right],$$

$$=1+\frac{1}{3.2^2}+\frac{1}{5.2^4}+\frac{1}{7\cdot2^6}+\ldots \quad \text{...(C)}$$

Now $$1=1\cdot0000$$

$$\frac{1}{3.2^2}=\cdot0833\ldots$$

$$\frac{1}{5.2^4}=\cdot0125\ldots\ldots$$

$$\frac{1}{7.2^6}=\cdot0022\ldots\ldots$$

$$\frac{1}{9.2^8}=\cdot0004\ldots\ldots$$

Add $$\overline{1\cdot0984\ldots\ldots},$$

so that $\log_e 3=1\cdot0984\ldots\ldots$

Since the last term considered above does not contribute anything to the third place of decimal, and the subsequent terms are going to be much smaller, therefore, we find that the value of $\log_e 3$ is $1\cdot098$ correct to three places of decimal.

$$\log_{10}3=\log_e 3\times\log_{10}e=1\cdot098\times\cdot43429=\cdot477,$$

correct to three places of decimal.

EXERCISE 8 (*c*)

1. Write the first four terms in the expansion of $\log_e(1+3x)$. For what values of x is the expansion valid?

2. Write the expansion of $\log_e(1-4x^2)$ as far as the term involving x^{10}. For what values of x is the expansion valid ?

3. Find the coefficient of x^{21} in the expansion of $\log_e(1+2x^3)$.

4. Prove that

$$\frac{x-1}{x+1}+\frac{1}{2}\cdot\frac{x^2-1}{(x+1)^2}+\frac{1}{3}\cdot\frac{x^3-1}{(x+1)^3}+\dots=\log_e x$$

5. Show that

$$\log_e\frac{1+3x}{1-2x}=5x-\frac{5}{2}x^2+\frac{35}{3}x^3-\frac{65}{4}x^4+\dots\dots$$

6. If α and β are the roots of $x^2-px+q=0$, show that

$$\log_e(1+px+qx^2)=(\alpha+\beta)x-\frac{\alpha^2+\beta^2}{2}x^2+\frac{\alpha^3+\beta^3}{3}x^3+\dots$$

7. Prove that the coefficient of x^n in the expansion of $\log_e(1+x+x^2)$ is $-\left(\frac{2}{n}\right)$ or $\frac{1}{n}$ according as n is, or is not a multiple of 3.

8. Show that

$$\frac{5}{1.2.3}+\frac{7}{3.4.5}+\frac{9}{5.6.7}+\dots\dots=3\log_e 2-1.$$

9. Sum the series

$$\frac{1}{2}-\frac{1}{2.2^2}+\frac{1}{3.2^3}-\frac{1}{4.2^4}+\dots\dots$$
$$+2\left(\frac{1}{5}+\frac{1}{3.5^3}+\frac{1}{5.5^5}+\dots\dots\right)$$

10. Show that

$$\log_e\sqrt{2}=\frac{1}{2}\left(\frac{1}{2}+\frac{1}{3}\right)-\frac{1}{4}\left(\frac{1}{2^2}+\frac{1}{3^2}\right)+\frac{1}{6}\left(\frac{1}{2^3}+\frac{1}{3^3}\right)+\dots\dots$$

TEST YOUR UNDERSTANDING VIII

In each of the following problems four alternatives are given out of which only one is correct. Put a tick mark (✓) against the correct alternative :

1. The sum of the series $1+\frac{1}{2\,!}+\frac{1}{4\,!}+\frac{1}{6\,!}+\dots\dots$ is

(a) e^2 (b) $e+e^{-1}$

(c) $\frac{1}{2}(e+e^{-1})$ (d) $\frac{1}{2}(e-e^{-1})$.

2. The sum of the series $1+\frac{1}{3\,!}+\frac{1}{5\,!}+\frac{1}{7\,!}+\ldots\ldots$ is

(a) e^{-1} (b) $\frac{1}{2}(e+e^{-1})$

(c) $\frac{1}{2}(e-e^{-1})$ (d) e^3.

3. The sum of the series $\frac{2}{3\,!}+\frac{4}{5\,!}+\frac{6}{7\,!}+\ldots\ldots$ is

(a) e (b) e^{-1}

(c) $2e$ (d) $2e^{-1}$.

4. The product $\left(1+\frac{1}{1\,!}+\frac{1}{2\,!}+\ldots\right)\left(1-\frac{1}{1\,!}+\frac{1}{2\,!}-\ldots\right)$ equals

(a) 1 (b) e^{-2}

(c) $-e^2$ (d) -1.

5. The coefficients of x^5 in the expansion of $e^{3x}+e^{-x}$ is

(a) $\frac{121}{60}$ (b) $\frac{61}{30}$

(c) $\frac{3}{2}$ (d) $\frac{2}{3}$.

6. The coefficient of x^6 in the expansion of e^{2x+3} is

(a) $\frac{32}{3}$ (b) $\frac{4}{45}e^3$

(c) $\frac{2^6\,.\,3^6}{6\,!}$ (d) $\frac{2^6+3^6}{6\,!}$.

7. The coefficient of x^5 in the expansion of $\log_e\frac{1-x}{1+x}$ is

(a) $\frac{2}{5\,!}$ (b) $\frac{2}{5}$

(c) $-\frac{2}{5\,!}$ (d) $-\frac{2}{5}$.

8. The coefficient of x^5 in the expansion of $\log_e(1-x^2)$ is

(a) $\frac{1}{5}$ (b) $-\frac{1}{5}$

(c) 0 (d) $-\frac{1}{5\,!}$.

9. The sum of the series $3x+\frac{3^2x^2}{2}+\frac{3^3x^3}{3}+\ldots\ldots$ is

(*a*) $\log_e(1-3x)$ (*b*) $\log_e\frac{1}{1-3x}$

(*c*) e^{3x} (*d*) $e^{3x}-1$.

10. The sum of the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\dots\dots$ is

(*a*) $\log_e\frac{1}{2}$ (*b*) e^{-1}

(*c*) 0 (*d*) $\log_e 2$.

REVIEW EXERCISE VIII

1. Show that
$$\frac{1^2}{2!}+\frac{1^2+2^2}{3!}+\frac{1^2+2^2+3^2}{4!}+\dots\dots=\frac{5}{6}e.$$

2. Find the value of $e^{-1/5}$ correct to four places of decimal.

3. Sum the series $\sum_{n=2}^{\infty} {}^nC_2\,\frac{4^{n-2}}{n!}$.

4. If $y=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots\dots$, and $|x|<1$, show that
$$x=y+\frac{y^2}{2!}+\frac{y^3}{3!}+\dots\dots$$

5. By using the series expansion for $\log_e 2$ or otherwise, show that $\cdot 6<\log_e 2<\cdot 8$

6. Show that the series $\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\dots\dots$ has the same sum as the series $\frac{1}{n+1}+\frac{1}{2(n+1)^2}+\frac{1}{3(n+1)^3}+\dots\dots$

7. Expanding $(e^x-1)^n$ in two different ways, show that
$$n^n-n(n-1)^n+\frac{n(n-1)}{2!}(n-2)^n+\dots=n!$$
(*Roorkee Entrance, 1986*)

8. Show that $\log_e\frac{1001}{999}=\cdot 002$, correct to three decimal places.

9. Show that $\log_e(n+1)-\log_e n=2\left[\frac{1}{2n+1}+\frac{1}{3(2n+1)^3}+\frac{1}{5(2n+1)^5}+\dots\right]$.

10. Show that $2\log_e n-\log_e(n+1)-\log_e(n-1)$
$$=\frac{1}{n^2}+\frac{1}{2n^4}+\frac{1}{3n^6}+\dots\dots$$

SUMMARY

1. The number e stands for the sum of the series

$$1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots\ldots$$

2. The number e is irrational. Its value is approximately 2·718218.
3. If x be any real number, then

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$$

4. $\log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\ldots$, provided $|x|<1$.

5. $\log_{10} e=\cdot43429\ldots\ldots$

HISTORICAL NOTE

The letter e was first used by Leonhard Euler (1707-1783) around 1727-28. It appeared in print for the first time in Euler's *Mechanica* (1736). The notation became standard in course of time. Liouville (1809-1882) showed in 1844 that neither e nor e^2 could be the root of a quadratic equation with integral coefficients. Charles Hermite (1822-1901) showed in 1873 that e could not be the root of *any* polynomial equation with integral coefficients.

In the 1618 edition of Edward Wright's translation of Napier's famous work *Descriptio* there is printed an appendix, probably written by Oughtred in which appears a statement equivalent to saying that $\log_e 10=2\cdot302584$. This is an indication that e as a base for logarithms had been recognized by that time. The idea of expressing a logarithm by means of a series appears to have been originated with James Gregory (1638-1675) and to have been elaborated by Nicolaus Mercator (1620-1687) around 1667 who discovered a special case of the logarithmic series for $\log_e(1+x)$.

□□

PART II : CO-ORDINATE GEOMETRY

★ ★ ★ ★

The Hindu scholars were quite advanced in their knowledge of geometry. The Sulvasutras contain rules for construction of right-angled triangles. Aryabhata in his *Aryabhatiyam* states a rule which gives the modern approximation 3·1416 for π.

★ ★ ★ ★

Plato was the centre of mathematical activity of his times. The following motto was inscribed over the doors of his school :

Let no one ignorant of geometry enter here.

RENE DESCARTES (1569—1650)

Rene Descartes, a philosopher and mathematician of the highest order, was born on March 31, 1596 at La Haye near Tours in France. He belonged to a wealthy family and was educated first at the Jesuit College at La Fleche and later on at Poiter where he studied law. As the creator of analytic geometry he has a permanent place in the history of mathematics. He published *La Geometrie* in which he outlined the methods of analytic geometry as one of the three appendices to his treatise *Discourse de la methode*, a treatise on philosophy in which he thought of giving illustrations of his general method. His creation of analytic geometry changed the entire course of Mathematics. He died on February 11, 1650 at Sweden.

CHAPTER 9

Cartesian System of Rectangular Co-Ordinates

9·1. INTRODUCTION

In earlier classes you have studied geometry. We shall devote some more time to the study of geometry but our approach now will be different. The method applied until now in the study of geometry was the *synthetic* method. We shall now be using the *analytic method.* The analytic method depends upon the use of algebra. Just as it is possible to set up a one-to-one correspondence between the set of real numbers and the set of points on a line, similarly it is also possible to set up a one-to-one correspondence between the set of ordered pairs of real numbers and the set of points in a plane. This correspondence enables us to look upon a geometric figure or a region of the plane as a set of ordered pairs of real numbers. We can perform algebraic operations on this set and interpret our results geometrically. The two real numbers associated with a point are called its co-ordinates. Because of the use of co-ordinates, the subject matter that we are going to study in this chapter as also in the next three chapters is called *co-ordinate geometry.* Because of the analytic method used, it is sometimes also called *analytical geometry.* The idea of using co-ordinates is due to the French mathematician Rene Descartes (1596-1650) and therefore, the system of co-ordinates that we shall use is often called the *system of cartesian co-ordinates.*

9·2. CO-ORDINATES SYSTEM IN A PLANE

We can represent real numbers graphically by the points of a straight line. Let us take a point O on a straight line and mark the number 0 on it. Let us also choose a unit of length.

On one side of O we mark positive numbers and on the other side negative numbers in such a manner that the number corresponding to any point is equal in absolute value to the distance from O to the point. We thus obtain a scale on the line. When the line is parallel to the printed lines of this page, as in Fig. 9·1, we usually mark the positive numbers to the right of O and negative numbers to the left of O.

The point O thus divides the straight line into two parts The part on which positive numbers are represented is called the

Fig. 9·1.

positive part, and the part, on which negative numbers are represented is called the negative part.

The distance between any two points of the line is equal to the difference of the numbers represented by those points. Thus, if the points P, Q represent the numbers x_1, x_2 respectively,

$$PQ = x_2 - x_1.$$

9·2·1. Co-ordinate Axes

In the plane, take two perpendicular straight lines X'OX, Y'OY. Take O as the zero point of both the straight lines. It is usual to draw X'OX parallel to the printed lines of the page and Y'OY perpendicular to the same. That part of X'OX which lies to the right of O is usually taken as positive and that part which lies to the left of O is taken as negative. Similarly, that part of Y'OY which is above the line X'OX is taken as positive and that part which is below X'OX is taken as negative.

The point O is called the **origin,** X'OX is called the **axis of X** (or simply **X-axis**) and Y'OY is called the **axis of Y** (or simply **Y-axis**). The axes divide the plane into four parts called **quadrants.** These are numbered I, II, III, IV as shown in Fig. 9·2.

The position of any point P in the plane can be fixed with reference to the axes. Through any point P draw PM parallel to YOY' and PN parallel to XOX'. Let the number at M in the scale X'OX, be x and that at N in the scale Y'OY be y. These numbers are called the **cartesian co-ordinates** of P relative to the axes X'OX and Y'OY, and the axes X'OX, Y'OY are called the axes of Co-ordinates. x is called the **abscissa** of P and y is called the **ordinate** of P.

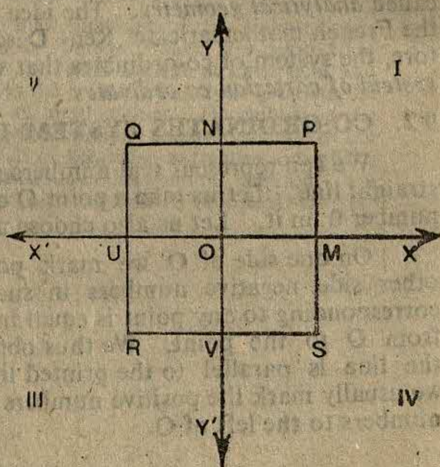


Fig. 9·2.

The abscissa x is equal in magnitude to the

distance NP (=OM) of P from the Y-axis ; it is positive for points on the right of the Y-axis and negative for points on the left of the Y-axis.

The ordinate y is equal in magnitude to the distance MP (=ON) of P from the X-axis ; it is positive for points above the X-axis and negative for points below the X-axis. For example, in Fig. 9·2, the point P has co-ordinates $x=3, y=4$, Q has co-ordinates $x=-3$, $y=4$, R has co-ordinates $x=-3$, $y=4$ and S has co-ordinates $x=3, y=-4$. Notice that the *co-ordinates of O are* $x=0$, $y=0$.

The point whose co-ordinates are x and y is represented by (x, y). To denote that a point P has co-ordinates x and y, we write $P(x, y)$. For example, (3, 4) is the point whose abscissa is 3 and ordinate is 4. Also, P(3, 4) indicates that the abscissa of P is 3 and that the ordinate of P is 4.

It is easy to see that :

$$\begin{aligned}
x\text{-axis} &= \{(x, y) : y=0\}.\\
y\text{-axis} &= \{(x, y) : x=0\}.\\
\text{Quadrant I} &= \{(x, y) : x > 0 \text{ and } y > 0\}.\\
\text{Quadrant II} &= \{(x, y) : x < 0 \text{ and } y > 0\}.\\
\text{Quadrant III} &= \{(x, y) : x < 0 \text{ and } y < 0\}.\\
\text{Quadrant IV} &= \{(x, y) : x > 0 \text{ and } y < 0\}.
\end{aligned}$$

9·2·2. Plotting of Points

The process of locating a point whose co-ordinates are given is called *plotting* of the point. Before learning how to plot a point, observe that if a point is in the first quadrant, then it is above x-axis and to the right of y-axis. Hence both its abscissa and ordinate are positive. If the point is in the second quadrant, it is above x-axis but to the left of y-axis so that its abscissa is negative and its ordinate is positive. Similar discussion would reveal that points in the the third quadrant have both the co-ordinates negative and those in the fourth quadrant have positive abscissa and negative ordinate.

To plot a point we make use of squared paper, also called graph paper, and available at any stationery shop. We take any two perpendicular straight lines X′OX and Y′OY as axes. To plot the point $P(x, y)$ we start from O and move along OX or OX′ according as x is positive or negative and count $|x|$ small divisions. Then we move parallel to y-axis upwards or downwards according as the ordinate y is positive or negative and count $|y|$ small divisions. The point thus obtained is the requiredpoint and we write $P(x, y)$ near it.

Suppose that we have to plot the points A(10, 5), B(−6, 8), C(−9, −7) and D(7, −8). Take two perpendicular rulings X′OX and Y′OY as axes of co-ordinates.

The point A (10, 5) is in the first quadrant both its abscissa and ordinate being positive. To plot this point we first move along OX, to rhe right of O since its abscissa is positive, and count 10 ($=|10|$) small divisions. Then since the ordinate is positive, we move parallel to the axis of Y upwards, and count 5 ($=|5|$) small divisions. Mark the point thus obtained as shown in Fig. 9·3 and then write A (10, 5) near it. The point B $(-6, 8)$ is in the second quadrant. Since the abscissa -6 is negative, count 6 ($=|-6|$) divisions from O along OX′ to the left of O and then count 8 divisions upwards because the ordinate 8 is positive. Mark the point so obtained as in Fig. 9·3.

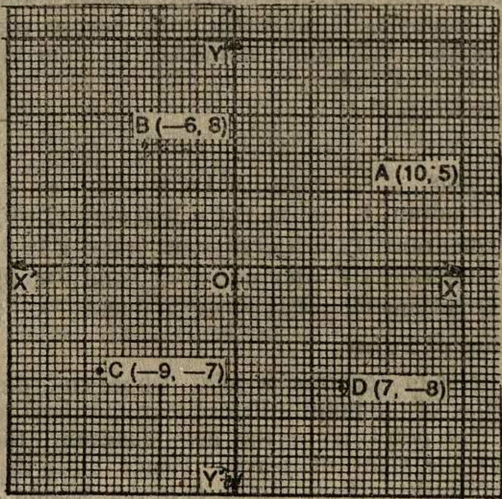


Fig. 9·3.

The point $(-9, -7)$ is in the third quadrant. Count 9 ($=|-9|$) divisions from O along OX′ to the left of O and then 7 ($=|-7|$) divisions downwards. Mark the point so obtained as in Fig. 9.3.

The point $(7, -8)$ is in the fourth quadrant. Count 7 divisions from O along OX to the right of O and then 8 divisions downwards. Mark the point so obtained as in Fig. 9·3.

9·2·3. Graph of a Condition

The set of points on the plane whose co-ordinates satisfy a given condition is called the graph of the condition. For example, consider the condition $x \geqslant 2$. The graph consists of the set of points (x, y) for which $x \geqslant 2$ (no matter what the value of y). It is shown shaded in Fig. 9.4 (a). Similarly the graph of the condition $|y| \leqslant 2$ is the set of points (x, y) for which $-2 \leqslant y \leqslant 2$ (no matter what the value of x). The graph is shown shaded in Fig. 9·4 (b).

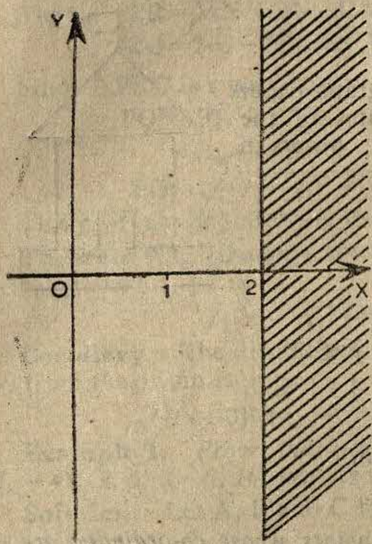


Fig. 9·4 (*a*).

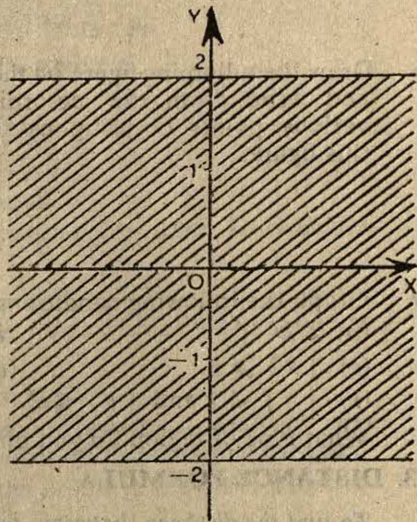
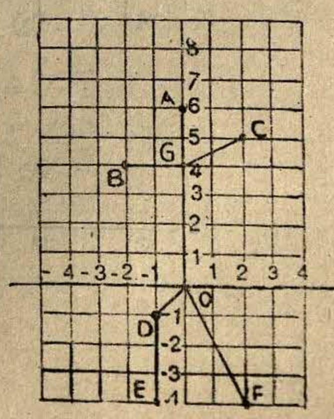


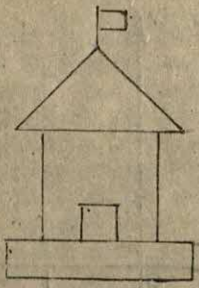
Fig. 9·4 (*b*).

EXERCISE 9 (*a*)

1. Plot the points $A(4, 0)$, $B(4, 4)$, $C(0, 4)$ and draw the lines OA, AB, BC and CO. What figure do you obtain ?
2. Plot the points $A\ (3, 3)$ and $B\ (-3, 3)$. Join OA, OB and BA. What figure do you obtain ?
3. Plot the points $O(0, 0)$, $A(1, 0)$, $B(4, 0)$, $C(2, 4)$, $D(-2, 4)$, $E(-4, 0)$, $F(-4, -7)$, $G(0, -7)$, $H(1, -7)$, $I(3, -7)$, $J(3\frac{1}{2}, -7)$, $K(4, -7)$, $L(4, -8)$, $M(-4, -8)$, $N(3, -6\frac{1}{2})$ and $R(3\frac{1}{2}, -6\frac{1}{2})$. Join EB, BC, CD, DE, OG, AH, FK, ML, LK, MF, IN, NR and JR. What do you obtain ?
4. Plot the points $A(7, 0)$, $B(7, 3)$, $C(0, 3)$, $D(0, 12)$, $E(4, 1)$, $F(0, 13)$, $G(-5, 0)$, $H(-3, -4)$, $I(7, -4)$, $J(8\frac{1}{2}, -1)$, $K(-4\frac{1}{2}, -1)$ and $L(9, 0)$. Join OA, AB, BC, OF, DE, EF, GL, KJ, HI, IL and HG. Name this figure.
5. Look at the adjoining scare-crow and write down the co-ordinates of the points A, B, C, D, E, F and G.



6. Draw the adjoining figure on the graph paper and write down the co-ordinates of the various points occuring as the corners in this figure.



7. Graph the following conditions :

   (a) $x \geqslant 1$, (b) $y \leqslant 2$,

   (c) $|x| \geqslant 3$, (d) $|y| \leqslant 3$,

   (e) $|x| < 1$ and $|y| \leqslant 2$.

   What type of region do you obtain ?

9·3. DISTANCE FORMULA

To find the distance between two points whose co-ordinates are given.

Let P and Q be the given points, and let their co-ordinates be (x_1, y_1) and (x_2, y_2) respectively.

Draw PM aud QN parallel to YO, to meet OX in M and N respectively. Draw PR parallel to OX to meet NQ in R.

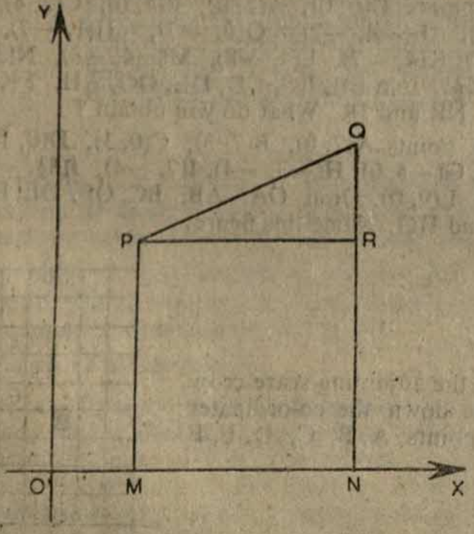


Fig. 9·5

Then $\quad PR=MN=ON-OM=x_2-x_1$

$RQ=NQ-NR=NQ-MP=y_2-y_1.$

Since $\triangle PRQ$ is right-angled at R, we have

$$PQ^2=PR^2+RQ^2,$$
$$=(x_2-x_1)^2+(y_2-y_1)^2,$$

or
$$PQ=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\ .$$

This gives us the following theorem.

Theorem 9·1. *Distance between two points whose co-ordinates are* (x_1, y_1) *and* (x_2, y_2) *is*

$$\sqrt{\{(x_2-x_1)^2+(y_2-y_1)^2\}}.$$

Corollary. The distance of the point whose co-ordinates are (x, y) from the origin is

$$\sqrt{\{(x-0)^2+(y-0)^2\}} \text{ or } \sqrt{(x^2+y^2)}$$

Example 1. *Prove that the points whose co-ordinates are* $(-3, -4), (2, 6), (-6, 10)$ *are the vertices of a right-angled triangle.*

Solution. Let A, B and C be the points in the order as given. Now

$$BC=\sqrt{\{(-6-2)^2+(10-6)^2\}}=\sqrt{80},$$
$$CA=\sqrt{\{(-3+6)^2+(-4-10)^2\}}=\sqrt{205},$$
$$AB=\sqrt{\{(2+3)^2+(6+4)^2\}}=\sqrt{125}.$$

Since $\quad CA^2=BC^2+AB^2,$

therefore, $\triangle ABC$ is right-angled, $\angle B$ being a right angle.

Example 2. *Prove that the points* $A(2, -1), B(3, 4), C(-2, 3$ *and* $D(-3, -2)$ *are the vertices of a rhombus.*

Solution.
$$AB=\sqrt{\{(3-2)^2+(4+1)^2\}}=\sqrt{26},$$
$$BC=\sqrt{\{(-2-3)^2+(3-4)^2\}}=\sqrt{26},$$
$$CD=\sqrt{\{(-3+2)^2+(-2-3)^2\}}=\sqrt{26},$$
$$DA=\sqrt{\{(2+3)^2+(-1+2)^2\}}=\sqrt{26}.$$

Since $\quad AB=BC=CD=DA,$

therefore, ABCD is a rhombus.

Example 3. *Prove that* $P(-1, 0)$, $Q(0, 3)$, $R(3, 2)$ *and* $S(2, -1)$ *are the vertices of a square.*

Solution.
$$PQ=\sqrt{\{(0+1)^2+(3-0)^2\}}=\sqrt{10},$$
$$QR=\sqrt{\{(3-0)^2+(2-3)^2\}}=\sqrt{10},$$
$$RS=\sqrt{\{(2-3)^2+(-1-2)^2\}}=\sqrt{10},$$
$$SP=\sqrt{\{-1-2)^2+(0+1)^2\}}=\sqrt{10}.$$

Since $\quad PQ=QR=RS=SP, \qquad \ldots(i)$

therefore, PQRS is a rhombus.

Also, $\quad PQ^2+QR^2=PR^2=20. \qquad \ldots(ii)$

From (*i*) we find that $\angle$PQR is a right angle.

Hence PQRS is a square.

Example 4. *Show that the points $A(2, 0)$, $B(1, 1)$, $C(0, 2)$ are collinear.*

Solution. $AB=\sqrt{\{(1-2)^2+(1-0)^2\}}=\sqrt{2}$,

$BC=\sqrt{\{(0-1)^2+(2-1)^2\}}=\sqrt{2}$,

$AC=\sqrt{\{(0-2)^2+(2-0)^2\}}=2\sqrt{2}$,

Since AB+BC=AC, the points A, B, C are collinear.

Example 5. *Find a point equidistant from the points $A(-3, -1)$ $B(-1, 3)$ and $C(6, 2)$.*

Solution. Let $P(x, y)$ be the point required.

Since $PA=PB$,

$\therefore \sqrt{(x+3)^2+(y+1)^2}=\sqrt{(x+1)^2+(y-3)^2}$

Squaring and cancelling we have

$$x+2y=0. \qquad ...(i)$$

Similarly, since PA=PC, we have

$$\sqrt{(x+3)^2+(y+1)^2}=\sqrt{(x-6)^2+(y-2)^2}.$$

Squaring and cancelling, we have

$$3x+y-5=0. \qquad ...(ii)$$

Solving (*i*) and (*ii*) simultaneously, we have

$$x=2,\ y=-1.$$

The required point is, therefore, $(2, -1)$.

Example 6. *ABC is an equilateral triangle. If the co-ordinates of the points B and C be $(1, 1)$ and $(-1, -1)$ respectively, find the co-ordinates of the possible positions of the point A.*

Solution. Let the co-ordinates of A be (x, y).

Since $AB=AC$,

$\therefore \sqrt{(x-1)^2+(y-1)^2}=\sqrt{(x+1)^2+(y+1)^2}.$

Squaring and cancelling, we have

$$x+y=0. \qquad ...(i)$$

Similarly, since $AB=BC$,

$\therefore \sqrt{(x-1)^2+(y-1)^2}=\sqrt{(-1-1)^2+(-1-1)^2}.$

Squaring and simplifying, we have

$$x^2+y^2-2x-2y-6=0. \qquad ...(ii)$$

From (*i*), we have

$$y=-x. \qquad ...(iii)$$

Substituting the value of y in (ii), we have

$$2x^2-6=0,$$

or

$$x=\pm\sqrt{3}.$$

When $x=\sqrt{3}, y=-\sqrt{3}$.

When $x=-\sqrt{3}, y=\sqrt{3}$.

Therefore, the required co-ordinates are

$$(\sqrt{3}, -\sqrt{3}) \text{ or } (-\sqrt{3}, \sqrt{3}).$$

EXERCISE 9 (*a*)

Find the distances berween the following pairs of points :

1. (3, 1), (6, 5).
2. (3, 7), (2, −5).
3. (−4, −16), (11, 8).
4. $(a, 0), (0, b)$.
5. $(at_1^2, 2at_1), (at_2^2, 2at_2)$.
6. $(a \cos \alpha, a \sin \alpha), (a \cos \beta, a \sin \beta)$.
7. Find the perimeter of the triangle whose vertices are (8, 4), (11, 0), (5, 0).
8. Prove that the points (1, 1), (−4, 4), (4, 6) are the vertices of an isosceles triangle.
9. Prove that the points (3, 3), (−3, −3) and $(-3\sqrt{3}, 3\sqrt{3})$ are the vertices of an equilateral triangle.
10. Prove that the triangle formed by the points (10, 8), (−2, 4) and (−11, 31) is right-angled.
11. Prove that the points (3, 7), (9, 9), (11, 3) are the vertices of a right-angled isosceles triangle.
12. Prove that the points (−1, 0), (3, 1), (2, 2) and (−2, 1) are the vertices of a parallelogram.
13. Show that the points (3, 1), (4, 6), (−1, 5) and (−2, 0) are the vertices of a rhombus.
14. Show that the points (0, 2), (1, 1), (4, 4) and (3, 5) are the vertices of a rectangle.
15. Show that the points (5, 3), (1, 2), (2, −2) and (6, −1) are the vertices of a square.
16. Show that the points (3, 4), (5, −2), (−1, 16) are collinear.
17. Show that the points (1, 3), (5, 1), (3, 2) are collinear.
18. Find a point equidistant from the three points P(10, 0), Q(−5, 3) and R(6, 6).
19. Find the centre of the circle passing through the points (0, 0), (−3, 3) and (5, 4).

20. Find the co-ordinates of two points whose distances from (2, 3) and 4 each and whose ordinates are equal to 5.

21. Find the co-ordinates of the point which is at a distance of 2 units from (5, 4) and at a distance of 10 units from (11, −2).

22. An equilateral triangle has one vertex at the point (3, 4) and another at the point (−2, 3). Find the co-ordinates of the third vertex.

9·4. AREA OF A TRIANGLE

To find the area of a triangle the co-ordinates of whose vertices are given.

Let ABC be a triangle and let the co-ordinates of its vertices A, B and C be (x_1, y_1), (x_2, y_2) and (x_3, y_3) respectively.

Draw AP, BQ, CR perpendiculars to OX (Fig. 9·6).

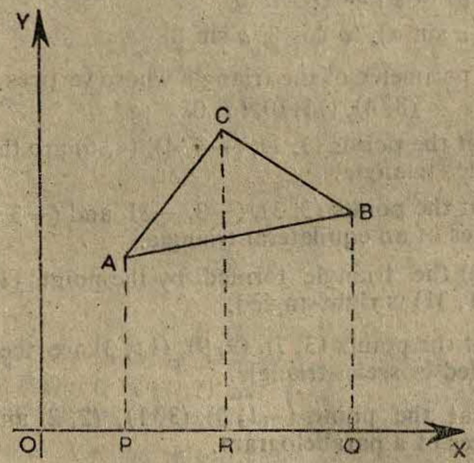


Fig. 9·6.

If Δ denotes the required area, then

$$\Delta = \text{trapezium APRC} + \text{trapezium CRQB} - \text{trapezium APQB},$$
$$= \tfrac{1}{2}PR(PA+RC) + \tfrac{1}{2}RQ(RC+QB) - \tfrac{1}{2}PQ(PA+QB),$$
$$= \tfrac{1}{2}(x_3-x_1)(y_1+y_3) + \tfrac{1}{2}(x_2-x_3)(y_3+y_2) - \tfrac{1}{2}(x_2-x_1)(y_1+y_2),$$
$$= \tfrac{1}{2}\{x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3\}, \qquad \ldots(i)$$
$$= \tfrac{1}{2}\{x_1(y_2-y_3) + x_2(y_3-y_1) + x_3(y_1-y_2)\}. \qquad \ldots(ii)$$

Note. The above formulae will give a positive expression for the area if in going round the triangle ABC in the order of the letters we have the area on the left hand (as in Fig. 9·6) ; a negative

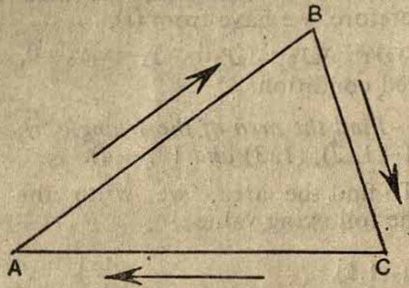


Fig. 9·7.

answer would be obtained if in going round the triangle ABC in the order of the letters we have the area on the right-hand (as in Fig. 9·7).

9·4·1. An Aid to Remember the Formula for the Area of a Triangle.

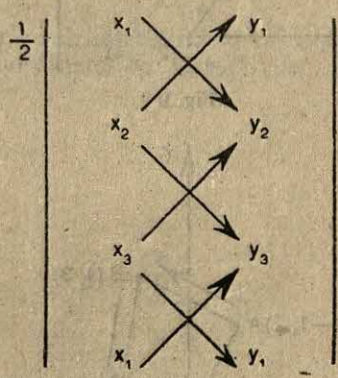


Fig. 9·8.

Write the co-ordinates of the vertices as shown above and join them by arrows as shown ; multiply the pairs of numbers connected by arrows ; add the products corresponding to numbers for which the arrow is directed downwards and subtract the products corresponding to numbers for which the arrow is directed upwards ; multiply the expression so obtained by $\frac{1}{2}$; the resulting product is the required area.

9·4·2. Condition Under which Three Given Points are Collinear

Let the co-ordinates of three given points A, B, C be (x_1, y_1), (x_2, y_2) and (x_3, y_3) respectively.

$$\triangle ABC=\tfrac{1}{2}(x_1y_2-x_2y_1+x_2y_3-x_3y_2+x_3y_1-x_1y_3). \qquad ...(i)$$

If the points A, B, C are collinear, the area of the $\triangle ABC$ must be zero. Therefore, we have from (i),

$$x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3 = 0,$$

which is the required condition.

Example 7. *Find the area of the triangle, the co-ordinates of whose vertices are $(-1, 2)$, $(1, 3)$ and $(2, -4)$.*

Solution. To find the area, we write the vertices as in Fig. 9·9, and get the following value :

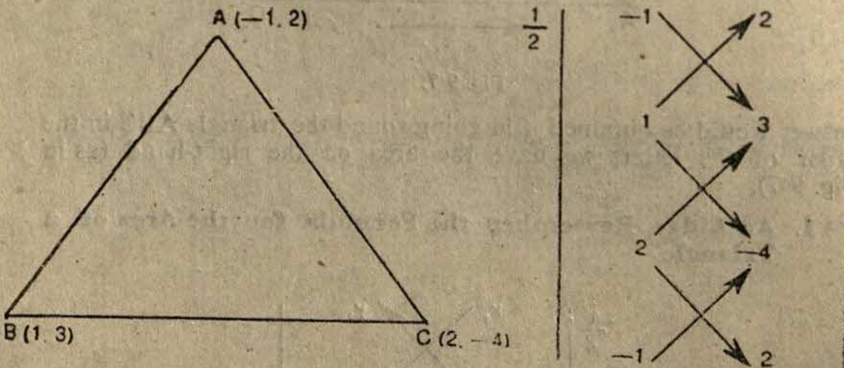


Fig. 9·9.

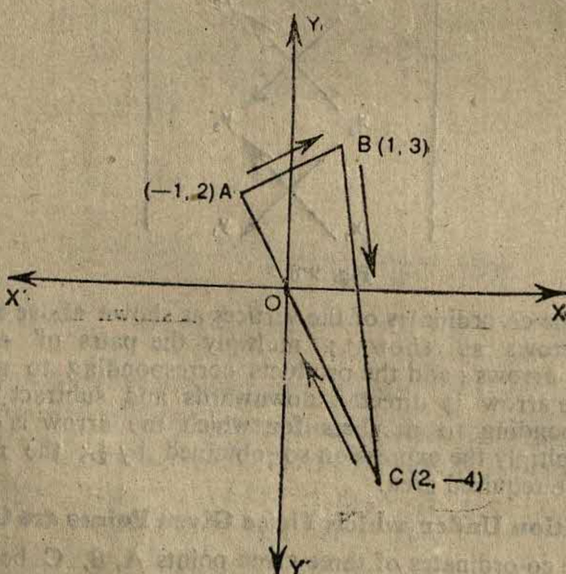


Fig. 9.10.

$$\triangle ABC = \tfrac{1}{2}\{(-1).3 - 1.2 + 1.(-4) - 2.3 + 2.2 - (-1)(-4)\},$$
$$= -15/2. \qquad ...(i)$$

Hence the required area is $7\frac{1}{2}$ square units.

Note. The negative sign in (i) shows that if we actually plot the points A, B, C to scale, then in going round the triangle ABC in the order of the letters, the area will always be on our right hand (see Fig. 9·10).

Example 8. *Show that the points* $(5, 1)$, $(1, -1)$ *and* $(11, 4)$ *are collinear.*

Solution. Area of the triangle formed by joining the points $(5, 1)$, $(1, -1)$ and $(11, 4)$

$$= \tfrac{1}{2}\{5.(-1) - 1.1 + 1.4 - 11.(-1) + 11.1 - 5.4\},$$
$$= 0.$$

Hence the given points are collinear.

9·4·3. Area of a Quadrilateral

To find the area of a quadrilateral the co-ordinates of whose vertices are given.

Let the vertices of a quadrilateral taken in order be A, B, C and D, and let their co-ordinates be respectively (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4).

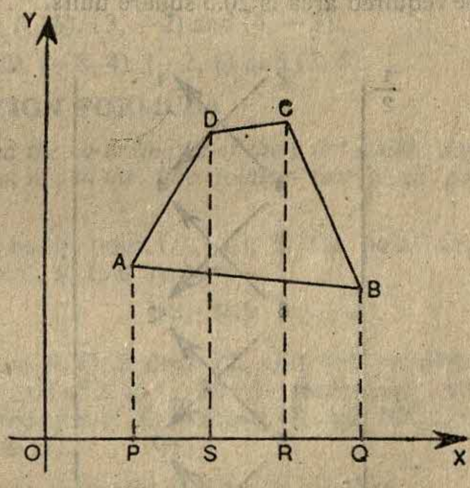


Fig. 9·11.

Draw AP, BQ, CR, DS perpendicular to the axis of X (Fig. 9·11).

The area of the quadrilateral

$$=\text{trapezium APSD}+\text{trapezium DSRC}+\text{trapezium CRQB}-\text{trapezium APQB},$$

$$=\tfrac{1}{2}PS(PA+SD)+\tfrac{1}{2}SR(SD+RC)+\tfrac{1}{2}RQ(RC+QB)-\tfrac{1}{2}PQ(PA+QB),$$

$$=\tfrac{1}{2}(x_4-x_1)(y_1+y_4)+\tfrac{1}{2}(x_3-x_4)(y_4+y_3)+\tfrac{1}{2}(x_2-x_3)(y_3+y_2)-\tfrac{1}{2}(x_2-x_1)(y_1+y_2),$$

$$=\tfrac{1}{2}\{(x_1y_2-x_2y_1)+(x_2y_3-x_3y_2)+(x_3y_4-x_4y_3)+(x_4y_1-x_1y_4)\}.$$

Note. If in any problem the vertices be not given in order, we may find the vertices in order by plotting the vertices to scale.

Example 9. *Find the area of the quadrilateral the co-ordinates of whose vertices taken in order are (2, 1), (4, 4), (6, −2), and (5, −7).*

Solution. The required area

$$=\tfrac{1}{2}\{(2.4-4.1+4.(-2)-6.(4)+6.(-7)-5.(-2)+5.1-2.(-7)\},$$

$$=-\frac{41}{2}.$$

Hence the required area is 20.5 square units.

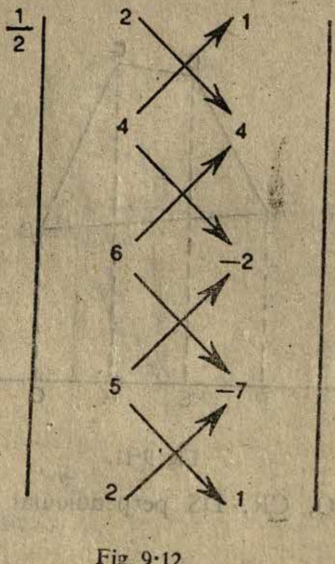


Fig. 9·12.

EXERCISE 9 (*b*)

Find the areas of the triangles the co-ordinates of whose vertices are :

1. $(2, 6), (-4, -2), (3, -1)$.
2. $(-2, 5), (7, -4), (3, 2)$.
3. $(5, 7), (-2, -1), (0, 8)$.
4. $(-2, -5), (5, 3), (3, 9)$.
5. $(c, a), (c-a, -a), (c+a, a)$.
6. $(ap^2, 2ap), (aq^2, 2aq), (ar^2, 2ar)$.
7. $(a \cos \alpha, a \sin \alpha), (a \cos \beta, a \sin \beta), (a \cos \gamma, a \sin \gamma)$.
8. $(cp, c/p), (cq, c/q), (cr, c/r)$.

Prove that the following sets of points are in a straight line.

9. $(-6, 9), (0, -9)$ and $(-2, -3)$.
10. $(5, 7), (-3, 9)$ and $(1, 8)$.
11. $(p+q, r), (q+r, p)$ and $(r+p, q)$.

Find the areas of the quadrilaterals the co-ordinates of whose vertices, taken in order, are

12. $(1, 1), (3, 4), (5, -2)$ and $(4, -7)$.
13. $(4, 5), (1, 6), (3, -2)$ and $(4, -3)$.
14. $(-5, 2), (-8, 4), (-2, 6)$ and $(3, 5)$.

9·5. SECTION FORMULA

To find the co-ordinates of the point which divides internally in a given ratio, $m : n$, the line joining two given points (x_1, y_1) and (x_2, y_2).

Let P be the point (x_1, y_1), Q the point (x_2, y_2), and R the required point, so that we have

$$PR : RQ :: m : n.$$

Through P, Q, R draw PL, QM, RN parallel to the axis of Y to meet the axis of X in L, M, N respectively and draw PS, RT parallel to the axis of X, to meet NR and MQ in S and T respectively (Fig. 9.13).

If the co-ordinates of R be (x, y), then

$$\begin{aligned}
PS &= LN = ON - OL = x - x_1, \\
RT &= NM = OM - ON = x_2 - x, \\
SR &= NR - NS = NR - LP = y - y_1, \\
TQ &= MQ - MT = MQ - NR = y_2 - y.
\end{aligned}$$

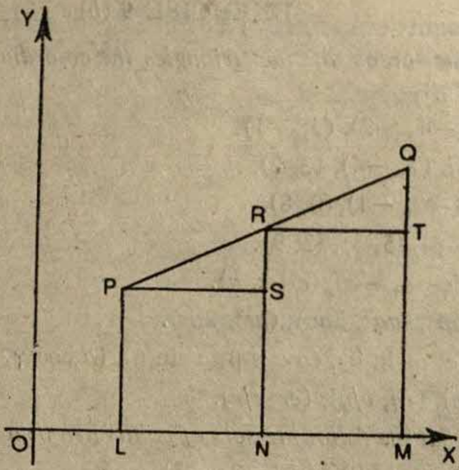


Fig. 9·13.

Since PS ∥ RT, and PQ meets them,
therefore, $\angle SPR = \angle TRQ$, (corres. $\angle s$).

Since RN ∥ QM and PQ meets them,
therefore, $\angle PRS = \angle RQT$ (corres. $\angle s$).

The triangles PRS, RQT are, therefore, equiangular and hence similar.

Therefore,

$$\frac{PS}{RT} = \frac{SR}{TQ} = \frac{PR}{RQ},$$

or
$$\frac{x - x_1}{x_2 - x} = \frac{y - y_1}{y_2 - y} = \frac{m}{n}.$$

Since
$$\frac{x - x_1}{x_2 - x} = \frac{m}{n},$$

therefore,
$$n(x - x_1) = m(x_2 - x),$$

or
$$x = \frac{mx_2 + nx_1}{m + n}. \qquad ...(i)$$

Since
$$\frac{y - y_1}{y_2 - y} = \frac{m}{n},$$

therefore,
$$n(y - y_1) = m(y_2 - y),$$

or
$$y = \frac{my_2 + ny_1}{m + p}. \qquad ...(ii)$$

From (*i*) and (*ii*) we find that the co-ordinates of the point which divides the join of (x_1, y_1) and (x_2, y_2) internally in the ratio $m : n$ are

$$\frac{mx_2+nx_1}{m+n} \text{ and } \frac{my_2+ny_1}{m+n}$$

Corollary. The co-ordinates of the middle-point of the line joining the points (x_1, y_1) and (x_2, y_2) are

$$\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}.$$

To find the co-ordinates of the point which divides externally in a given ratio, m : n ($m \neq n$) *the line joining two given points* (x_1, y_1) *and* (x_2, y_2).

Let P be the point (x_1, y_1), Q the point (x_2, y_2) and R the required point, so that we have

$$PR : QR :: m : n.$$

Through P, Q, R draw PL, QM, RN parallel to the axis of Y

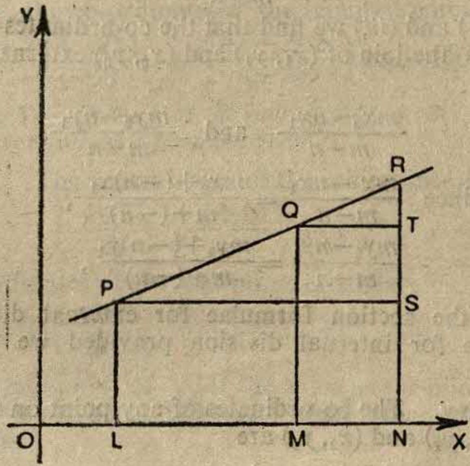


Fig. 9·14.

meet the axis of X in L, M, N respectively and draw PS, QT parallel to the axis of X to meet NR in S and T respectively (Fig. 9·14).

From similar triangles PSR, QTR we have

$$\frac{PS}{QT}=\frac{SR}{TR}=\frac{PR}{QR}. \qquad ...(i)$$

Now $$\frac{PR}{QR}=\frac{m}{n},$$

$$\frac{PS}{QT}=\frac{LN}{MN}=\frac{ON-OL}{ON-OM}=\frac{x-x_1}{x-x_2},$$

$$\frac{SR}{TR}=\frac{NR-NS}{NR-NT}=\frac{NR-LP}{NR-MQ}=\frac{y-y_1}{y-y_2}.$$

Therefore, (*i*) may be written as

$$\frac{x-x_1}{x-x_2}=\frac{y-y_1}{y-y_2}=\frac{m}{n}.$$

Since $\dfrac{x-x_1}{x-x_2}=\dfrac{m}{n}$,

therefore, $n(x-x_1)=m(x-x_2)$,

or
$$x=\frac{mx_2-nx_1}{m-n}. \qquad ...(ii)$$

Since $\dfrac{y-y_1}{y-y_2}=\dfrac{m}{n}$,

therefore, $n(y-y_1)=m(y-y_2)$,

or
$$y=\frac{my_2-ny_1}{m-n}. \qquad ...(iii)$$

From (*ii*) and (*iii*) we find that the co-ordinates of the point which divides the join of (x_1, y_1) and (x_2, y_2) externally in the ratio $m : n$ are

$$\frac{mx_2-nx_1}{m-n} \quad \text{and} \quad \frac{my_2-ny_1}{m-n}.$$

Note. Since $\dfrac{mx_2-nx_1}{m-n}=\dfrac{mx_2+(-n)x_1}{m+(-n)}$,

$$\frac{my_2-ny_1}{m-n}=\frac{my_2+(-n)y_1}{m+(-n)},$$

we find that the section formulae for external division are the same as those for internal division provided we take the ratio negatively.

Corollary. The co-ordinates of any point on the line joining the points (x_1, y_1) and (x_2, y_2) are

$$\frac{\lambda x_2+x_1}{\lambda+1},\ \frac{\lambda y_2+y_1}{\lambda+1},\ \text{where } \lambda\neq-1.$$

For, let P, Q be the points whose co-ordinates are (x_1, y_1) and (x_2, y_2) and let R (x, y) be any point on the straight line PQ. Let R divide PQ in the ratio $\lambda : 1$ (R may divide the line PQ internally or externally).

Since R divides the join of (x_1, y_1), (x_2, y_2) in the ratio $\lambda : 1$, the co-ordinates of R are

$$\left(\frac{\lambda x_2+x_1}{x+1},\ \frac{\lambda y_2+y_1}{\lambda+1}\right).$$

Example 10. *Find the co-ordinates of the point which divides the join of the points P(1, 2) and Q(3, 4) in the ratio 1 : 2 (i) internally, (ii) externally.*

Solution. (*i*) The co-ordinates of the point which divides the join of (x_1, y_1), (x_2, y_2) in the ratio $m : n$ are

$$\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}.$$

Here $(x_1, y_1)=(1, 2)$,

$(x_2, y_2)=(3, 4)$,

$m : n=1 : 2.$

$\therefore$ The co-ordinates of the required point are

$$\left(\frac{1.3+2.1}{1+2}, \frac{1.4+2.2}{1+2}\right) \text{ or } \left(\frac{5}{3}, \frac{8}{3}\right).$$

(*ii*) Here $(x_1, y_1)=(1, 2)$,

$(x_2, y_2)=(3, 4)$,

$m : n=1 : -2.$

Therefore, the co-ordinates of the required point are

$$\left(\frac{1.3+(-2).1}{1+(-2)}, \frac{1.4+(-2).2}{1+(-2)}\right) \text{ or } (-1, 0).$$

Example 11. *Prove that the points A(4, 6), B(7, 7), C(10, 10), D(7, 9) are the vertices of a parallelogram.*

Solution. The co-ordinates of the mid-point of AC are

$$\left(\frac{4+10}{2}, \frac{6+10}{2}\right) \text{ or } (7, 8).$$

The co-ordinates of the mid-point of BD are

$$\left(\frac{7+7}{2}, \frac{7+9}{2}\right) \text{ or } (7, 8).$$

Since the mid-point of AC is the same as the mid-point of BD, therefore, the diagonals AC, BD bisect each other. Hence ABCD is a parallelogram.

Example 12. *The points A(8, 5), B(−7, −5), C(−5, 5) are three of the vertices of a parallelogram ABCD. Find the co-ordinates of the point D.*

Solution. Let the co-ordinates of D be (x, y).

The co-ordinates of the mid-point of AC are

$$\left(\frac{8-5}{2}, \frac{5+5}{2}\right) \text{ or } \left(\frac{3}{2}, 5\right).$$

The co-ordinates of the mid-point of BD are

$$\left(\frac{-7+x}{2}, \frac{-5+y}{2}\right).$$

Since the diagonals of a parallelogram bisect each other, therefore, the mid-point of AC is also the mid-point of BD.

Therefore, $\left(\frac{3}{2}, 5\right)$ and $\left(\frac{-7+x}{2}, \frac{-5+y}{2}\right)$ are the co-ordinates of the same point.

Therefore, $\frac{3}{2}=\frac{-7+x}{2}$, $5=\frac{-5+y}{2}$,

or $\qquad x=10, y=15.$

Thus the co-ordinates of D are (10, 15).

Example 13. *Find the ratio in which the join of (3, 6) and (7, 13) is divided by (11, 20).*

Solution. Let the points be called A, B, C respectively in the order given.

Let C(11, 20) divide the join of A(3, 6) and B(7, 13) in the ratio $\lambda : 1$.

The co-ordinates of the point dividing the join of A and B in the ratio $\lambda : 1$ are

$$\left(\frac{7\lambda+3}{\lambda+1}, \frac{13\lambda+6}{\lambda+1}\right).$$

If this is the same as the point C, we have

$$11=\frac{7\lambda+3}{\lambda+1}, \; 20=\frac{13\lambda+6}{\lambda+1}. \qquad \ldots(i)$$

The first of the relations (i) gives $\lambda=-2$.

Therefore, the required ratio is $2 : -1$.

Hence (11, 20) divides the join of (3, 6) and (7, 13) externally in the ratio 2 : 1.

Note. The second of the relations (i) would have also given us the same value of λ. In fact, the correctness of the result can always be checked by verifying whether both the relations give the same value of λ.

Example 14. *Prove that the points A(27, −7), B(3, −4), C(−5, −3) are collinear.*

Solution. The co-ordinates of any point on the join of A(27, −7) and B(3, −4) are

$$\left(\frac{3\lambda+27}{\lambda+1}, \frac{-4\lambda-7}{\lambda+1}\right).$$

If A, B, C are collinear, then there must be some value of λ for which

$$\left(\frac{3\lambda+27}{\lambda+1}, \frac{-4\lambda-7}{\lambda+1}\right)=(-5, -3),$$

or $$\frac{3\lambda+27}{\lambda+1}=-5,\quad \frac{-4\lambda-7}{\lambda+1}=-3. \qquad \ldots(i)$$

If A, B, C are collinear, *both* the relations (i) must be satisfied by the same value of λ. The first of these relations gives

$$3\lambda+27=-5(\lambda+1),$$

or $$\lambda=-4.$$

By actual substitution we find that $\lambda=-4$ satisfies the second relation also.

Hence the points A, B, C are collinear.

Example 15. *Find the ratio in which the join of (6, 8) and (10, 5) is divided by the axis of X.*

Solution. Let the given points be named A and B respectively. The co-ordinates of a point C, which divides the join of A and B in the tatio $\lambda : 1$ are

$$\left(\frac{10\lambda+6}{\lambda+1}, \frac{5\lambda+8}{\lambda+1}\right).$$

If C lies on the axis of X, its y-co-ordinate must be zero.

Therefore, $$\frac{5\lambda+8}{\lambda+1}=0,$$

or $$\lambda=-\frac{8}{5}.$$

Therefore, the required ratio is

$$-\frac{8}{5} : 1 \text{ or } 8 : -5.$$

Hence the X-axis divides the join of the given points externally in the ratio 8 : 5.

9·6. CENTROID OF A TRIANGLE

To show that the medians of a triangle are concurrent.

Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be the vertices of a triangle. The co-ordinates of D, the mid-point of BC, are

$$\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right).$$

Let $G(x, y)$ be the point which divides AD in the ratio 2 : 1. Then

$$x=\frac{2.\frac{1}{2}(x_2+x_3)+1.x_1}{2+1},$$

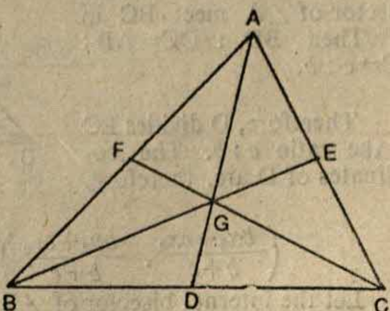


Fig. 9·15.

$$=\frac{x_1+x_2+x_3}{3},$$

$$y=\frac{2.\frac{1}{2}(y_2+y_3)+1.y_1}{2+1}=\frac{y_1+y_2+y_3}{3}.$$

Since the expressions $\frac{x_1+x_2+x_3}{3}$, $\frac{y_1+y_2+y_3}{3}$, remain unchanged when the suffixes 1, 2, 3 are cyclically interchanged, therefore, the point

$$G\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$

lies on the medians BE and CF also and divides each of them in the ratio 2 : 1.

Hence the medians of a triangle are concurrent.

Note. Recall that the point of concurrence of the medians of a triangle is called the *centroid* of the triangle.

Example 16. *Find the centroid of the triangle whose vertices are (—5, 7), (1, —1) and (1, 6) respectively.*

Solution. The co-ordinates of the centroid are

$$\left(\frac{-5+1+1}{3}, \frac{7-1+6}{3}\right) \text{ or } (-1, 4).$$

9.7. INCENTRE OF A TRIANGLE

To show that the internal bisectors of the angles of a triangle are concurrent.

Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be the vertices of a triangle and let a, b, c be the lengths of the sides BC, CA, AB respectively.

Let AD, the internal bisector of $\angle A$ meet BC in D. Then BD : DC=AB : AC=$c : b$.

Therefore, D divides BC in the ratio $c : b$. The co-ordinates of D are, therefore,

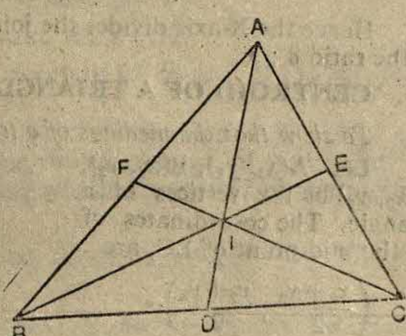


Fig. 9·16.

$$\left(\frac{bx_2+cx_3}{b+c}, \frac{by_2+cy_3}{b+c}\right).$$

Let the internal bisector of $\angle B$ meet AD in I. Then

DI : IA=BD : BA.

But since D divides BC in the ratio $c : b$,

$$BD=\frac{ac}{b+c}.$$

Therefore, $DI : IA=\frac{ac}{b+c} : c=a : (b+c)$.

Therefore, the co-ordinates of I are

$$\left(\frac{a.x_1+(b+c).\frac{bx_2+cx_3}{b+c}}{a+(b+c)}, \frac{a.y_1+(b+c).\frac{by_2+cy_3}{b+c}}{a+(b+c)}\right),$$

or

$$\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right). \qquad ...(i)$$

The symmetry of the expression in (*i*) shows that I is a point on the internal bisector of $\angle C$ also. Hence the internal bisectors of the angles of a triangle are concurrent.

Note. Recall that the point of concurrence of the internal bisectors of the angles of a triangle is called the *in-centre* of the triangle. The co-ordinates of the in-centre of a triangle are

$$\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right),$$

where the symbols have the usual meaning.

Example 17. *Find the co-ordinates of the in-centre of the triangle whose vertices are $A(-36, 7)$, $B(20, 7)$, $C(0, -8)$.*

Solution. The lengths of the sides of the $\triangle ABC$ are

$$a=BC=\sqrt{(0-20)^2+(-8-7)^2}=25,$$
$$b=CA=\sqrt{(-36-0)^2+(7+8)^2}=39,$$
$$c=AB=\sqrt{(20+36)^2+(7-7)^2}=56.$$

The co-ordinates (x, y) of the in-centre are given by

$$x=\frac{ax_1+bx_2+cx_3}{a+b+c},$$
$$=\frac{25(-36)+39.20+56.0}{25+39+56}=-1,$$
$$y=\frac{ay_1+by_2+cy_3}{a+b+c},$$
$$=\frac{25.7+39.7+56(-8)}{25+39+56}=0.$$

Hence the in-centre is the point $(-1, 0)$.

9·8. CHOICE OF AXES

Most of the theorems of geometry can be proved by the methods of co-ordinate geometry. A suitable choice of the axes

of the co-ordinates makes the proofs simple and elegant. The following examples will illustrate the method.

Example 18. *If D be the middle point of the side BC of a triangle ABC, prove that*

$$AB^2+AC^2=2(AD^2+BD^2).$$

Solution. Take D as the origin, DC as the axis of X, and the straight line thorugh D perpendicular to BC as the axis of Y.

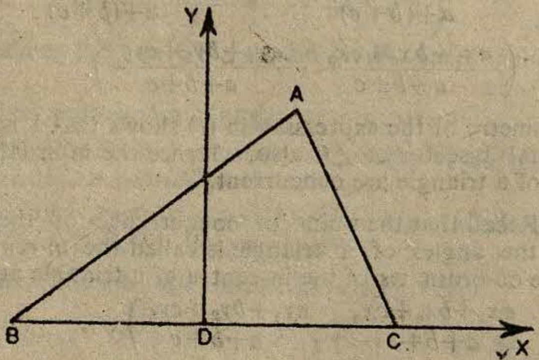


Fig. 9·17.

If $BC=2l$, the co-ordinates of B and C are $(-l, 0)$ and $(l, 0)$ respectively. Let (x, y) be the co-ordinates of A.

Then
$$AB^2=(x+l)^2+y^2,$$
$$AC^2=(x-l)^2+y^2,$$
$$AD^2=x^2+y^2,$$
$$BD^2=l^2.$$

Therefore,

$$AB^2+AC^2=\{(x+l)^2+y^2\}+\{(x-l)^2+y^2\},$$
$$=2\{x^2+l^2+y^2\},$$
$$=2\{(x^2+y^2)+l^2\},$$
$$=2(AD^2+BD^2).$$

Example 19. *Prove that the sum of the squares of the sides of a parallelogram is equal to the sum of the squares on the diagonals.*

Solution. Let the vertex A of a parallelogram ABCD be taken as the origin, the side AB be taken as the axis of X, and let the perpendicular from A on AB be taken as the axis of Y. Let the co-ordinates of B be $(b, 0)$ and those of D be (c, d).

Since AC and BD bisect each other, the co-ordinates of C are $(b+c, d)$. See Fig. 9·18.

Now, $AB^2=b^2$, $BC^2=c^2+d^2$, $CD^2=b^2$, $AD^2=c^2+d^2$,
$AC^2=(b+c)^2+d^2$, $BD^2=(b-c)^2+d^2$.

Therefore, $AB^2+BC^2+CD^2+DA^2=2(b^2+c^2+d^2)$.

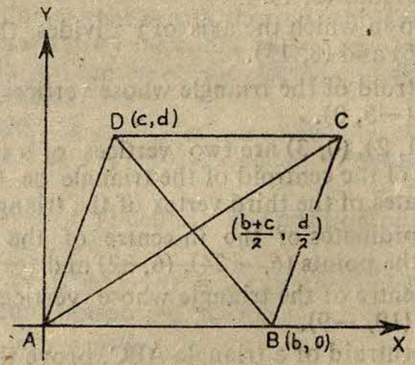


Fig. 9·18.

$$AC^2+BD^2=\{(b+c)^2+d^2\}+\{(b-c)^2+d^2\},$$
$$=2\{b^2+c^2+d^2\}.$$

Hence $AB^2+BC^2+CD^2+DA^2=AC^2+BD^2$.

EXERCISE 9 (*c*)

Find the co-ordinates of the point which

1. divides the line joining the points (8, 10) and (7, 6) internally in the ratio 4 : 3.
2. divides the line joining the points (−7, 8) and (4, −3) externally in the ratio 5 : 7.
3. divides the line joining the points (−2, 1) and (5, −4) externally in the ratio 3 : 2.
4. Find the co-ordinates of the points which divide the line joining (2, 3) and (−4, 5) internally and externally in the ratio 2 : 3.
5. Find the lengths of the medians of the triangle whose angular points are (1, 2), (0, 3) and (−1, −2).
6. Find the co-ordinates of the points of trisection of the line joining the points (−5, −5) and (25, 10).
7. Find the co-ordinates of the points that divide the line joining the points (35, −20) and (5, −10) into four equal parts.
8. Show that the points (6, −1), (7, 3), (8, 2), (7, −2) are the vertices of a parallelogram.
9. Find the ratio in which the join of (−1, 1) and (−9, −7) is divided by (−4, −2).
10. In what ratio does the point (−5, −20) divide the join of the points (1, −2) and (4, 7) ?

11. Show that the three points $(-1, -2)$, $(4, 1)$ and $(9, 4)$ lie on a straight line and find the ratio of the distances of the second from the other two.
12. Find the ratio in which the axis of x divides the join of the points $(-1, 3)$ and $(3, 7)$.
13. Find the ratio in which the axis of y divides the join of the points $(4, -6)$ and $(8, 13)$.
14. Find the centroid of the triangle whose vertices are $(21, -2)$, $(15, 10)$ and $(-5, 0)$.
15. The points $(1, 2)$, $(0, 3)$ are two vertices of a triangle. If the co-ordinates of the centroid of the triangle be $(-1, -2)$, find the co-ordidates of the third vertex of the triangle.
16. Find the co-ordinates of the in-centre of the triangle whose vertices are the points $(6, -24)$, $(6, 32)$ and $(-9, 12)$.
17. Find the in-centre of the triangle whose vertices are $(3, -8)$, $(3, -2)$ and $(10, -9)$.
18. If G be the centroid of a triangle ABC, prove that
$$3(GA^2+GB^2+GC^2)=BC^2+CA^2+AB^2.$$
19. Prove that the straight line joining the mid-points of two sides of a triangle is half the third side.
20. ABCD is a rectangle and P is any point in the plane of the rectangle. Prove that
$$PA^2+PC^2=PB^2+PD^2.$$

9·9. LOCUS AND ITS EQUATION

When a point moves so that it always satisfies a given condition or conditions, the path traced out by it is called its **locus** under these conditions. Let us consider a few examples.

(1) Let O be a given point in the plane of the paper and let a point P move on the paper so that its distance from O is constant and is equal to a. All the positions of the moving point must lie on a circle whose centre is O and radius is a. This circle is, therefore, the 'locus' of P when it moves under the condition that its distance from O is equal to a constant a.

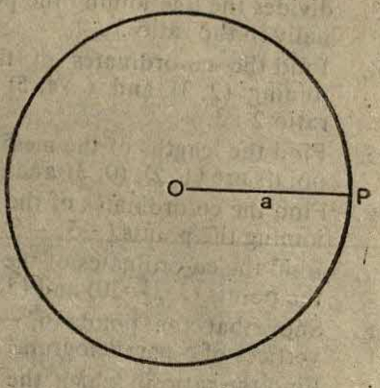


Fig. 9·19.

(2) Let A and B be two fixed points in the plane of the paper and let a point P move on the paper so that its distances from A and B are equal. All the positions of the moving point must lie on the right-bisector of AB.

The right-bisector of AB is, therefore, the 'locus' of P when it moves under the condition that its distances from A and B are equal.

9·9·1. Equation to a Locus

If a locus and an equation are such that (1) every point on the locus has co-ordinates that satisfy the equation and (2) every point whose co-ordinates satisfy the equation lies on the locus, then the equation is said to represent the locus and the locus is said to represent the equation.

Example 20. *A point P moves so that its distances from the points A(3, 4) and B(−4, −3) are equal. Find the equation to its locus.*

Solution. If (x, y) denotes *any* position of the point P, then since PA=PB, we have

$$\sqrt{(x-3)^2+(y-4)^2}=\sqrt{(x+4)^2+(y+3)^2}.$$

Squaring and transposing, we have

$$x+y=0,$$

which is the equation to the locus.

Example 21. *A point moves so that its distance from the point (−1, 1) is always twice its distance from the point (0, 3). Find the equation to its locus.*

Solution. Let (x, y) be any point which satisfies the given condition. We then, have

$$\sqrt{(x+1)^2+(y-1)^2}=2\sqrt{x^2+(y-3)^2}.$$

Squaring both sides, we have

$$(x+1)^2+(y-1)^2=4\{x^2+(y-3)^2\},$$

or
$$3x^2+3y^2-2x-22y+34=0,$$

which is the required locus.

9·9·2. Point on a Locus

If a point lies on a locus, its co-ordinates satisfy the equation to the locus. Therefore, to determine whether a point lies on a given locus, we substitute the co-ordinate of the point in the equation to the locus and find out whether the equation is satisfied.

Example 22. *Show that the origin lies on the locus of the equation $x^2+y^2-3x-4y=0$.*

Solution. Substituting $x=0$, $y=0$ in the given equation we find that the given equation is satisfied by these values. Therefore, the point (0, 0) lies on the locus of the given equation.

EXERCISE 9 (*d*)

1. Find the equation to the locus of a point which moves so that it is always at a distance 3 units from the point (−6, 1).
2. Find the equation to the locus of a point which moves so that it is equidistant from the points (3, −6) and (−4, 7).

3. Find the equation to the locus of a point which moves so that its distance from the point $(-4, 0)$ is three times its distance from the point $(0, -1)$.

4. Find the equation to the locus of a point which moves so that the sum of its distances from the points $(-2, 0)$ and $(2, 0)$ is 6 units.

5. A point P moves so that

$$PC^2 - PB^2 = AC^2 - AB^2.$$

If A, B, C be the points $(2, 3)$, $(-1, 1)$, $(2, -3)$ respectively, find the equation to the locus of P.

6. A point moves so that the sum of the squares of its distances from the points $(1, 2)$ and $(2, 1)$ is constant. Find the equation to the locus of P.

7. A point moves so that its distance from the axis of x is four times its distance from the point $(0, 2)$. Find the equation to the locus of P.

8. A point moves so as to form a triangle of area 17 units with the points $(-3, 2)$ and $(-4, -3)$. Show that the equation to its locus is

$$5x - y + 17 = \pm 34.$$

9. A point moves so that the ratio of its distances from the points $(-a, 0)$ and $(a, 0)$ is $2 : 3$. Find the equation to its locus.

TEST YOUR UNDERSTANDING IX

In each of the following problems four alternatives are given. Put a tick-mark (✓) against the correct alternative :

1. The triangle having vertices $A(-2, -7)$, $B(-4, 1)$ and $C(2, -6)$ is
   (*a*) equilateral (*b*) right-angled
   (*c*) acute-angled (*d*) obtuse-angled.

2. Q, R, S are the points $(0, 3)$, $(4, 0)$ and $(2, -4)$. The point P such that PQRS is a parallelogram, is
   (*a*) $(2, 1)$ (*b*) $(1, 2)$
   (*c*) $(-2, -1)$ (*d*) $(-1, -2)$.

3. P and Q are points on the line joining $M(-2, 5)$ and $N(3, 1)$ such that $MP = PQ = QN$. The mid-point of PQ is
   (*a*) $(4, 2)$ (*b*) $(1, 3)$
   (*c*) $(\frac{1}{2}, 3)$ (*d*) $(2, 4)$.

4. A and B are the points $(-3, 4)$ and $(2, 1)$ respectively. The point C on AB produced such that $AC = 2BC$, is
   (*a*) $(-2, 7)$ (*b*) $(7, 3)$
   (*c*) $(4, 2)$ (*d*) $(7, -2)$.

5. If the points $(8, k)$, $(2, -2)$ and $(-2, -5)$ are collinear, the value of k is

(*a*) 5 (*b*) $\frac{5}{2}$

(*c*) 3 (*d*) $-\frac{3}{2}$.

6. The centroid of the triangle having $(1, 2)$, $(3, 1)$ and $(5, -6)$ as vertices, is

(*a*) $(-1, 3)$ (*b*) $(3, -1)$

(*c*) $(9, -3)$ (*d*) $\left(\frac{9}{2}, -\frac{3}{2}\right)$.

7. The locus of the point which moves so that its distance from the x-axis is twice its distance from the y-axis, is

(*a*) $2x-y=0$ (*b*) $x-2y=0$

(*c*) $x+2y=0$ (*d*) $2x+y=0$.

8. The locus of the point the product of whose distances from the axes of co-ordinates is 4, is

(*a*) $|xy|=4$ (*b*) $4x=y$

(*c*) $4y=x$ (*d*) $xy=16$.

9. The vertices of a triangle are A(1, 4), B(3, −9) and C(−5, 2). The length of the median drawn from B to the opposite side is

(*a*) 13 (*b*) 7

(*c*) 8 (*d*) 9.

10. The triangle having A (−2, −2), B (−1, 2) and C (3, 1) as vertices is

(*a*) scalene (*b*) right-angled

(*c*) equilateral (*d*) obtuse-angled.

REVIEW EXERCISE IX

1. Find the distance between the points $(a, 0)$ and $(0, b)$.
2. Prove that the points (a, a), $(-a, -a)$ and $(-a\sqrt{3}, a\sqrt{3})$ are the vertices of an equilateral triangle.
3. Prove that the points $(4, 2)$, $(5, 7)$, $(0, 6)$ and $(-1, 1)$ are the vertices of a rhombus.
4. Prove that the points $(3, -2)$, $(6, 1)$, $(3, 4)$, and $(0, 1)$ are the vertices of a square.
5. Prove that the points $(a, b+c)$, $(b, c+a)$, and $(c, a+b)$ are collinear.
6. Show that the points $(6, 2)$, $(-2, -4)$, $(5, -5)$, and $(-1, 3)$ are concyclic.

[**Hint.** Show that there is a point equidistant from the given points.]

7. Find the co-ordinates of the points each of which is at a distance of 5 units from (4, 5) and at a distance of 13 units from (6, 13).
8. Find the co-ordinates of the points of trisection of the line segment joining the points P(1, 2) and Q(3, -2).
9. The line joining the points (9, -5) and (-5, 9) is divided into four equal parts. Find the co-ordinates of the points of the section.
10. Prove that the points (1, 3), (3, 5), (4, 6), and (7, 9) are collinear. In what ratio is the line joining the first and the third divided by the (*i*) second, (*ii*) fourth ?
11. Prove that the middle point of the line joining the points (-5, 12) and (9, -2) is a point of trisection of the line joining the points (-8, -5) and (7, 10).
12. The point (2, 6) is the intersection of the diagonals of a parallelogram two of whose corners are the points (7, 6) and (10, 2). Find the co-ordinates of the remaining corners.
13. Two vertices of a triangle are (x_1, y_1) and (x_2, y_2) and its centroid is (x, y). Find the co-ordinates of the third vertex.
14. ABCD is a parallelogram. Show that D, a point of trisection of AC, and the mid-point of AB are collinear.
15. Prove that the lines joining the middle points of opposite sides of a quadrilateral and the line joining the middle points of its diagonals meet in a point and bisect one another.
16. If G be the centroid of a triangle and O be any other point, prove that

$$OA^2+OB^2+OC^2=GA^3+GB^2+GC^2+3GO^2.$$

17. Find the area of the triangle whose vertices are

$$(apq,\ a(p+q)),\ (aqr,\ a(q+r)),\ \text{and}\ (arp,\ a(r+p)).$$

18. Show that the points $(0, 3a)$, $(3b, 0)$ and $(b, 2a)$ are collinear.
19. Find the area of the quadrilateral whose vertices are the points (-2, -3), (3, 4), (1, -2) and (-1, 5).
20. Show that the points (0, -1), (2, 1), (0, 3), and (-2, 1) are the corners of a square.

SUMMARY

1. The distance between the points (x_1, y_1) and (x_2, y_2) is

$$\sqrt{[(x_2-x_1)^2+(y_2-y_1)^2]}.$$

2. The point which divides the join of the points (x_1, y_1) and (x_2, y_2) in the ratio $m : n$, is

$$\left(\frac{mx_2+nx_1}{m+n},\ \frac{my_2+ny_1}{m+n}\right).$$

3. The centroid of the triangle whose vertices are the points (x_1, y_1), (x_2, y_2), and (x_3, y_3), is

$$\left(\frac{x_1+x_2+x_3}{3},\ \frac{y_1+y_2+y_3}{3}\right).$$

4. The in-centre of the triangle whose vertices are the points (x_1, y_1), $(x_2, y_2,)$ and (x_3, y_3) and whose sides are of lengths a, b and c, is

$$\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right).$$

5. The area of the triangle whose vertices are the points (x_1, y_1), (x_2, y_2), and (x_3, y_3) is

$$\tfrac{1}{2}\,[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)].$$

or

$$\tfrac{1}{2}\,[(x_1y_2-x_2y_1)+(x_2y_3-x_3y_2)+(x_3y_1-x_1y_3)].$$

6. The area of the quadrilateral whose vertices taken in order are (x_1, y_1), (x_2, y_2), (x_3, y_3), and (x_4, y_4), is

$$\tfrac{1}{2}\,[(x_1y_2-x_2y_1)+(x_2y_3-x_3y_2)+(x_3y_4-x_4y_3)+(x_4y_1-x_1y_4)].$$

HISTORICAL NOTE

Rene Descartes (1596-1650), a French philosopher, was the first to study geometrical problems with the help of equations. His only work on geometry, *La geometrie* was published in 1637 as an appendix to a philosophical treatise.

Pierre de Fermat (1601-1655), a French jurist also made fundamental contributions to the study of geometry by analytical methods. He wrote in Latin a very short treatise on analytical geometry, entitled *Introduction to plane and solid loci.* Though it consisted of only eight pages, yet the fundamental principle of analytical geometry was put forth very clearly and precisely in these eight pages. Fermat wrote : *whenever in a final equation two unknown quantities are found, we have a locus.* He showed that equations of the first degree represent straight lines and equations of the second degree represent circles, ellipses, parabolas and hyperbolas.

It was rather unfortunate that Fermat's treatise was published 14 years after his death, and 50 years after it had been written. Descartes' work had already appeared forty years earlier, and therefore, analytic geometry came to be known as Cartesian geometry. The credit which should have gone at least partly, to Fermat went almost entirely to Descartes. It must, however, be said that it was the influence of Descartes and not of Fermat which was responsible for the spread of analytical geometry.

Newton amplified Descartes' method to the study of the general theory of curves. Leonhard Euler (1707-1783) extended Cartesian geometry to the space of three dimensions.

The credit for giving to co-ordinate geometry its present shape goes to the French mathematician Gaspard Monge (1746-1818). He fully realized and appreciated the power of analytical geometry, and used it to study the elementary geometry of straight lines and circles (and of course, planes and spheres as well).

PIERRE DE FERMAT (1601-1655)

Pierre-Simon De Fermat, son of a leather merchant, was born in August 1601, at Beaumont-de-Lomagne, France. He had his early education at home. He studied law at Toulouse, and in 1631, he was made Counciller for the parliament of Toulouse. He devoted his leisure time mostly to mathematics. Fermat is regarded as one of the discoverers of analytic geometry. His *Introduction to Loci* was not published during his life-time and therefore, the credit for discovering analytic geometry usually goes to Descartes but it is now established that he had discovered the same method as Descartes well before the appearance of Descartes' *La Geometrie.*

Fermat made important contributions to Number Theory. He proved that every prime number of the form $4n+1$ can be written in one and only one way as a sum of two squares. He claimed to have proved that there do not exist any non-zero integers x, y, z such that $x^3+y^3=z^3$. This proposition, known as Fermat's Last Theorem, has defied all attempts by the greatest of mathematicians, and remains still unproved.

CHAPTER 10

Straight Lines

10·1. INTRODUCTION

In this chapter we shall apply the methods of co-ordinate geometry to study straight lines. We shall study various forms of the equation of a straight line, such as point-slope form, slope-intercept form, intercept form etc. We shall also devote out attention to finding angle between two straight lines, point of intersection of two straight lines, distance of a point from a straight line, co-ordinates of orthocentre and circumcentre of a triangle etc. Pair of straight lines represented by the homogeneous equation of the second degree will also be studied. We shall also obtain a necessary and sufficient condition for the general equation of the second degree to represent a pair of straight lines.

10·2. VARIOUS FORMS OF THE EQUATION OF A STRAIGHT LINE

In this section we shall study various forms of the equation of a straight line.

10·2·1. Equation of the Axes of Co-ordinates

The distance of any point P on y-axis from y-axis is zero. Hence the abscissa of every point on the y-axis is zero. Also, if the abscissa of a point is zero, it must lie on the y-axis so that the equation of the y-axis is $x=0$.

Similary, observe that the ordinate of a point Q is zero if and only if the point R lies on the x-axis. Consequently, the equation of the x-axis is $y=0$.

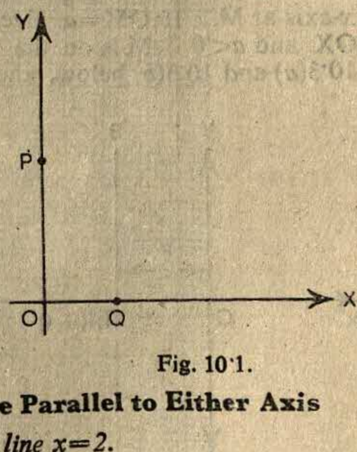


Fig. 10·1.

10·2·2. Equation of a Straight Line Parallel to Either Axis

Example 1. *Trace the straight line $x=2$.*

Solution. The line $x=2$ is the collection of those and only those points whose abscissa is 2. Let us plot some point whose abscissa is 2. There are lots of such points. ......$(2,-2)$, $(2,-1)$, $(2, 0)$, $(2, 1)$, $(2, 2)$, ...are only some of them. All these seem to be forming a pattern. The pattern is obviously a straight line parallel to the y-axis. What is the distance of any point on this line from the x-axis ? The answer is '2 units'. Hence x, the abscissa of every point on this line is 2. Thus $x=2$ for every point on this line. As already seen, any point with abscissa 2 lies on this line. Hence this is the locus whose equation is $x=2$. Drawing

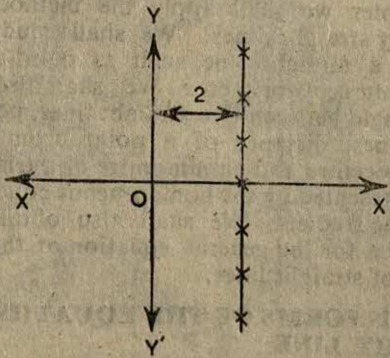


Fig. 10·2.

this line is also called tracing this line.

Let BA be a straight line parallel to the y-axis meeting the x-axis at M. If OM$=a$ algebraically (meaning, $a>0$, if M is on OX and $a<0$ if M is on the negative part of the x-axis ; see Fig. 10·3(a) and 10·3(b) below, then the abscissa of every point on BA,

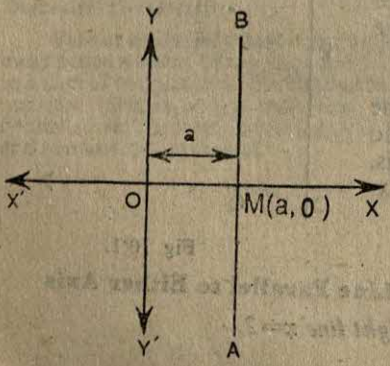


Fig. 10·3 (a).

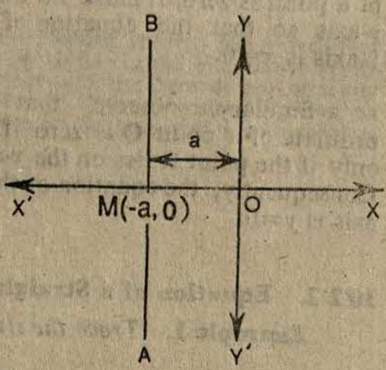


Fig. 10.3 (b).

and of no other point, is exactly a. Hence the equation of BA is $x=a$.

Conversely, every equation of the form $x=a$ represents a straight line parallel to OY because every point on the locus of $x=a$ is at the same distance (a) from the y-axis and the locus consequently must be a straight line parallel to y-axis.

Consider now a straight line CD papallel to the x-axis meeting the y-axis at N. Let ON$=b$ algebraically. Then the ordinates of all points on CD, and of no other point, are b. Hence the equation of CD is $y=b$.

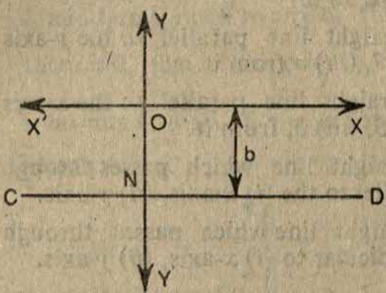


Fig. 10·4 (a).

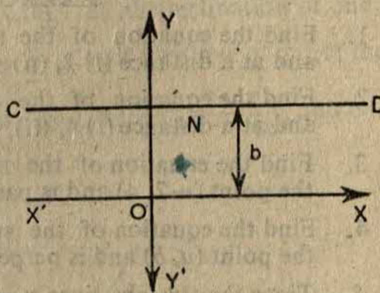


Fig. 10·4 (b).

Conversely, every equation of the form $y=b$ represents a straignt line parallel to the x-axis because every point on the locus of $y=b$ must be at the constant distance b from the x-axis.

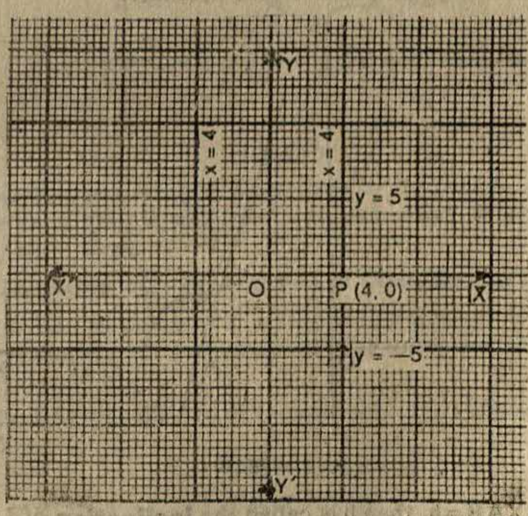


Fig. 10·5.

Example 2. *Trace the straight lines $x=4$, $y=5$, $x=-4$, $y=-5$ on squared paper.*

Solution. Take a pair of perpendicular lines X′OX, Y′OY as the axes of co-ordinates.

Plot the point P (4, 0). Through P draw a straight line parallel to Y′OY. This is the trace of the straight line $x=4$. Similarly, we may trace the remaining straight lines. The traces are as shown in Fig. 10·5.

EXERCISE 10 (*a*)

1. Find the equation of the straight line parallel to the y-axis and at a distance (*i*) 2, (*ii*) -7, (*iii*) a from it.
2. Find the equation of the straight line parallel to the x-axis and at a distance (*i*) 4, (*ii*) -5, (*iii*) b, from it.
3. Find the equation of the straight line which passes through the point $(-2, 4)$ and is parallel to the (*i*) x-axis, (*ii*) y-axis.
4. Find the equation of the straight line which passes through the point (a, b) and is perpendicular to (*i*) x-axis, (*ii*) y-axis.
5. Trace the straight lines $x=2$, $y=4$, $x=-3$, $y=-6$ on squared paper.

10·2·3. Slope of a Non-vertical Line

Let l be a line in the xy-plane meeting the x-axis at A and let B be a point on l with positive y-co-ordinate.

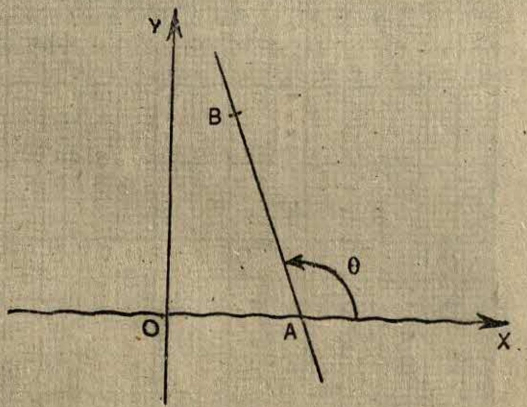


Fig. 10·6.

If $\angle XAB=\theta$, then θ is called the **inclination** of l. The inclination of a line may be an acute angle, right angle or obtuse angle.

Definition 10·1. *If* θ *is the inclination of a line, and* $\theta \neq 90°$, *then tan* θ *called the* **slope** *of the line.*

If the inclination of a line is an acute angle, the slope is positive. If the inclination of a line is 90° (*i.e.*, if the line is parallel to the y-axis), the slope is not defined. If the inclination of a line is an obtuse angle, its slope is negative.

10·2·4. Parallel and Perpendicular Lines

Let l_1 and l_2 be two perpendicular straight lines in the xy-plane. If l_1 be parallel to the x-axis, then l_2 is perpendicular to l_1 if and only if l_2 is parallel to y-axis. Let us suppose that neither of the two lines l_1 and l_2 is parallel to any of the axes. If the inclination of one of them is θ, that of the other is $\theta \pm \frac{\pi}{2}$, *i.e.*, if the slope of one of them is $m = \tan\theta$, that of the other is $\tan\left(\theta \pm \frac{\pi}{2}\right) = -\frac{1}{m}$, and conversely.

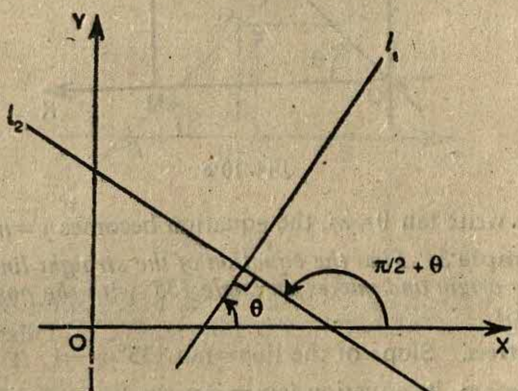


Fig. 10·7.

Therefore, *two straight lines, neither of which is parallel to any of the axes, are perpendicular to each other if and only if the product of their slopes is* -1.

10·2·5. Equation of a Straight Line through the Origin

Let R be any point on a straight line through O which makes an angle θ (measured positively in the counter-clockwise sense) with OX (Fig. 10·8).

Draw RM parallel to YO and let (x, y) be the co-ordinates of R. Then

$$OM = x,\ MR = y,$$

$$y = MR = OM \tan\theta,$$

$$= x \tan\theta.$$

Thus all points (x, y) on the line satisfy the equation

$$y = x \tan \theta.$$

Clearly, every point satisfying the above relation lies on the given line. Therefore, $y = x \tan \theta$ is the equation of the straight line OR.

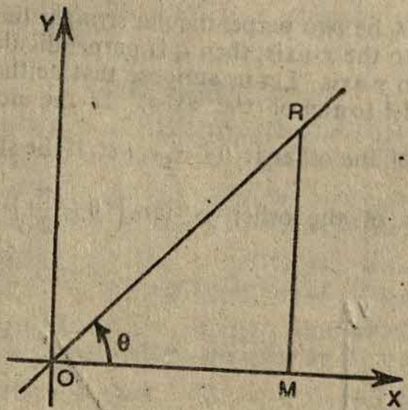


Fig. 10·8.

If we write $\tan \theta = m$, the equation becomes $y = mx$.

Example 3. *Find the equation of the straight line which passes through the origin and makes an angle 135° with the positive direction of the x-axis.*

Solution. Slope of the line $= \tan 135° = -1$.

Therefore, its equation is $y = -x$.

EXERCISE 10 (*b*)

1. Find the equation of the straight line which passes through the origin and makes an angle
   (*i*) 30°, (*ii*) 60°, (*iii*) 120°,
   (*iv*) 135°, (*v*) 150°, (*vi*) 210°,
   with the x-axis.
2. What angles do the lines
   (*i*) $y = x$, (*ii*) $y = \sqrt{3}x$, (*iii*) $y = -\sqrt{\frac{1}{3}}x$,
   make with the x-axis ?
3. Trace the lines
   (*i*) $y = -x$, (*ii*) $y = \frac{1}{2}x$,
   on squared paper.

4. Find the equations of the straight lines which bisect the angles between the co-ordinate axes.

5. Find the equations of the straight lines which trisect the angles between the co-ordinate axes.

10·9·6. Equation of a Straight Line (Slope-intercept Form)

Let AB be a straight line which makes an angle XRB$=\theta$ with OX (measured positively in the counter-clockwise direction) and cuts OY in Q. Let OQ$=c$.

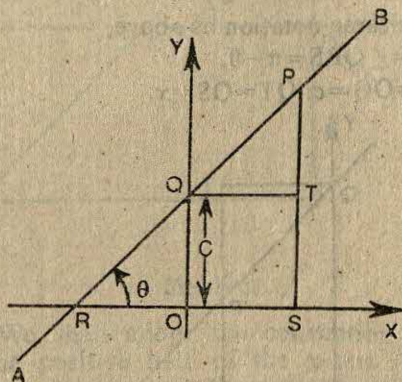


Fig. 10·9.

Let the co-ordinates of any point P on AB be (x, y). Through P draw PS parallel to YO meeting OX in S and QT parallel to OX meeting PS in T.

Since QT ∥ OX and AB meets them,

therefore, $\angle TQP=\angle XRP=\theta.$

Since PS ∥ YO and QT ∥ OX,

therefore,

$$\angle PTQ=\frac{\pi}{2}.$$

Also, $ST=OQ=c,\ QT=OS=x,$

$$y=SP=ST+TP=OQ+TP,$$
$$=c+QT\tan\theta=c+x\tan\theta.$$

Therefore, the equation of the straight line is

$$y=x\tan\theta+c, \qquad ...(i)$$

or

$$y=mx+c, \qquad ...(ii)$$

where $m=\tan\theta$ is the slope of the straight line.

Hence *the equation of the straight line whose slope is 'm' and which makes an intercept 'c' on the y-axis is*

$$y = mx + c.$$

The above form of the equation of a straight line is known as the **slope-intercept form.**

In Fig. 10·9 we have considered a case where θ is a positive acute angle and c is positive. We shall now consider case where, (*i*) θ is an obtuse angle, (*ii*) c is negative. We shall see that the equation of a straight line can be put in the same form as above even in these cases.

(*i*) In Fig. 10·10, θ is an obtuse angle.

With the same notation as above,

$\angle TQP = \angle QRS = \pi - \theta$,

$ST = OQ = c$, $QT = OS = x$.

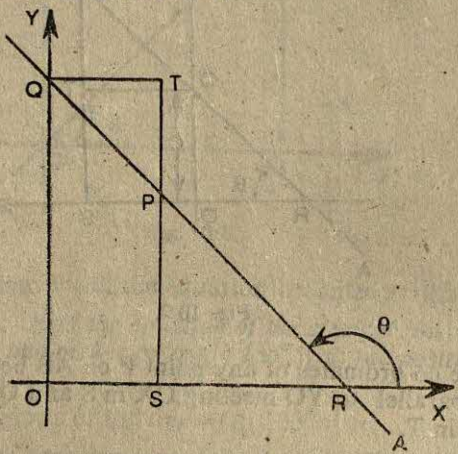


Fig. 10·10.

$$\begin{aligned} y &= SP = ST - PT, \\ &= OQ - QT \tan \angle TQP \\ &= c - x \tan(\pi - \theta), \\ &= x \tan \theta + c, \\ &= mx + c, \text{ as before.} \end{aligned}$$

(*ii*) In Fig. 10·11, OQ is on the negative side of the y-axis so that $OQ = c$ and

$\therefore \quad QO = -c.$

$$\begin{aligned} y &= SP = TP - TS, \\ &= QT \tan \angle TQP - QO, \\ &= x \tan \theta - (-c), \\ &= x \tan \theta + c. \end{aligned}$$

Now $c<0$ means : OQ is on the negative side of the y-axis and (the numerical value of) the distance OQ is $-c$.

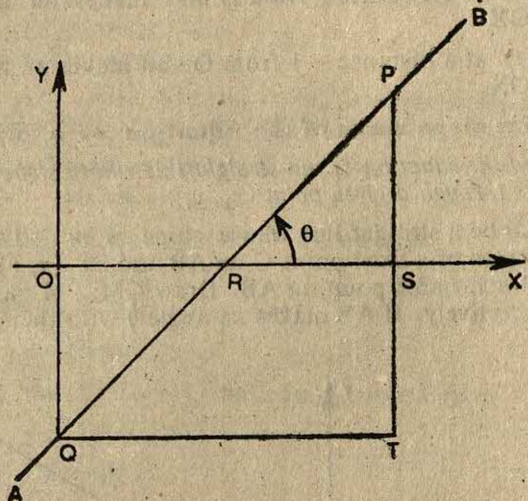


Fig. 10·11.

Remark. We shall adopt the convention that distances measured along the positive half of the x-axis (resp. y-axis) are denoted positively and those along the negative half of the x-axis (resp. y-axis) are denoted negatively.

Example 4. *Find the equation of the straight line which makes an angle of 60° with OX and makes an intercept 2 on the y-axis.*

Solution. Let the equation of the straight line be

$$y=mx+c.$$

Since $m=\tan 60°=\sqrt{3}$ and $c=2$, the required equation is

$$y=\sqrt{3}x+2.$$

Example 5. *What angle does the straight line $y=-\sqrt{3}x+1$ make with the x-axis?*

Solution. Comparing the given equation with $y=mx+c$, we have

$$m=-\sqrt{3}=\tan 120°.$$

Hence the given straight line makes an angle of 120° with the x-axis.

EXERCISE 10 (c)

Find, in each of the following cases, the equation of the straight line which :

1. cuts OY at a distance 2 from O and makes an angle of 45° with OX.

2. cuts OY at a distance 3 from O and makes an angle of 135° with OX.

3. cuts OY at a distance -1 from O and makes an angle of 135° with OX.

10·2·7. Point-slope Form of the Equation of a Straight Line

To find the equation of the straight line whose slope is 'm' and which passes through a given point (x_1, y_1).

Let AB be a straight line whose slope is 'm'. Let (x_1, y_1) be the co-ordinates of a fixed point C on AB and let (x, y) be the co-ordinates of a variable point on AB. Draw CM, PM parallel to OX and YO respectively. If AB makes an angle θ with the x-axis, then

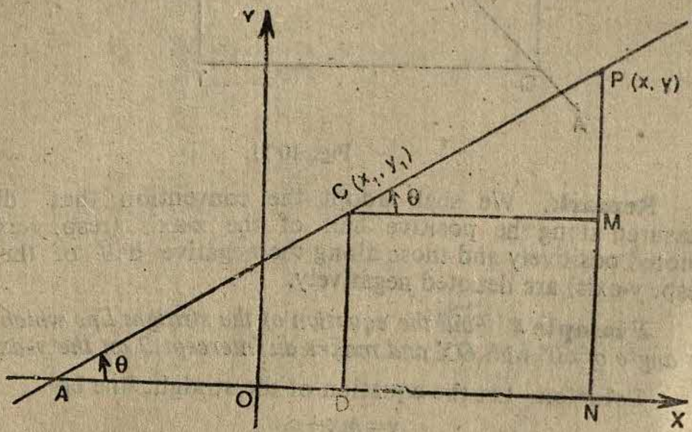


Fig. 10·12

$$\angle PCM=\theta,\ MP=CM\tan\theta.$$

But $\quad MP=y-y_1,\ CM=x-x_1.$

Thus we have,

$$y-y_1=(x-x_1)\tan\theta, \quad \ldots(i)$$

or $\quad y-y_1=m(x-x_1), \quad \ldots(ii)$

where $\quad m=\tan\theta.$

Equation (ii) is known as the **point-slope form** of the equation of a straight line.

Aliter. The equation of the straight line whose slope is 'm' and which makes an intercept 'c' on the y-axis is

$$y=mx+c. \quad \ldots(i)$$

If (*i*) passes through (x_1, y_1), we have

$$y_1 = mx_1 + c. \qquad \cdots(ii)$$

Subtracting (*ii*) from (*i*), we have

$$y - y_1 = m(x - x_1),$$

which is the required equation.

Example 6. *Find the equation of the straight line through the point (−1, −2) making an angle of 135° with the x-axis.*

Solution. Let the equation of the straight line be

$$y = mx + c. \qquad \cdots(i)$$

Since (*i*) passes through $(-1, -2)$, we have

$$-2 = m(-1) + c. \qquad \cdots(ii)$$

Subtracting (*ii*) from (*i*), we have

$$y + 2 = m(x + 1). \qquad \cdots(iii)$$

Since $m = \tan 135° = -1$, the required equation is

$$y + 2 = -(x + 1),$$

or $$x + y + 3 = 0.$$

EXERCISE 10 (*d*)

Find, in each of the following cases, the equation of the straight line which :

1. passes through the point (1, 2) and makes an angle of 45° with OX.
2. passes through the point (−2, 3) and makes an angle of 60° with OX.
3. passes through the point (3, −2) and makes an angle of 120° with OX.
4. passes through the point (−4, 0) and makes an angle of 150° with OX.
5. passes through the point (0, −1) and makes an angle of 135° with OX.
6. passes through the point (p, q) and makes an angle $\frac{\pi}{2} + \alpha$ with OX.

10·2·8. Two-point Form of the Equation of a Straight Line

To find the equation of the straight line which passes through two given points (x_1, y_1) *and* (x_2, y_2).

Let A, B be the two given points (x_1, y_1), (x_2, y_2) and P any point (x, y) on the line. Draw AM, BN parallel to OX, and AN,

PM parallel to YO, thus forming two right-angled triangles ABN and PAM. Since BN ∥ AM and PB cuts them,

$$\angle ABN = \angle PAM \text{ (corres. angles).}$$

It follows that $\triangle$s ABN and PAM are equiangular and hence similar.

Therefore, $\dfrac{MP}{NA} = \dfrac{AM}{BN}$.

But $AM = x - x_1$, $MP = y - y_1$,

$BN = x_1 - x_2$, $NA = y_1 - y_2$.

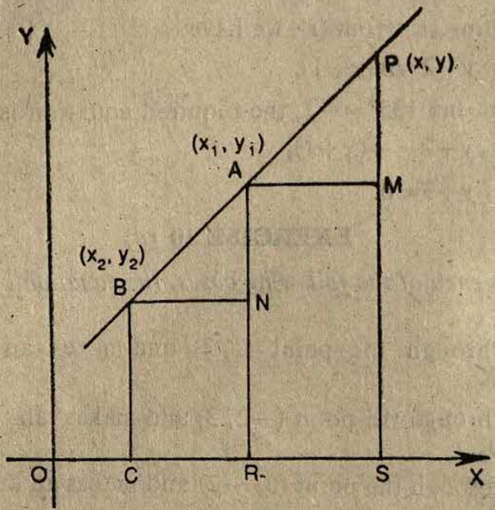


Fig. 10·13.

Therefore, $\dfrac{y-y_1}{y_1-y_2} = \dfrac{x-x_1}{x_1-x_2}$,

or $\dfrac{y-y_1}{y_2-y_1} = \dfrac{x-x_1}{x_2-x_1}$,

is the required equation.

Aliter. Let the required equation be

$$y = mx + c, \qquad ...(i)$$

Since the straight line (*i*) passes through the points (x_1, y_1) and (x_2, y_2), we have

$$y_1 = mx_1 + c, \qquad ...(ii)$$
$$y_2 = mx_2 + c. \qquad ...(iii)$$

Subtracting (*ii*) from (*i*) and (*iii*), we have

$$y-y_1=m(x-x_1), \qquad \text{...}(iv)$$

$$y_2-y_1=m(x_2-x_1). \qquad \text{...}(v)$$

Dividing (*iv*) by (*v*), we have

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1},$$

which is the required equation.

Corollary. From (*v*), we have

$$m=\frac{y_2-y_1}{x_2-x_1}.$$

Hence the slope of the straight line joining the points (x_1, y_1) and (x_2, y_2) is $(y_2-y_1)/(x_2-x_1)$.

Aliter. Since the points (x, y), (x_1, y_1), (x_2, y_2) are in a straight line, the area of the triangle formed by them is zero.

Therefore, $\frac{1}{2}\{x(y_1-y_2)+x_1(y_2-y)+x_2(y-y_1)\}=0$,

or $x(y_1-y_2)-x_1(y-y_1+y_1-y_2)+x_2(y-y_1)=0$,

or $(x-x_1)(y_1-y_2)+(x_2-x_1)(y-y_1)=0$,

or $(x_2-x_1)(y-y_1)=(x-x_1)(y_2-y_1)$,

or $$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}.$$

which is the required equation.

Example 7. *Find the equation of the straight line passing through the points (1, 2) and (5, 7).*

Solution. Since the equation of the straight line joining the points (x_1, y_1) and (x_2, y_2) is

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1},$$

therefore, the equation of the straight line joining the points (1, 2) and (5, 7) is

$$\frac{y-2}{7-2}=\frac{x-1}{5-1},$$

or $$\frac{y-2}{5}=\frac{x-1}{4},$$

or $4(y-2)=5(x-1)$,

or $5x-4y+3=0$.

Example 8. *The vertices of a triangle are A(2, 3), B(−4, 5), C(6, −7). Find the equation of the median through A.*

Solution. The co-ordinates of the mid-point D of BC are

$$\left(\frac{-4+6}{2}, \frac{5-7}{2}\right), \textit{i.e.}, (1, -1).$$

The equation of the straight line passing through the points A(2, 3) and D(1, −1) is

$$\frac{y-3}{-1-3}=\frac{x-2}{1-2},$$

or
$$\frac{y-3}{-4}=\frac{x-2}{-1},$$

or
$$y-3=4(x-2),$$

or
$$4x-y-5=0.$$

Example 9. *Prove that the points (1, 4), (3, −2) and (−3, 16) are in a straight line and find the equation of the straight line through them.*

Solution. We shall find the equation of the straight line joining the points (1, 4) and (3, −2), and then show that the point (−3, 16) lies on this straight line.

The equation of the straight line joining the points (1, 4) and (3, −2) is

$$\frac{y-4}{-2-4}=\frac{x-1}{3-1},$$

or
$$3x+y-7=0. \qquad ...(i)$$

Putting $x=-3$, $y=16$ in (i) we find that the left-hand side of (i) becomes zero. Therefore, (−3, 16) lies on the straight line (i). Hence the given points are collinear and the equation of the straight line through them is

$$3x+y-7=0.$$

Example 10. *In what ratio is the line joining (1, 2) and (4, 3) divided by the line joining (2, 3) and (4, 1) ?*

Solution. The equation of the straight line joining the points (2, 3) and (4, 1) is

$$\frac{y-3}{1-3}=\frac{x-2}{4-2},$$

or
$$x+y-5=0. \qquad ...(i)$$

Let the line joining (1, 2) and (4, 3) meet the line (i) at the point R and let R divide the join of (1, 2) and (4, 3) in the ratio $k:1$.

Since R divides the join of (1, 2) and (4, 3) in the ratio $k:1$, the co-ordinates of R are

$$\left(\frac{4k+1}{k+1}, \frac{3k+2}{k+1}\right).$$

Since R lies on (i), we have

$$\frac{4k+1}{k+1}+\frac{3k+2}{k+1}-5=0,$$

or $$(4k+1)+(3k+2)-5(k+1)=0,$$

or $$k=1.$$

Hence R divides the join of (1, 2) and (4, 3) in the ratio 1 : 1, *i.e.*, the join of (1, 2) and (4, 3) is bisected by the join of (2, 3) and (4, 1).

EXERCISE 10 (*e*)

Find the equations to the straight lines passing through the following pairs of points :

1. $(0, 0)$ and $(3, -3)$.
2. $(4, 5)$ and $(6, 7)$.
3. $(-1, -2)$ and $(-4, -5)$.
4. $(7, 1)$ and $(5, -2)$.
5. $(a, 0)$ and $(0, b)$.
6. $(0, a)$ and $(-b, 0)$.
7. $(am_1^2, 2am_1)$ and $(am_2^2, 2am_2)$.
8. $\left(am_1, \dfrac{a}{m_1}\right)$ and $\left(am_2, \dfrac{a}{m_2}\right)$.
9. $(a \cos \theta_1, a \sin \theta_1)$ and $(a \cos \theta_2, a \sin \theta_2)$.
10. $(a \cos \theta_1, b \sin \theta_1)$ and $(a \cos \theta_2, b \sin \theta_2)$.

Find the equations to the sides of the triangles the co-ordinates of whose vertices are respectively :

11. $(1, 3)$, $(-7, 6)$ and $(5, -1)$.
12. $(0, 4)$, $(3, 6)$ and $(-8, -2)$.
13. $(5, 2)$, $(-9, -3)$ and $(-3, -5)$.
14. $(1, 2)$, $(-2, -1)$ and $(3, -2)$.
15. The vertices of a triangle are $(-2, -3)$, $(1, -2)$ and $(3, 4)$. Find the equations of the medians.
16. Find the equations of the medians of a triangle, the co-ordinates of whose vertices are $(-1, 6)$, $(-3, -9)$ and $(5, -8)$.
17. Find the equation of the straight line which bisects the join of $(1, 2)$ and $(3, -4)$ and also bisects the join of $(3, -7)$ and $(5, -3)$.
18. Prove that the points $(0, -1)$, $(1, 1)$ and $(3, 7)$ lie on a straight line and find the equation of the straight line through them.
19. Prove that the points $(3, 7)$, $(6, 5)$ and $(15, -1)$ are collinear and find the equation of the straight line joining them.

20. Find the ratio in which the join of (1, 2) and (3, 4) is divided by the join of (−5, −1) and (9, 7).

10·2·9. Distance Form of the Equation of a Straight Line

To obtain the equation of a straight line through the point (x_1, y_1) *and making an angle* θ *with the x-axis in the form*

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r.$$

Let a straight line pass through a given point $A(x_1, y_1)$ and make an angle θ with OX.

Let P be a point (x, y) on the line at a distance r from A. Draw AM, PM parallel to OX and YO respectively. Then

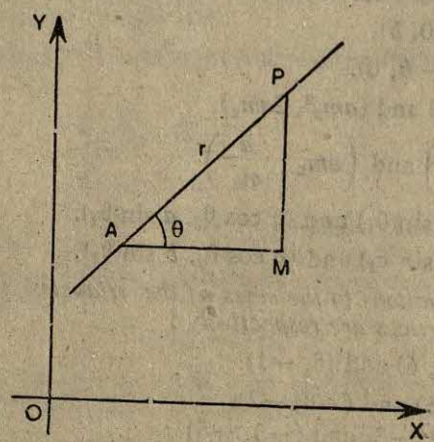


Fig. 10·14.

$$AM=AP\cos\theta=r\cos\theta,$$
$$MP=AP\sin\theta=r\sin\theta.$$

But $AM=x-x_1$ and $MP=y-y_1$,

therefore, $x-x_1=r\cos\theta$,

and $y-y_1=r\sin\theta$.

Thus we have

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r \qquad ...(i)$$

as the equation of the straight line.

Corollary. From (i), we have

$$x=x_1+r\cos\theta,$$
$$y=y_1+r\sin\theta. \qquad ...(ii)$$

The equations (*ii*) give the co-ordinates of any point on the line in terms of its distance r from the point (x_1, y_1). It is on account of this fact that the above equation is known as the **distance form** of the equation of a straight line.

Notes. 1. Equation (*i*) is also called the **'symmetrical form'** of the equation of a straight line.

2. Since (*ii*) expresses the co-ordinates of any point on the given line in terms of a parameter 'r', therefore, it is also called the **parametric form** of the equation of a straight line.

Example 11. Find the co-ordinates of the points each of which is at a distance $\sqrt{2}$ units from the point (1, 2) and lies on the straight line through this point and inclined at an angle of 45° with the x-axis.

Solution. The distance form of the equation of the straight line which passes through (1, 2) and makes an angle of 45° with the x-axis is

$$\frac{x-1}{\cos 45°}=\frac{y-2}{\sin 45°}=r,$$

or
$$\frac{x-1}{1/\sqrt{2}}=\frac{y-2}{1/\sqrt{2}}=r.$$

When $r=\sqrt{2}, x=2, y=3.$

When $r=-\sqrt{2}, x=0, y=1.$

Hence the required points are (2, 3) and (0, 1).

EXERCISE 10 (*f*)

1. Obtain the equation of the straight line through the point (−2, −1) and inclined at an angle of 60° with the x-axis in the distance form.
2. Obtain the parametric equations of the straight line through the point (−4, 0) and inclined at an angle of 135° with the x-axis.
3. Obtain, in symmetrical form, the equation of the straight line through the point (0, −2) and inclined at an angle of 30° with the x-axis.
4. Find the co-ordinates of the points each of which is at a distance 3 units from the point (−2, −1) and lies on the straight line through the point and inclined at an angle of 60° with the x-axis.
5. Find the co-ordinates of the points each of which is at a distance $2\sqrt{2}$ units from the point (1, 0) and lies on the straight line through the point and inclined at an angle of 135° with the x-axis.

10·2·9. Equation of a Straight Line in Terms of the Intercepts it Makes on the Co-ordinate Axes (Intercept Form)

Let AB be a straight line which cuts OX, OY in A, B respectively so that OA$=a$, OB$=b$. Let the co-ordinates of any point P on the straight line be (x, y).

Through P draw PM parallel to YO. Then

$$MP=y,\ OM=x,\ MA=OA-OM=a-x.$$

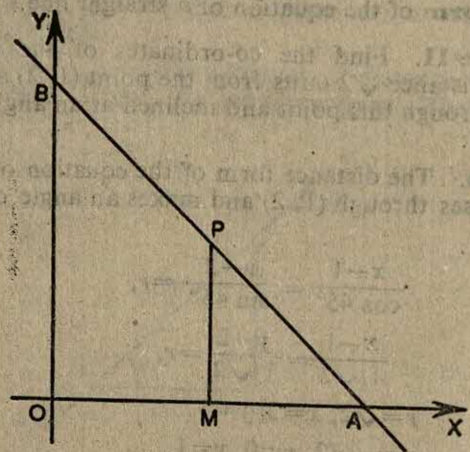


Fig. 10·15.

The triangles AMP and AOB are similar.

Therefore, $$\frac{MA}{OA}=\frac{MP}{OB},$$

or $$\frac{a-x}{a}=\frac{y}{b},$$

or $$1-\frac{x}{a}=\frac{y}{a},$$

or $$\frac{x}{a}+\frac{y}{b}=1.$$

Aliter. The co-ordinates of A are $(a, 0)$ and the co-ordinates of B are $(0, b)$. Since A, P, B are in a straight line, the area of the triangle APB is zero.

Therefore, $$\tfrac{1}{2}[ay+bx-ab]=0,$$

or $$ay+bx=ab,$$

or $$\frac{x}{a}+\frac{y}{b}=1.$$

Aliter. Through P draw PM || YO and PN || XO. Join PO.
Then $MP=y$, $NP=x$.

$$\triangle POA=\tfrac{1}{2}OA.MP=\tfrac{1}{2}ay,\ \triangle BOP=\tfrac{1}{2}OB.NP=\tfrac{1}{2}bx.$$

$$\triangle BOA=\tfrac{1}{2}OA.OB=\tfrac{1}{2}ab.$$

Since $\triangle BOA=\triangle BOP+\triangle POA$,

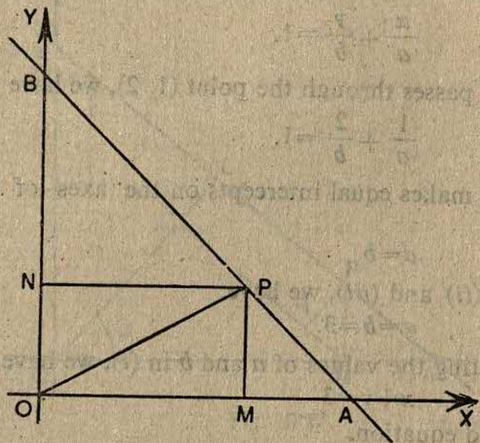


Fig. 10·16.

therefore,

$$\tfrac{1}{2}ab=\tfrac{1}{2}bx+\tfrac{1}{2}ay,$$

or $$\frac{x}{a}+\frac{y}{b}=1.$$

The above form of the equation of a straight line is called the **intercept form.**

Figs. 10·15 and 10·16 have been drawn for the case when the intercepts on both the axes are positive. By drawing figures for the cases when the intercepts are not all positive, it can be seen that the result holds for all cases.

Example 12. *Find the equation of the straight line which makes intercepts —6, 4 on the axes of co-ordinates.*

Solution. Putting $a=-6$, $b=4$ in the equation

$$\frac{x}{a}+\frac{y}{b}=1,$$

we have $$\frac{x}{-6}+\frac{y}{4}=1,$$

or $$2x-3y+12=0,$$

which is the required equation.

Example 13. *Find the equation to the straight line which passes through the point (1, 2) and makes equal intercepts on the axes of co-ordinates.*

Solution. Let the equation to the straight line be

$$\frac{x}{a}+\frac{y}{b}=1. \qquad ...(i)$$

Since (*i*) passes through the point (1, 2), we have

$$\frac{1}{a}+\frac{2}{b}=1. \qquad ...(ii)$$

Since (*i*) makes equal intercepts on the axes of co-ordinates, therefore,

$$a=b \qquad ...(iii)$$

Solving (*ii*) and (*iii*), we have

$$a=b=3. \qquad ...(iv)$$

Substituting the values of a and b in (*i*), we have

$$x+y=3$$

as the required equation.

EXERCISE 10 (g)

Find the equation to the straight line :

1. cutting off intercepts 4 and 5 from the axes.
2. cutting off intercepts —3 and 2 from the axes.
3. cutting off intercepts —2 and —6 from the axes.
4. cutting off intercepts a and $-a$ from the axes.
5. Find the equation of the straight line which passes through the point (—1, —2) and cuts off equal intercept on the axes.
6. Find the equation of the straight line which passes through the point (—4, 1) and has intercepts on the axes equal in magnitude but opposite in sign.

10·2·10. Perpendicular Form of the Equation of a Straight Line

Equation of a straight line in terms of the length of the perpendicular upon it from the origin and the angle which that perpendicular makes with the x-axis.

Let OL$=p$ be the perpendicular from the origin to the given straight line and let OL make an angle LOX$=\alpha$ with OX.

Let P (x, y) be any point on the straight line. Draw the ordinate PM, MN perpendicular to OL, and PR perpendicular to MN.

Since MN and PL are both perpendicular to OL, therefore, MN $\parallel$ PL. Since MN is perpendicular to both RP and NL, therefore, NL $\parallel$ RP. Therefore, NLPR is a rectangle, so that NL=RP.

Also, $\angle PMR = 90° - \angle RMO = 90° - (90° - \alpha) = \alpha$.

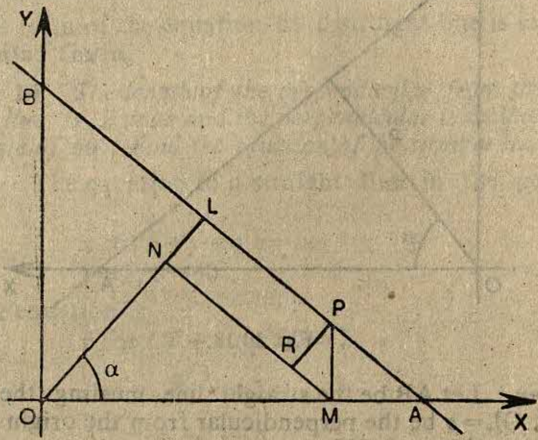


Fig. 10·17.

Now $\quad p = OL = ON + NL,$

$= ON + RP,$

$= OM \cos \angle MON + MP \sin \angle PMR,$

$= x \cos \alpha + y \sin \alpha.$

Hence $x \cos \alpha + y \sin \alpha = p$, is the required equation.

Notes. 1. p is always measured positively and α is measured positively from 0° to 360°.

2. The reader may verify by drawing figures with the line OL falling in other quadrants and see that the equation holds in every case.

Aliter. If the straight line makes intercepts OA, OB on the axes, then

$$OA = p \sec \alpha,$$
$$OB = p \operatorname{cosec} \alpha.$$

The equation of the straight line which makes intercepts $p \sec \alpha, p \operatorname{cosec} \alpha$ on the axes is

$$\frac{x}{p \sec \alpha} + \frac{y}{p \operatorname{cosec} \alpha} = 1,$$

or $$x \cos \alpha + y \sin \alpha = p,$$
which is the required equation.

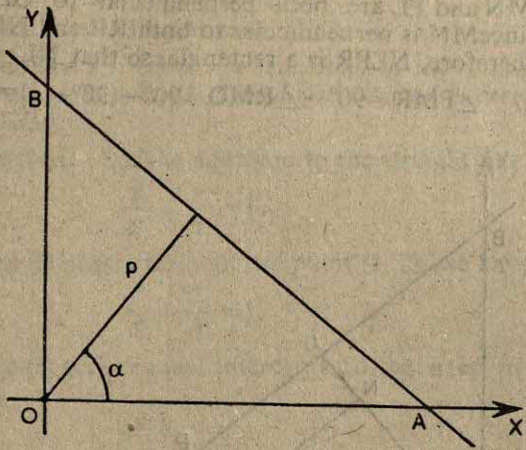


Fig. 10·18.

Aliter. Let AB be the straight line meeting the axes in A and B. Let OL$=p$ be the perpendicular from the origin to the given straight line and let $\angle$LOX$=\alpha$.

Let P(x, y) be any point on the straight line. Draw PM perpendicular on OX.

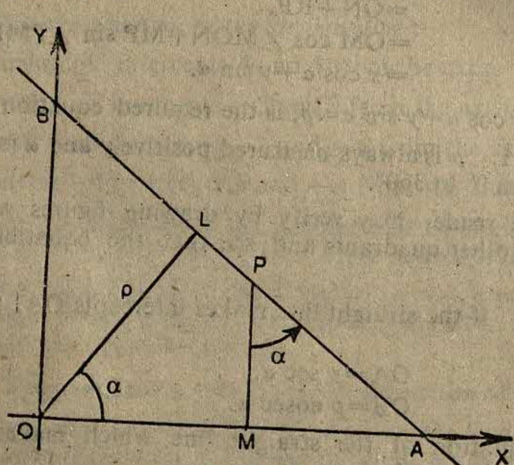


Fig. 10·19.

Since $$AB \perp OL, \; MP \perp OX,$$
therefore, $$\angle APM = \angle XOL = \alpha.$$

Now $\qquad p=OL,$

$$=OA \cos \alpha,$$
$$=(OM+MA) \cos \alpha,$$
$$=(x+MP \tan \alpha) \cos \alpha,$$
$$=(x+y \tan \alpha) \cos \alpha,$$
$$=x \cos \alpha+y \sin \alpha.$$

Hence $p=x \cos \alpha+y \sin \alpha$ is the required equation.

The above form of the equation of a straight line is known as the **perpendicular form.**

Example 14. *The length of the perpendicular from the origin on a straight line is 2 units and the perpendicular is inclined to the x-axis at an angle of 60°. Find the equation of the straight line.*

Solution. The equation to a straight line in the perpendicular form is

$$x \cos \alpha+y \sin \alpha=p.$$

Here $\qquad p=2, \alpha=60°.$

Hence the equation is

$$x \cos 60°+y \sin 60°=2,$$

or
$$x \cdot \frac{1}{2}+y \cdot \frac{\sqrt{3}}{2}=2,$$

or
$$x+y\sqrt{3}=4.$$

EXERCISE 10 (*h*)

1. The length of the perpendicular from the origin on a straight line is 3 units and the perpendicular is inclined to the x-axis at an angle of 45°. Find the equation to the straight line.
2. The length of the perpendicular from the origin on a straight line is 4 units and the perpendicular is inclined to the x-axis at an angle of 135°. Find the equation to the straight line.
3. The length of the perpendicular from the origin on a straight line is 2 units and the perpendicular is inclined to the x-axis at an angle of 120°. Find the equation of the straight line.
4. The length of the perpendicular from the origin on a straight line is 3 units and the perpendicular is inclined to the x-axis at an angle of 150°. Find the equation to the straight line.

10·3. THE GENERAL FORM OF THE EQUATION OF A STRAIGHT LINE

In the last section we obtained the equation of a straight line in various forms. We found that the equation of a straight line always turned out to be of the first degree. We shall now prove the converse, namely *every equation of the first degree represents a straight line.*

Let $$Ax+By+C=0 \qquad ...(i)$$
be an equation of the first degree and let (x_1, y_1), (x_2, y_2) be two points P, Q on the locus of (i), so that

$$Ax_1+By_1+C=0,$$
$$Ax_2+By_2+C=0. \qquad ...(ii)$$

A point R which divides the join of P and Q in the ratio $k:1$ has co-ordinates

$$\frac{x_1+kx_2}{1+k}, \frac{y_1+ky_2}{1+k}.$$

This point lies on (i) provided

$$\frac{A(x_1+kx_2)}{1+k}+\frac{B(y_1+ky_2)}{1+k}+C=0,$$

or $$A(x_1+kx_2)+B(y_1+ky_2)+C(1+k)=0,$$

or $$(Ax_1+By_1+C)+k(Ax_2+By_2+C)=0. \qquad ...(iii)$$

Because of (ii), the condition (iii) is satisfied for all values of k. Therefore, if any two points P, Q are taken on the locus, then every point R on the straight line PQ is also on the locus. Hence the locus is a straight line.

Aliter. Let the equation be

$$Ax+By+C=0. \qquad ...(i)$$

Let $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ be any three points on the locus of (i). Then, we must have

$$Ax_1+By_1+C=0, \qquad ...(ii)$$
$$Ax_2+By_2+C=0, \qquad ...(iii)$$
$$Ax_3+By_3+C=0. \qquad ...(iv)$$

Solving the equations (ii) and (iii), we have

$$\frac{A}{y_1-y_2}=\frac{B}{x_2-x_1}=\frac{C}{x_1y_2-x_2y_1}=k, \text{ (say)}$$

or $$A=k(y_1-y_2),\ B=k(x_2-x_1),\ C=k(x_1y_2-x_2y_1).$$

Substituting the values of A, B, C in (iv), we have

$$k(y_1-y_2)\,x_3+k(x_2-x_1)y_3+k(x_1y_2-x_2y_1)=0,$$

or $$k\{x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\}=0,$$

or $$x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)=0.$$

Therefore, the points P, Q, R are in a straight line. Since P, Q, R are any three points on the locus, it follows that the locus is a straight line.

10·3·1. Reduction of the equation Ax+By+C=0 to various forms

We have obtained several different forms for the equation of a straight line, *e.g.*,

(*i*) $Ax+By+C=0$,

(*ii*) $y=mx+c$,

(*iii*) $x/a+y/b=1$,

(*iv*) $x \cos \alpha+y \sin \alpha-p=0$.

We shall show that all these forms are equivalent. In order to do this, we shall show that (*i*) can be reduced to any one of the forms (*ii*), (*iii*) or (*iv*).

(*i*) **Reduction of the equation Ax+By+C=0 to slope-intercept form**

Let $B \neq 0$. The equation

$$Ax+By+C=0$$

may then be written as

$$y=-\frac{A}{B}x+\frac{C}{-B}, \qquad \ldots(i)$$

which is of the form $y=mx+c$. From (*i*) we see that

$$Ax+By+C=0$$

has its slope equal to $-\dfrac{A}{B}$ and that it makes an intercept $-\dfrac{C}{B}$ on the y-axis.

If $B=0$, the equation may be written as $x=-\dfrac{C}{A}$, which represents a straight line parallel to the y-axis and at a distance $-\dfrac{C}{A}$ from it.

Example 15. *Find the equation of the straight line joining the points (1, 1) and (−4, 3) and reduce it to point-slope form.*

Solution. The equation of the straight line joining the points (1, 1) and (−4, 3) is

$$\frac{y-1}{3-1}=\frac{x-1}{-4-1},$$

or
$$\frac{y-1}{2}=\frac{x-1}{-5},$$

or
$$2x+5y-7=0,$$

or
$$5y=-2x+7,$$

or
$$y=-\frac{2}{5}x+\frac{7}{5},$$

which is the required equation.

(*ii*) **Reduction of the equation Ax+By+C=0 to intercept form**

Let $C \neq 0$. The equation

$$Ax+By+C=0, \qquad \ldots(i)$$

may then be written as

$$Ax+By=-C,$$

or
$$\frac{A}{-C}x+\frac{B}{-C}y=1,$$

or
$$\frac{x}{-C/A}+\frac{y}{-C/B}=1 \text{ (provided } A\neq 0,\ B\neq 0) \ldots(ii)$$

which is of the form

$$\frac{x}{a}+\frac{y}{b}=1. \qquad \ldots(iii)$$

Comparing (*ii*) and (*iii*) we find that if none of A, B, and C is zero, then

$$Ax+By+C=0$$

represent a straight line that makes intercepts $\frac{C}{-A}$, $\frac{C}{-B}$ on the co-ordinate axes.

Note. If $C\neq 0$, $A=0$, we find from (*i*) that the equation represents a straight line parallel to the x-axis and at a distance $-\frac{C}{B}$ from it.

If $C \neq 0$, $B=0$, the equation represents a straight line parallel to the y-axis and at a distance $-\frac{C}{A}$ from it.

If $C=0$, the equation represents a straight line through the origin.

Example 16. *Find the equation of the straight line that passes through the point* $(-5, -7)$ *and has slope* $-\frac{1}{2}$. *Reduce it to intercept form.*

Solution. The equation of the straight line which passes through the point $(-5, -7)$ and whose slope is $-\frac{1}{2}$, is

$$y+7=-\tfrac{1}{2}(x+5),$$

or
$$x+2y+19=0,$$

or
$$x+2y=-19,$$

or
$$\frac{x}{-19}+\frac{2y}{-19}=1,$$

or
$$\frac{x}{-19}+\frac{y}{-19/2}=1,$$

which is the required equation.

Example 17. *Reduce the equation $3x-2y=6$ to intercept form and trace it.*

Solution. Dividing the equation

$$3x-2y=6,$$

throughout by 6, we have

$$\frac{x}{2}+\frac{y}{-3}=1.$$

Therefore, the given straight line makes an intercept 2 units on x-axis and -3 units on the y-axis.

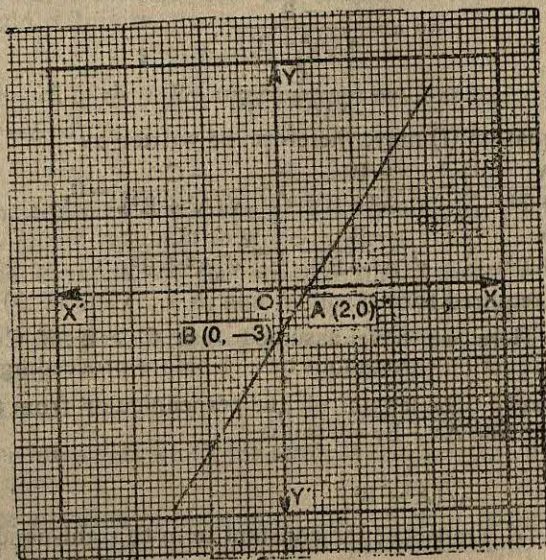


Fig. 10·20.

On squared paper, take a pair of perpendicular lines X'OX and Y'OY as the axes of co-ordinates. Plot the points A (2, 0) and B(0, −3). Join the points A, B by a straight line. AB is the required straight line.

(*iii*) **Reduction of the equation Ax+By+C=0 to perpendicular form**

If the length of the perpendicular from the origin on

$$Ax+By+C=0 \quad \cdots(i)$$

be p and if α be the angle which the perpendicular makes with the x-axis, then (i) must be the same as

$$x\cos\alpha+y\sin\alpha-p=0. \quad \ldots(ii)$$

Comparing (*i*) and (*ii*), we have

$$\frac{\cos\alpha}{A}=\frac{\sin\alpha}{B}=-\frac{p}{C},$$

i.e.,

$$\frac{p}{C}=\frac{\cos\alpha}{-A}=\frac{\sin\alpha}{-B}$$

$$=\frac{\sqrt{(\cos^2\alpha+\sin^2\alpha)}}{\sqrt{(A^2+B^2)}}=\frac{1}{\sqrt{(A^2+B^2)}}.$$

Hence

$$\cos\alpha=\frac{-A}{\sqrt{(A^2+B^2)}},$$

$$\sin\alpha=\frac{-B}{\sqrt{(A^2+B^2)}},$$

and

$$p=\frac{C}{\sqrt{(A^2+B^2)}}.$$

The equation (*i*) may, therefore, be written as

$$\frac{-A}{\sqrt{(A^2+B^2)}}x+\frac{-B}{\sqrt{(A^2+B^2)}}y+\frac{-C}{\sqrt{(A^2+B^2)}}=0, \qquad ...(iii)$$

or

$$\frac{-A}{\sqrt{(A^2+B^2)}}x+\frac{-B}{\sqrt{(A^2+B^2)}}y=\frac{C}{\sqrt{(A^2+B^2)}},$$

which is the required form.

Note. Here we have assumed that $C>0$; if $C<0$, we multiply the equation throughout by -1 before proceeding further.

Corollary. The length of the perpendicular from the origin on the straight line $Ax+By+C=0$ is

$$\frac{|C|}{\sqrt{(A^2+B^2)}}.$$

Rule. (*i*) Transpose the constant term to right-hand side.

(*ii*) Make the constant term positive if it is not already so (this may be done by multiplying throughout by -1, if necessary).

(*iii*) Divide throughout by

$$\sqrt{\{(\text{coeff. of } x)^2+(\text{coeff. of } y)^2\}}.$$

Example 18. *Reduce the equation*

$$x+\sqrt{3}y+4=0$$

to perpendicular form and hence find the length of the perpendicular from the origin on the straight line.

Solution. Transposing the constant term to right-hand side, the equation

$$x+\sqrt{3}y+4=0,$$

may be written as

$$x+\sqrt{3}y=-4. \qquad ...(i)$$

Multiplying throughout by -1, we have

$$-x-\sqrt{3}y=4. \qquad \ldots(ii)$$

Dividing throughout by

$\sqrt{\{(-1)^2+(-\sqrt{3})^2\}}$, *i.e.*, by 2, we have

$$-\frac{1}{2}x+-\frac{\sqrt{3}}{2}y=2,$$

or
$$x\cos\frac{4\pi}{3}+y\sin\frac{4\pi}{3}=2, \qquad \ldots(iii)$$

which is the required form.

Since $p=2$, it follows that the length of the perpendicular from the origin is 2 units.

EXERCISE 10 (*i*)

Reduce each of the following equations to slope-intercept form :

1. $2x+3y-6=0$.
2. $x-2y+5=0$.
3. $4x-5y+1=0$.
4. $3x+7y-4=0$.
5. Find the equation of the straight line joining the points $(2, -1)$ and $(1, 7)$ and reduce it to slope-intercept form.
6. Find the equation of the straight line passing through the points $(4, 3)$ and $(1, -6)$ and reduce it to intercept form.

Reduce each of the following equations to intercept form :

7. $3x-y+4=0$.
8. $x-2y=1$.
9. $y=2x+7$.
10. $bx+ay=ab$.
11. Find the intercepts which the straight line $3x+2y-6=0$ makes on the axes.
12. Find the equation of the straight line which is inclined to the x-axis at an angle of $60°$ and passes through the point $(-4, -3)$. Reduce it to intercept form.

Reduce each of the following equations to perpendicular form :

13. $x+y\sqrt{3}+5=0$.
14. $\sqrt{3}x-y-1=0$.
15. Reduce the equation $5x+12y-6=0$ to perpendicular form and hence find out the length of the perpendicular from the origin on the straight line.

10·4. INTERSECTION OF STRAIGHT LINES

10·4·1. Point of Intersection of Two Straight Lines

Let the equations of two straight lines AB and CD be

$$ax+by+c=0, \qquad \ldots(i)$$

$$a'x+b'y+c'=0, \qquad \ldots(ii)$$

respectively.

Let P be the point of intersection of AB and CD, and let the co-ordinates of P be (x_1, y_1).

Since P lies on both the straight lines, its co-ordinates must satisfy the equations (*i*) and (*ii*). Therefore,

$$ax_1 + by_1 + c = 0, \quad ...(iii)$$
$$a'x_1 + b'y_1 + c' = 0. \quad ...(iv)$$

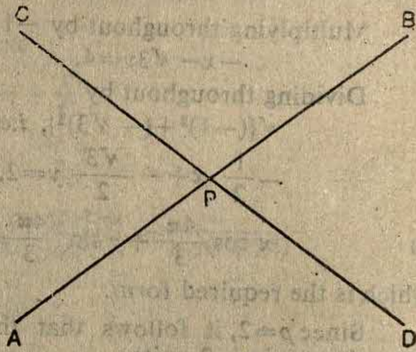


Fig. 10·21.

Solving equations (*iii*) and (*iv*) by the cross-multiplication method, we have

$$\frac{x_1}{bc'-b'c} = \frac{y_1}{ca'-c'a} = \frac{1}{ab'-a'b}.$$

Therefore,

$$x_1 = \frac{bc'-b'c}{ab'-a'b}, \; y_1 = \frac{ca'-c'a}{ab'-a'b},$$

are the co-ordinates of the point of intersection of the given straight lines.

Example 19. *Find the point of intersection of the straight lines $3x+4y-11=0$ and $x-5y+9=0$.*

Solution. Let (x_1, y_1) be the co-ordinates of the point of intersection of the given straight lines. Then (x_1, y_1) must satisfy the equations of both the straight lines.

Therefore,

$$3x_1+4y_1-11=0, \quad ...(i)$$
$$x_1-5y_1+9=0. \quad ...(ii)$$

By the cross-multiplication method, we have

$$\frac{x_1}{4.9-(-11)(-5)} = \frac{y_1}{-11.1-3.9} = \frac{1}{3(-5)-4.1},$$

or

$$\frac{x_1}{-19} = \frac{y_1}{-38} = \frac{1}{-19}.$$

Therefore, $\quad x_1=1, y_1=2.$

Hence the co-ordinates of the point of intersection are (1, 2).

10·4·2. Consistent, Inconsistent and Dependent Equations

Let us consider the system of equations

$$a_1x+b_1y+c_1=0 \; (a_1, b_1 \text{ not both } 0), \quad ...(i)$$
$$a_2x+b_2y+c_2=0 \; (a_2, b_2 \text{ not both } 0). \quad ...(ii)$$

If $A=\{(x, y) : a_1x+b_1y+c_1=0\}$,
$B=\{(x, y) : a_2x+b_2y+c_2=0\}$

denote the solution-sets of (*i*) and (*ii*) respectively, what can be said about $A \cap B$?

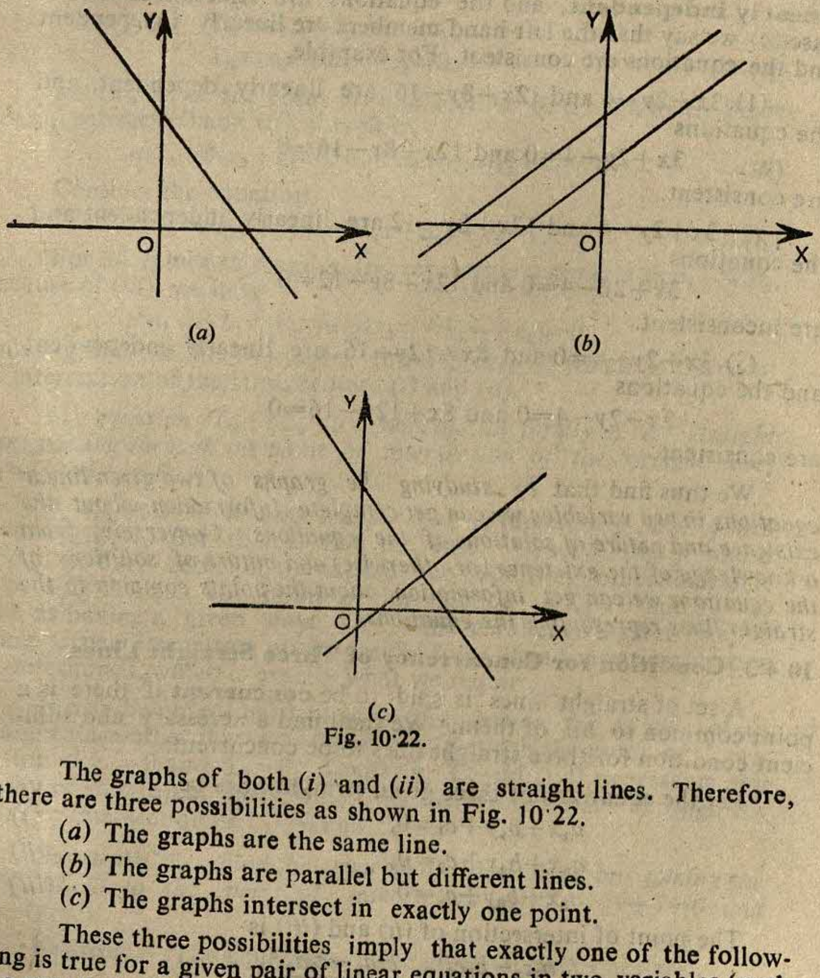


Fig. 10·22.

The graphs of both (*i*) and (*ii*) are straight lines. Therefore, there are three possibilities as shown in Fig. 10·22.

(*a*) The graphs are the same line.

(*b*) The graphs are parallel but different lines.

(*c*) The graphs intersect in exactly one point.

These three possibilities imply that exactly one of the following is true for a given pair of linear equations in two variables (such as equations (*i*) and (*ii*) above).

(*a*) The solution-sets of the two equations are equal, and their intersection consists of all those ordered pairs that are found in either one of the given solution-sets.

(*b*) The intersection of the two solution-sets is the empty set.

(*c*) The intersection of the two solution-sets contains exactly one ordered pair.

In case (*a*), we say that the left-hand members ofthe two linear equations in x and y are **linearly dependent**, and the equations are **consistent** ; in case (*b*) we say that the left-hand members are **linearly independent**, and the equations are **inconsistent** ; in case (*c*) we say that the left-hand members are linearly independent and the equations are consistent. For example,

(1) $3x+2y-4$ and $12x+8y-16$ are linearly dependent, and the equations

$$3x+2y-4=0 \text{ and } 12x+8y-16=0$$

are consistent.

(2) $3x+2y-4$ and $12x+8y-12$ are linearly independent and the equations

$$3x+2y-4=0 \text{ and } 12x+8y-12=0$$

are inconsistent.

(3) $3x+2y-4=0$ and $8x+12y-16$ are linearly independent and the equations

$$3x+2y-4=0 \text{ and } 8x+12y-16=0$$

are consistent.

We thus find that *by studying the graphs of two given linear equations in two variables we can get complete information about the existence and nature of solutions of the equations. Conversely, from a knowledge of the existence (or otherwise) and nature of solutions of the equations we can get information about the points common to the straight lines representing the equations.*

10·4·3. Condition for Concurrency of Three Straight Lines

A set of straight lines is said to be **concurrent** if there is a point common to all of them. We shall find a necessary and sufficient condition for three straight lines to be concurrent.

Let the equations of three given straight lines be

$$a_1x+b_1y+c_1=0, \qquad \ldots(i)$$

$$a_2x+b_2y+c_2=0, \qquad \ldots(ii)$$

$$a_3x+b_3y+c_3=0. \qquad \ldots(iii)$$

The point of intersection of (*ii*) and (*iii*) is

$$\left(\frac{b_2c_3-b_3c_2}{a_2b_3-a_3b_2}, \frac{c_2a_3-c_3a_2}{a_2b_3-a_3b_2}\right).$$

The three straight lines are concurrent if and only if this point lies on (*i*) also, that is, if and only if

$$a_1\left[\frac{b_3c_3-b_2c_2}{a_2b_3-a_3b_2}\right]+b_1\left[\frac{c_2a_3-c_3a_2}{a_2b_3-a_3b_2}\right]+c_1=0,$$

or $\qquad a_1(b_2c_3-b_3c_2)+b_1(c_2a_3-c_3a_2)+c_1(a_2b_3-a_3b_2)=0.$

10·4·4. Family of Straight Lines Passing through the Point of Intersection of Two Straight Lines

Suppose we are given a pair of intersecting straight lines whose equations are

$$L_1 \equiv a_1x+b_1y+c_1=0, \qquad \ldots(i)$$

$$L_2 \equiv a_2x+b_2y+c_2=0. \qquad \ldots(ii)$$

Let (x_1, y_1) be the point of intersection of (*i*) and (*ii*). Since (x_1, y_1) satisfies (*i*) and (*ii*), therefore,

$$a_1x_1+b_1y_1+a_1=0 \text{ and } a_2x_1+b_2y_1+c_2=0. \qquad \ldots(iii)$$

Consider the equation

$$p(a_1x+b_1y+c_1)+q(a_2x+b_2y+c_2)=0. \qquad \ldots(iv)$$

For all values of p and q, (*iv*) represents a straight line. Also, because of (*iii*), we have

$$p(a_1x_1+b_1y_1+c_1)+q(a_2x_1+b_2y_1+c_2)=0,$$

showing that the line (*iv*) passes through (x_1, y_1) which is the point of intersection of the straight lines (*i*) and (*ii*).

The equation $pL_1+qL_2=0$ represents the family of all straight lines passing through the point of intersection of the straight lines $L_1=0$ and $L_2=0$.

It appears as if two *independent* parameters p and q are involved here. But this is not the case. Observe that p and q cannot be both zero. If we impose one condition (other than that of passing through the point of intersection of $L_1=0$ and $L_2=0$) such as having a given slope, or being parallel to a given line, or being perpendicular to a given straight line, then the ratio $p : q$ can be determined, which is exactly what we require.

It may be noted that $L_1+kL_2=0$ represents the family of all straight lines other than $L_2=0$ that pass through the point of intersection of $L_1=0$ and $L_2=0$. Similarly, $L_2+kL_1=0$ represents the family of all straight lines other than $L_1=0$ that pass through the point of intersection of $L_1=0$ and $L_2=0$.

Example 20. *Find the equation of the straight line joining the point (2, 3) to the point of intersection of $2x+3y+1=0$ and $3x-4y=8$.*

Solution. The equations of the given straight lines are

$$2x+3y+1=0, \qquad \ldots(i)$$

$$3x-4y-8=0. \qquad \ldots(ii)$$

The equation of any straight line [except (*ii*)] through the point of intersection of (*i*) and (*ii*) is

$$(2x+3y+1)+k(3x-4y-8)=0. \qquad \ldots(iii)$$

If (*iii*) passes through the point (2, 3),

$$(2.2+3.3+1)+k(3.2-4.3-8)=0,$$

or $$14-14k=0,$$

or $$k=1.$$

Putting $k=1$ in (*iii*), we have

$$(2x+3y+1)+(3x-4y-8)=0,$$

or $$5x-y-7=0, \qquad ...(iv)$$

as the required equation.

Aliter. Let (x_1, y_1) be the co-ordinates of the point of intersection of the given straight lines.

Then $$2x_1+3y_1+1=0, \qquad ...(i)$$

$$3x_1-4y_1-8=0. \qquad ...(ii)$$

Solving (*i*) and (*ii*) by the cross-multiplication method, we have

$$\frac{x_1}{3(-8)-1(-4)}=\frac{y_1}{1(3)-2(-8)}=\frac{1}{2(-4)-3(3)},$$

or $$\frac{x_1}{-20}=\frac{y_1}{19}=\frac{1}{-17}.$$

Therefore, the point of intersection of the given straight lines is

$$\left(\frac{20}{17}, -\frac{19}{17}\right).$$

The equation of the straight line joining the points (2, 3) and $\left(\frac{20}{17}, -\frac{19}{17}\right)$ is

$$\frac{y-3}{-\frac{19}{17}-3}=\frac{x-2}{\frac{20}{17}-2},$$

or $$-\frac{14}{17}(y-3)=-\frac{70}{17}(x-2),$$

or $$y-3=5(x-2),$$

or $$5x-y-7=0.$$

EXERCISE 10 (*j*)

Find the co-ordinates of the points of intersection of the straight lines whose equations are

1. $x-3y+4=0$ and $7x+4y+8=0$.
2. $x+2y+1=0$ and $y=x+7$.
3. $x+5y=13$ and $y+5x=13$.

4. $y=m_1x+\dfrac{a}{m_1}$ and $y=m_2x+\dfrac{a}{m_2}$.

5. $x\cos\theta_1+y\sin\theta_1=a$ and $x\cos\theta_2+y\sin\theta_2=a$.

6. $t_1y=x+at_1^2$ and $t_2y=x+at_2^2$.

7. $\dfrac{x}{a}+\dfrac{y}{b}=1$ and $\dfrac{x}{b}+\dfrac{y}{a}=1$.

8. Find the co-ordinates of the vertices of a triangle, the equations of whose sides are

$$y+x-6=0,\ 3y-x+2=0 \text{ and } 3y=5x+2.$$

9. The equations of the sides of a triangle are $3x+y+4=0$, $3x-5y+34=0$ and $3x-2y+1=0$. Find the co-ordinates of the vertices.

10. Show that the lines $5x+3y=7$, $3x-5y=11$ and $x+2y=0$ meet in a point.

11. Show that the lines $x-y=6$, $4x-3y-20=0$ and $6x+5y+8=0$ are concurrent.

12. For what value of k are the three lines $x-2y+1=0$, $5x-2y+3=0$ and $5x-9y+k=0$ concurrent ?

13. For what value of a are the lines $3x+7y=10$, $8x-3y=5$ and $ax-2y+1=0$, concurrent ?

14. Find the equation of the straight line which passes through the point (6, 4) and also through the point of intersection of the straight lines $x-y-1=0$ and $2x-3y+1=0$.

15. Find the equation of the straight line joining the origin to the point of intersection of the lines

$$5x-4y-7=0 \text{ and } x+2y-3=0.$$

16. Find the equations of the medians of the triangle the equations of whose sides are $3x+2y+6=0$, $2x-5y+4=0$ and $x-3y-6=0$.

17. Find the equation of the straight line passing through the intersection of $x+2y+3=0$ and $2x-3y-6=0$, and the intersection of $3x-4y, 12=0$ and $4x+5y+15=0$.

10·5. ANGLE BETWEEN TWO STRAIGHT LINES

Let the two straight lines be AB_1 and AB_2, meeting the x-axis in B_1 and B_2. Let the equations of AB_1 and AB_2 be

$$y=m_1x+c_1, \quad \ldots(i)$$

and

$$y=m_2x+c_2, \quad \ldots(ii)$$

respectively.

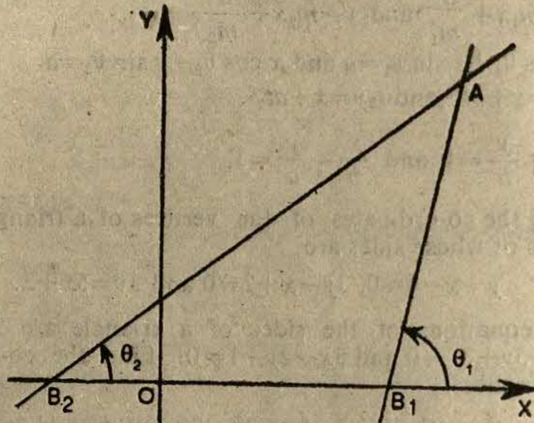


Fig. 10 23.

If θ_1, θ_2 be the angles that the lines make with OX, then

$$\tan\theta_1=m_1 \text{ and } \tan\theta_2=m_2.$$

Now $\angle B_2AB_1=\theta_1-\theta_2$.

$\therefore$ $\tan\angle B_2AB_1=\tan(\theta_1-\theta_2)$,

$$=\frac{\tan\theta_1-\tan\theta_2}{1+\tan\theta_1\tan\theta_2},$$

$$=\frac{m_1-m_2}{1+m_1m_2}. \quad \text{...}(iii)$$

Since $\angle B_2AB_1$ is either θ or $\pi-\theta$,

$$\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|.$$

Hence the required angle is

$$\tan^{-1}\left|\frac{m_1-m_2}{1+m_1m_2}\right|. \quad \text{...}(iv)$$

Corollary. The find the angle between the straight lines whose equations are

$$A_1x+B_1y+C_1=0,\ A_2x+B_2y+C_2=0.$$

If m_1, m_2 be the slopes of the straight lines, then

$$m_1=-\frac{A_1}{B_1},\ m_2=-\frac{A_2}{B_2}.$$

Therefore, the acute angle θ between the straight lines

$$=\tan^{-1}\left|\frac{m_1-m_2}{1+m_1m_2}\right|,$$

$$=\tan^{-1}\left|\frac{\left(-\frac{A_1}{B_1}\right)-\left(-\frac{A_2}{B_2}\right)}{1+\left(-\frac{A_1}{B_1}\right)\left(-\frac{A_2}{B_2}\right)}\right|.$$

$$=\tan^{-1}\left|\frac{A_2B_1-A_1B_2}{A_1A_2+B_1B_2}\right|.$$

Example 21. *Find the angle between the straight lines whose equations are $3x+y-7=0$ and $x+2y+9=0$.*

Solution. Let m_1, m_2 be the slopes of the given straight lines.

Then $m_1=-3,\ m_2=-\frac{1}{2}$.

If θ be the acute angle between the straight lines, then

$$\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|$$

$$=\left|\frac{-3-(-\frac{1}{2})}{1+(-3)(-\frac{1}{2})}\right|=1,$$

or $\theta=45°$.

Hence the angle between the straight lines is 45°.

10·5·1. Condition under which Two Straight Lines are Parallel

Let the equations of two given straight lines be

$$y=m_1x+c_1,\ y=m_2x+c_2.$$

If θ be the acute angle between the given straight lines, then

$$\tan\theta=\left|\frac{m_2-m_1}{1+m_1m_2}\right|.$$

If the straight lines are parallel, then

$$\theta=0,\ i.e.,\ m_2-m_1=0 \text{ or } m_1=m_2,$$

which is the required condition, which we noted earlier as well.

Corollaries. 1. The straight lines

$$A_1x+B_1y+C_1=0,\ A_2x+B_2y+C_2=0$$

are parallel provided

$$-\frac{A_1}{B_1}=-\frac{A_2}{B_2},$$

or

$$\frac{A_1}{B_1}=\frac{A_2}{B_2}. \quad ...(i)$$

2. *Family of straight lines parallel to a given straight line.* The straight lines $Ax+By+C=0$ and $Ax+By+K=0$ are parallel, because they satisfy condition (*i*). Hence the equation of any straight line parallel to $Ax+By+C=0$ is $Ax+By+K=0$. In other words, *the equation* $Ax+By+K=0$, *where K is a parameter,*

represents the family of all straight lines parallel to the straight line $Ax+By+C=0$.

A straight line is completely determined by two independent conditions. In the present case, one of these two conditions is that the straight line is parallel to $Ax+By+C=0$. The other condition can be used to find the value of K.

3. The equation of the striaght line passing through the point (x_1, y_1) and parallel to the straight line

$$Ax+By+C=0$$

is

$$A(x-x_1)+B(y-y_1)=0.$$

For, (*ii*) represents a straight line passing through (x_1, y_1) and its slope is $-A/B$, which is equal to the slope of the straight line $Ax+By+C=0$.

Example 21. *Find the equation to the straight line which passes throngh the point (1, 2) and is parallel to the straight line* $3x-2y+4=0$.

Solution. The equation of any straight line parallel to

$$3x-2y+4=0$$

is

$$3x-2y+k=0. \qquad \text{...}(i)$$

If (*i*) passes through (1, 2), we have

$$3.1-2.2+k=0,$$

or

$$k=1.$$

Substituting the value of k in (*i*), we have

$$3x-2y+1=0,$$

as the required equation.

Aliter. The equation of any straight line through the point (1, 2) is

$$y-2=m(x-1). \qquad \text{...}(i)$$

If (*i*) is parallel to the straight line

$$3x-2y+4=0, \qquad \text{...}(ii)$$

the slopes of (*i*) and (*ii*) must be equal.

Therefore,

$$m=\frac{3}{2}. \qquad \text{...}(iii)$$

Substituting the value of m in (*i*), we have

$$y-2=\frac{3}{2}(x-1),$$

or

$$2(y-2)=3(x-1),$$

or

$$3x-2y+1=0,$$

as the required equation.

10·5·2. Condition under which two Straight Lines are Perpendicular to Each Other.

Let the equations of two given straight lines be

$$y=m_1x+c_1,\ y=m_2x+c_2.$$

If θ be the angle between the given straight lines, then

$$\tan\theta=\left|\frac{m_2-m_1}{1+m_1m_2}\right|.$$

If the straight lines are perpendicular, then $\theta=90°$, *i.e.*,

$$1+m_1m_2=0 \text{ or } m_1m_2=-1,$$

which is the required condition.

Hence *two straight lines are perpendicular provided the product of their slopes is —1.*

Corollaries. 1. The straight lines

$$A_1x+B_1y+C_1=0 \text{ and } A_2x+B_2y+C_2=0$$

are perpendicular provided the product of their slopes is -1, *i.e.*, provided

$$\left(-\frac{A_1}{B_1}\right)\left(-\frac{A_2}{B_2}\right)=-1,$$

or $$A_1A_2+B_1B_2=0. \qquad ...(i)$$

2. *Family of straight lines perpendicular to a given straight line.* The straight lines $Ax+By+C=0$, $Bx-Ay+K=0$ are perpendicular to each other because they satisfy the condition (*i*). Hence the equation of any straight line perpendicular to $Ax+By+C=0$ is $Bx-Ay+K=0$. In other words *the equation $Bx-Ay+K=0$, where K is a parameter, represents the family of all straight lines perpendicular to the straight line* $Ax+By+C=0$.

As you already know, a straight line is completely determined by two independent conditions. In the present case one of these two conditions is that the straight line is perpendicular to $Ax+By+C=0$. The other condition can be used to find the value of K.

3. The equation of the straight line passing through the point (x_1, y_1) and perpendicular to the straight line $Ax+By+C=0$ is

$$B(x-x_1)-A(y-y_1)=0. \qquad ...(ii)$$

For, (*ii*) represents a straight line passing through (x_1, y_1) and its slope is B/A which is the negative reciprocal of the slope of the line $Ax+By+C=0$.

Example 23. *Find the equation of the straight line which passes through the point (—1, 3) and is perpendicular to the straight line $4x+3y+1=0$.*

Solution. The equation of any straight line perpendicular to

$$4x+3y+1=0,$$

is $$3x-4y+k=0. \qquad ...(i)$$

If (i) passes through $(-1, 3)$, we have

$$3(-1)-4.3+k=0,$$

or $$k=15.$$

Substituting the value of k in (i), we have

$$3x-4y+15=0,$$

as the required equation.

Aliter. The equation of any straight line through the point $(-1, 3)$ is

$$y-3=m(x+1). \qquad ...(i)$$

If (i) is perpendicular to the straight line

$$4x+3y+1=0, \qquad ...(ii)$$

the product of the slopes of (i) and (ii) must be equal to -1, *i.e.*,

$$m\left(-\frac{4}{3}\right)=-1,$$

or $$m=\frac{3}{4}. \qquad ...(iii)$$

Substituting the value of m in (i), we have

$$y-3=\frac{3}{4}(x+1),$$

or $$3x-4y+15=0,$$

as the required equation.

Example 24. *Find the equation of the straight line that passes through the point of intersection of the straight lines $2x+y-3=0$, $x-2y+1=0$, and is parallel to the straight line $y-x+2=0$.*

Solution. The equation of any straight line (other than $x-2y+1=0$ passing through the point of intersection of the straight lines $2x+y-3=0$ and $x-2y+1=0$ is

$$(2x+y-3)+k(x-2y+1)=0,$$

or $$x(2+k)+y(1-2k)+(-3+k)=0. \qquad ...(i)$$

The straight line (i) is parallel to

$$y-x+2=0 \qquad ...(ii)$$

provided the slopes of (i) and (ii) are equal, *i.e.*, provided

$$-\frac{2+k}{1-2k}=1,$$

or $$-2-k=1-2k,$$

or $$k=3. \qquad ...(iii)$$

Substituting the value of k in (i), we have

$$5x-5y=0,$$

or $$x-y=0,$$

as the required equation.

Aliter. The point of intersection of the straight lines

$$2x+y-3=0, \qquad \ldots(i)$$

and

$$x-2y+1=0, \qquad \ldots(ii)$$

is given by $\dfrac{x}{1-6}=\dfrac{y}{-3-2}=\dfrac{1}{-4-1}$.

or $\qquad x=1, y=1.$

Therefore, the point of intersection of (i) and (ii) is (1, 1).

The equation of the straight line which passes through the point (1, 1) and is parallel to the straight line

$$y-x+2=0$$

is

$$(y-1)-(x-1)=0,$$

or

$$y-x=0.$$

Example 25. *The equations of the sides BC, CA, AB of a triangle ABC are*

$$3x-5y+34=0, \qquad 3x-2y+1=0, \qquad 3x+y+4=0.$$

Find the equation of the straight line through A perpendicular to BC.

Solution. The equation of any straight line passing through A (other than AB), is

$$(3x-2y+1)+k(3x+y+4)=0,$$

or

$$x(3+3k)+y(-2+k)+1+4k=0. \qquad \ldots(i)$$

Slope of the line $(i)=-\dfrac{3+3k}{-2+k}$. $\qquad \ldots(ii)$

Also, slope of the line $BC=\dfrac{3}{5}$.

If the line (i) is perpendicular BC, we have

$$-\frac{3+3k}{-2+k}\cdot\frac{3}{5}=-1,$$

or

$$3(3+3k)=5(-2+k),$$

or

$$k=-\frac{19}{4}. \qquad \ldots(iii)$$

Substituting the value of k in (i), we have

$$x\left(3-\frac{57}{4}\right)+y\left(-2-\frac{19}{4}\right)+1-19=0,$$

or

$$5x+3y+8=0$$

as the required equation.

Aliter. A is the point of intersection of the straight lines

$$3x-2y+1=0,$$

$$3x+y+4=0.$$

Therefore, the co-ordinates of A are given by

$$\frac{x}{-8-1}=\frac{y}{3-12}=\frac{1}{3+6}.$$

Therefore, A is the point $(-1, -1)$.

The equation of any straight line perpendicular to BC is

$$5x+3y+k=0. \qquad ...(i)$$

If (i) passes through the point $A(-1, -1)$, we have

$$5(-1)+3(-1)+k=0,$$

or

$$k=8.$$

Substituting the value of k in (i), we have

$$5x+3y+8=0,$$

as the required equation.

18·5·3. Orthocentre of a triangle

To show that the three perpendiculars drawn from the vertices of a triangle upon the opposite sides are concurrent.

Let (x_1, y_1), (x_2, y_2), (x_3, y_3) be the vertices A, B, C of a triangle ABC, and AD, BE, CF the perpendiculars from them on the opposite sides.

The slope of $BC=\dfrac{y_3-y_2}{x_3-x_2}$.

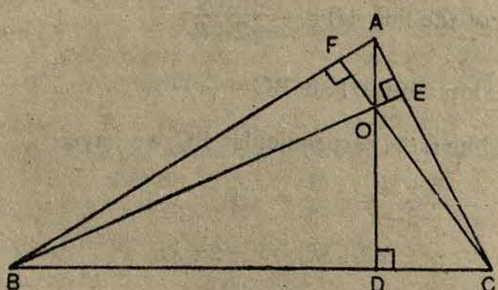


Fig. 10·24.

Since AD is perpendicular to BC, therefore the slope of

$$AD=-\frac{x_3-x_2}{y_3-y_2}.$$

The equation of AD is, therefore,

$$y-y_1=-\frac{x_3-x_2}{y_3-y_2}(x-x_1),$$

i.e.,

$$(y-y_1)(y_3-y_2)+(x_3-x_2)(x-x_1)=0,$$

or

$$x(x_3-x_2)+y(y_3-y_2)-x_1(x_3-x_2)-y_1(y_3-y_2)=0. \qquad ...(i)$$

Similarly the equations of BE and CF are

$$x(x_1-x_3)+y(y_1-y_3)-x_2(x_1-x_3)-y_2(y_1-y_3)=0, \qquad \cdots(ii)$$

$$x(x_2-x_1)+y(y_2-y_1)-x_3(x_2-x_1)-y_3(y_2-y_1)=0. \qquad \cdots(iii)$$

If the expressions on the left-hand sides of (*i*), (*ii*) and (*iii*) be denoted by L_1, L_2, L_3 respectively, then we find that

$$L_1+L_2+L_3=0,$$

which shows that $L_3=0$ is the same as $L_1+L_2=0$, *i.e.*, $L_3=0$ passes through the point of intersection of $L_1=0$ and $L_2=0$.

Hence $L_1=0$, $L_2=0$, $L_3=0$ are concurrent.

Recall that the point of concurrence of the three perpendiculars drawn from the vertices of a triangle upon the opposite sides is called the orthocentre of the triangle. The co-ordinates of the orthocentre O in the above discussion can be found by solving equations (*i*) and (*ii*).

The following example will illustrate the method of finding the co-ordinates of the orthocentre of a triangle.

Example 26. *Find the co-ordinates of the orthocentre of the triangle whose vertices are A(2, 3), B(3, 4) and C(6, 8).*

Solution. The slope of BC is $\frac{8-4}{6-3}$, *i.e.*, $\frac{4}{3}$. Therefore, the slope of any line perpendicular to BC is $-\frac{3}{4}$.

The equation of the perpendicular from A to BC is

$$y-3=-\frac{3}{4}(x-2),$$

i.e.,
$$3x+4y-18=0. \qquad \cdots(i)$$

Again, the slope of CA is $\frac{3-8}{2-6}$, *i.e.*, $\frac{5}{4}$.

Therefore, the slope of any line perpendicular to CA is $\frac{-4}{5}$.

The equation of the perpendicular from B to CA is

$$y-4=\frac{-4}{5}(x-3),$$

i.e.,
$$4x+5y-32=0. \qquad \cdots(ii)$$

Solving (*i*) and (*ii*) we find that the co-ordinates of the orthocentre of the triangle ABC are (38, −24).

Check. The equation of the perpendicular from C to AB is $x+y-14=0$ which passes through (38, −24).

10·5·4. Circumcentre of a Triangle

To show that the perpendicular bisectors of the sides of a triangle are concurrent.

Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be the vertices of a triangle. Let D, E and F be the mid-points of BC, CA and AB respectively. The co-ordinates of D, E and F are

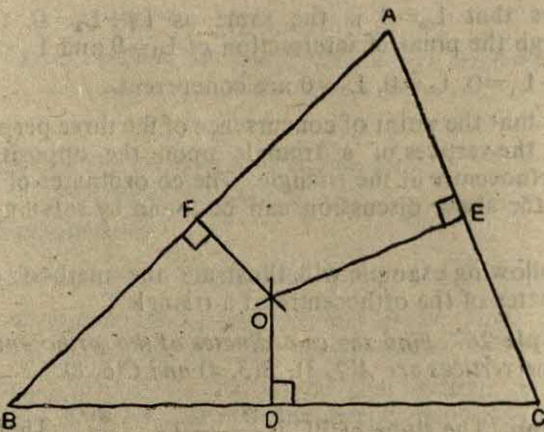


Fig. 10·25.

$$\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right), \left(\frac{x_3+x_1}{2}, \frac{y_3+y_1}{2}\right), \text{ and}$$

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) \text{ respectively.}$$

The equation of the perpendicular bisector of BC, *i.e.*, the line through D perpendicular to BC is

$$y-\frac{y_2+y_3}{2}=\left(-\frac{x_3-x_2}{y_3-y_2}\right)\left(x-\frac{x_2+x_3}{2}\right),$$

or $$x(x_2-x_3)+y(y_2-y_3)-\tfrac{1}{2}(x_2^2-x_3^2+y_2^2-y_3^2)=0. \qquad ...(i)$$

Similarly, the equations of the perpendicular bisectors of CA and AB are

$$x(x_3-x_1)+y(y_3-y_1)-\tfrac{1}{2}(x_3^2-x_1^2+y_3^2-y_1^2)=0, \qquad ...(ii)$$

and $$x(x_1-x_2)+y(y_1-y_2)-\tfrac{1}{2}(x_1^2-x_2^2+y_1^2-y_2^2)=0 \qquad ...(iii)$$

respectively. If the expressions on the left-hand sides of (*i*), (*ii*) and (*iii*) be denoted by L_1, L_2 and L_3 respectively, then we find that

$$L_1+L_2+L_3=0,$$

showing that $L_3=0$ is the same as $L_1+L_2=0$, *i.e.*, $L_3=0$ passes through the point of intersection of $L_1=0$ and $L_2=0$.

Hence the three lines $L_1=0$, $L_2=0$, $L_3=0$ are concurrent.

Let O be the point of concurrence. Since O lies on the perpendicular bisector of BC, therefore, OB=OC.

Again, since O lies on the perpendicular bisector of CA, therefore, OC=OA. Thus we have OA=OB=OC, so that if we draw a circle with centre O, and radius OA, then it will pass through all the three vertices of the triangle. Recall that the point of concurrence of the perpendicular bisectors of the sides of a triangle is called the circumcentre of the triangle, and the distance of the circumcentre from any of the vertices of the triangle is called the circumradius of the triangle. The co-ordinates of the circumcentre of the triangle in the above discussion can be found by solving equations (*i*) and (*ii*).

The following example will illustrate the method of finding the co-ordinates of the circumcentre of a triangle.

Example 27. *Find the co-ordinates of the circumcentre of the triangle whose vertices are A(2, 3), B(3, 4) and C(6, 8).*

Solution. The co-ordinates of the mid-point (say D) of BC are $\left(\frac{3+6}{2}, \frac{4+8}{2}\right)$, *i.e.*, $\left(\frac{9}{2}, 6\right)$. The slope of BC is $\frac{8-4}{6-3}$, *i.e.*, $\frac{4}{3}$, and therefore, the slope of the perpendicular bisector of BC is $-\frac{3}{4}$. The equation of the perpendicular bisector of BC is

$$y-6=-\frac{3}{4}\left(x-\frac{9}{2}\right),$$

i.e., $$6x+8y-75=0. \qquad \ldots(i)$$

Again, the co-ordinates of the mid-point (say E) of CA are $\left(\frac{6+2}{2}, \frac{8+3}{2}\right)$, *i.e.*, $\left(4, \frac{11}{2}\right)$. Since the slope of CA is $\frac{3-8}{2-6}$, *i.e.*, $\frac{5}{4}$, therefore, the slope of the perpendicular bisector of CA is $-\frac{4}{5}$. The equation of the perpendicular bisector of CA is

$$y-\frac{11}{2}=-\frac{4}{5}(x-4),$$

i.e., $$8x+10y-87=0. \qquad \ldots(ii)$$

Solving (*i*) and (*ii*), we find that the co-ordinates of the circumcentre of the triangle ABC are $\left(-\frac{27}{2}, \frac{39}{2}\right)$.

Aliter. Let (h, k) be the required centre and let R be the radius of the circumcircle of the triangle. Since the distance of the

centre from each of the points (2, 3), (3, 4) and (6, 8) is the same, namely R, therefore,

$$(h-2)^2+(k-3)^2=R^2, \quad ...(i)$$

$$(h-3)^2+(k-4)^2=R^2, \quad ...(ii)$$

$$(h-6)^2+(k-8)^2=R^2. \quad ...(iii)$$

From (i) and (ii), we have

$$h+k-6=0. \quad ...(iv)$$

From (ii) and (iii), we have

$$6h+8k-75=0. \quad ...(v)$$

Solving (iv) and (v) for h and k, we have

$$h=-\frac{27}{2},\ k=\frac{39}{2}.$$

The co-ordinates of the circumcentre are, therefore,

$$\left(-\frac{27}{2}, \frac{39}{2}\right).$$

Remark. On substituting the values of h and k in (i), we find that $R=\frac{5}{2}\sqrt{82}$. Therefore, the radius of the circumcircle of the given triangle is $\frac{5}{2}\sqrt{82}$.

EXERCISE 10 (k)

Find the angles between the pairs of straight lines :

1. $x+y\sqrt{3}=1$ and $\sqrt{3}x-y=2$.
2. $y+3x+1=0$ and $3y+x-1=0$.
3. $x-4y-1=0$ and $6x-y+3=0$.
4. $y=2x+5$ and $2x+4y+3=0$.
5. $y=x+1$ and $(2+\sqrt{3})\,x+y=2$.
6. The equations of the sides of a triangle are $y=4$, $y=x\sqrt{3}+1$, $y=-x\sqrt{3}+2$. Prove that the triangle is equilateral.
7. Find the equation of the straight line through (3, 4) parallel to the straight line $x+2y=1$.
8. Find the equation of the straight line through $(-1, 0)$ parallel to the straight line $3y-4x=6$.
9. Find the equation of the straight line through $(0, -3)$ perpendicular to the straight line $2x+3y=4$.
10. Find the equation of the straight line through (2, 4) perpendicular to the straight line $5x-7y=1$.
11. Find the equation of the straight line parallel to the straight line $2x-y=4$ and passing through the intersection of the straight lines $3x+y=7$ and $3y=2x-5$.

12. Find the equation of the straight line parallel to the straight line $x+2y=1$ and passing through the points of intersection of the straight lines $x-y=4$ and $3x+y=7$.
13. Find the equation of the straight line at right angles to the line $5x-2y+7=0$ and passing through the intersection of the lines $x+2y+1=0$ and $y=x+7$.
14. Find the equation of the straight line perpendicular to the line $2x-3y+1=0$ and passing through the point of intersection of the lines $x+y-4=0$ and $3x+2y-1=0$.
15. The equations of the sides of a triangle are

$$y+x-6=0,\ 3y-x+2=0,\ 3y=5x+2.$$

Find the co-ordinates of its orthocentre.
16. Write down the equations of the perpendiculars from the origin to the lines $x+5y=13$, $5x+y=13$ and find the equation of the line joining the feet of the perpendiculars.

10.6. DISTANCE OF A POINT FROM A LINE

To find the length of the perpendicular from a given point on a given straight line.

Let the equation of the given straight line be

$$ax+by+c=0$$

and let the co-ordinates of the given point P be (x_1, y_1).

If the given straight line meets the axes of co-ordinates in the point A and B, then the co-ordinates of A are

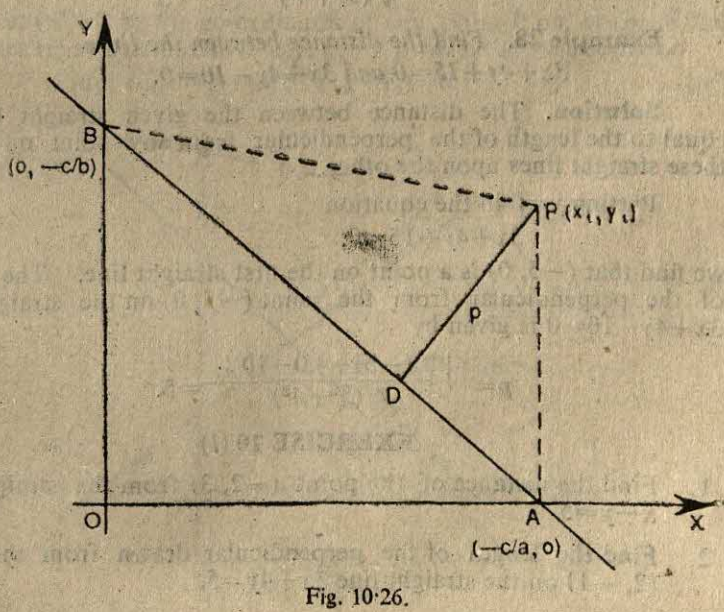


Fig. 10·26.

$(-c/a, 0)$ and those of B are $(0, -c/b)$. Join PA and PB and draw PD $\perp$ AB. If p denotes the length of PD, then

$$\tfrac{1}{2}\, p.\, AB = \Delta APB.$$

Therefore, $$p = \frac{2\Delta\, APB}{AB}. \qquad ...(i)$$

Since the co-ordinates of the points A, P, B are $(-c/a, 0)$, (x_1, y_1) and $(0, -c/b)$ respectively,

therefore, $$\Delta APB = \left| \frac{c}{2ab}(ax_1 + by_1 + c) \right|.$$

Also, $$AB = \sqrt{(c^2/a^2 + c^2/b^2)}.$$

Substituting the above values of ΔAPB and AB in (i), we have

$$p = \frac{|\, ax_1 + by_1 + c \,|}{\sqrt{(a^2 + b^2)}}$$

Corollary. The length of the perpendicular from the origin on the straight line $ax + by + c = 0$ is $\dfrac{|\, c \,|}{\sqrt{(a^2 + b^2)}}$.

Example 28. *Find the length of the perpendicular from the point $(1, -3)$ on the straight line $3x + 4y + 5 = 0$.*

Solution. If p denotes the required length, then

$$p = \frac{|\, 3.1 + 4\,(-3) + 5 \,|}{\sqrt{(3^2 + 4^2)}} = \frac{4}{5}.$$

Example 28. *Find the distance between the lines*

$$3x + 4y + 15 = 0 \text{ and } 3x + 4y - 10 = 0.$$

Solution. The distance between the given straight lines is equal to the length of the perpendicular from any point on one of these straight lines upon the other.

Putting $y = 0$ in the equation

$$3x + 4y + 15 = 0,$$

we find that $(-5, 0)$ is a point on the first straight line. The length of the perpendicular from the point $(-5, 0)$ on the straight line $3x + 4y - 10 = 0$ is given by

$$p = \frac{|\, 3\,(-5) + 4.0 - 10 \,|}{\sqrt{(3^2 + 4^2)}} = 5.$$

EXERCISE 10 (*l*)

1. Find the distance of the point $(-2, 3)$ from the straight line $x - y = 5$.

2. Find the length of the perpendicular drawn from the point $(2, -1)$ on the straight line $3x + 4y = 5$.

3. Find the length of perpendicular from the point (3, 4) upon the straight line $8x+15y+1=0$.
4. Find the distance of the point (4, 2) from the line joining the points (4, 1) and (2, 3).
5. Find the distance of the point (2, 3) from each of the straight lines $3x+4y-28=0$, $4x-3y+11=0$ and $5x+12y-20=0$.
6. Show that the perpendiculars let fall from the point (1, −1) upon the two straight lines $3x-4y=2$ and $5x-12y=4$ are equal.
7. The equations of the sides of a triangle are $3x+y+4=0$, $3x-5y+34=0$ and $3x-2y+1=0$. Find the lengths of the altitudes.
8. Find the lengths of the perpendiculars from the vertices on the opposite sides of the triangle whose vertices are the points (0, 0), (1, −1) and (3, 2).
9. Find the distance between the lines $9x+40y+21=0$ and $9x+40y-20=0$.
10. Find the distance between the straight lines $3x+4y+5=0$ and $3x+4y+20=0$.

10·7. PAIR OF STRAIGHT LINES

Consider the two straight lines represented by the equations:

$$a_1x+b_1y+c_1=0, \quad \ldots(i)$$

and

$$a_2x+b_2y+c_2=0. \quad \ldots(ii)$$

Let (x', y') be the co-ordinates of any point P on either of the two lines (*i*) and (*ii*). Obviously, (x', y') will satisfy the equation

$$(a_1x+b_1y+c_1)\,(a_2x+b_2y+c_2)=0, \quad \ldots(iii)$$

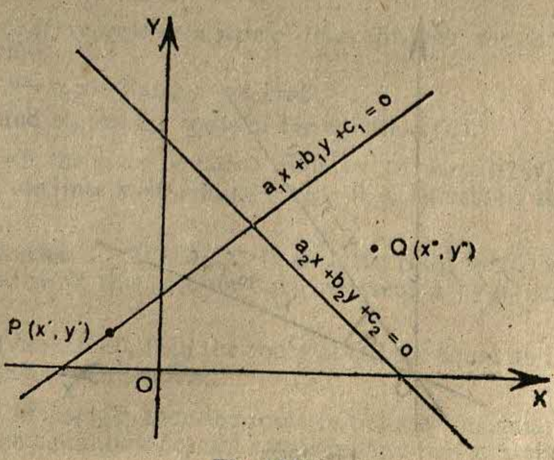


Fig. 10·27.

because if P lies on (*i*), then $a_1x'+b_1y'+c_1=0$ and if P lies on (*ii*), then $a_2x'+b_2y'+c_2=0$, and so in either case (x', y') when substituted in the left-hand side of (*iii*) will make it zero.

Also, if a point Q with co-ordinates (x'', y'') does not lie on any one of the lines (*i*) and (*ii*), then

$$a_1x''+b_1y''+c_1\neq 0 \text{ and } a_2x''+b_2y''+c_2\neq 0,$$

and so (x'', y'') when substituted in the left-hand side of (*iii*), cannot make it zero.

Thus equation (*iii*) is satisfied by the co-ordinates of all points which lie on (*i*) or on (*ii*) and is not satisfied by the co-ordinates of any other point. Hence (*iii*) represents the pair of straight lines (*i*) and (*ii*), that is, (*iii*) is the joint equation of the lines (*i*) and (*ii*). It is possible, therefore, that a single equation can represent two lines. In fact, if $F(x, y)$ is a quadratic expression in x, y which can be resolved into two real linear factors, then $F(x, y)=0$ represents two straight lines obtained by equating to zero the two linear factors of $F(x, y)$.

10·8. PAIR OF STRAIGHT LINES THROUGH THE ORIGIN

We know that the equation of a straight line through the origin is a homogeneous equation of the first degree, and conversely every homogeneous equation of the first degree represents a straight line through the origin. To represent a pair of straight lines through the origin, we need a homogeneous equation of the second degree. In fact, as we shall see, the equation of a pair of straight lines through the origin is a homogeneous equation of the second degree, and conversely every homogeneous equation of the second degree represents a pair of straight lines through the origin.

Theorem 10·1. *The equation of a pair of straight lines through the origin is a homogeneous equation of the second degree.*

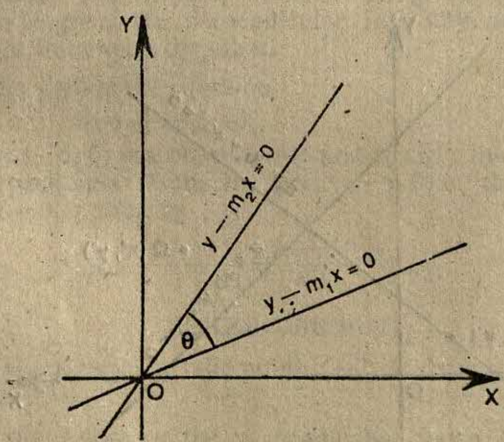


Fig. 10·28.

Proof. Let $y-m_1x=0$,

and $y-m_2x=0$

be two lines through the origin.

Then the equation of the given pair of lines is

$$(y-m_1x)\,(y-m_2x)=0,$$

i.e., $$m_1m_2x^2-(m_1+m_2)\,xy+y^2=0,$$

which is a homogeneous equation of the second degree.

Theorem 10·2. *Every homogeneous equation of the second degree represents a pair of straight lines through the origin.*

Proof. Consider the homogeneous equation of the second degree

$$ax^2+2hxy+by^2=0. \qquad ...(i)$$

Now if $b\neq 0$,

$$ax^2+2hxy+by^2=b\left(\frac{a}{b}x^2+\frac{2h}{b}xy+y^2\right),$$

$$=b(y-m_1x)(y-m_2x),$$

where $$m_1+m_2=\frac{-2h}{b},$$

$$m_1m_2=\frac{a}{b},$$

so that m_1, m_2 are the roots of the equation

$$m^2+\frac{2h}{b}m+\frac{a}{b}=0,$$

i.e., $$bm^2+2hm+a=0. \qquad ...(ii)$$

Hence (i) represents a pair of lines through the origin whose equations are

$$y-m_1x=0 \text{ and } y-m_2x=0,$$

where m_1 and m_2 are the roots of the equation (ii).

If $b=0$, the given equation reduces to $x(ax+2hy)=0$, which represents the lines $x=0$ and $ax+2hy=0$, both passing through the origin.

Remarks. 1. If $h^2-ab>0$, then the roots of (ii) are real and distinct and so in this case the lines represented by (i) are real and distinct.

2. If $h^2-ab=0$, then the roots of (ii) are equal and so in this case the lines are coincident.

3. If $h^2-ab<0$, then the roots of (ii) are imaginary and so in this case the equation does not represent any real straight lines.

10·9. ANGLE BETWEEN THE LINES $ax^2+2hxy-by^2=0$

Suppose $y-m_1x=0$ and $y-m_2x=0$ are the lines represented by the equation

$$ax^2+2hxy+by^2=0.$$

Then as discussed earlier, if $b\neq 0$,

$$m_1+m_2=\frac{-2h}{b},\quad m_1m_2=\frac{a}{b}.$$

If θ is the acute angle between the two lines, then from Fig. 10·29,

$$\tan\theta=|\tan(\theta_2-\theta_1)|,$$

$$=\left|\frac{\tan\theta_2-\tan\theta_1}{1+\tan\theta_1\tan\theta_2}\right|,$$

$$=\left|\frac{m_2-m_1}{1+m_1m_2}\right|,$$

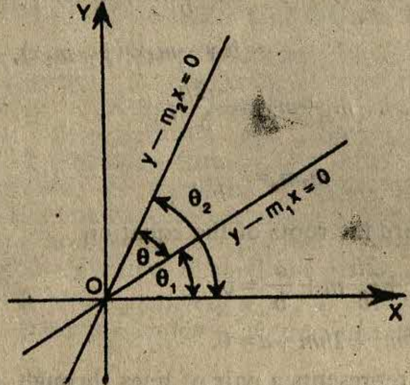


Fig. 10·29.

$$=\frac{\sqrt{\{(m_1+m_2)^2-4m_1m_2\}}}{|1+m_1m_2|},$$

$$=\frac{\sqrt{\left\{\left(-\frac{2h}{b}\right)^2-\frac{4a}{b}\right\}}}{\left|1+\frac{a}{b}\right|},$$

$$=\frac{\sqrt{(4h^2-4ab)}}{|b+a|},$$

$$=\frac{2\sqrt{h^2-ab}}{|a+b|}$$

$$\therefore \qquad \theta=\tan^{-1}\frac{2\sqrt{(h^2-ab)}}{|a+b|}$$

If $b=0$, then from Fig. 10·30,

$$\tan\theta=\tan(90°-\theta_1),$$
$$=\cot\theta_1,$$
$$=2h/a, \text{ provided } a\neq 0.$$

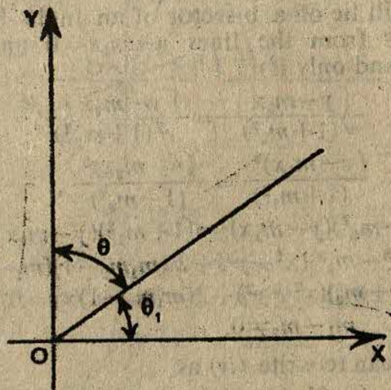


Fig. 10·30.

If $a=b=0$, the lines reduce to the axes of co-ordinates, the angle between which is 90°.

Hence we have the following :

Theorem 10.3. *The angle between the lines*

$ax^2+2hxy+by^2=0$ *is* $\tan^{-1}\dfrac{2\sqrt{h^2-ab}}{|a+b|}$.

Corollaries. 1. The lines are perpendicular $\Leftrightarrow m_1m_2=-1$,

$$\Leftrightarrow \frac{a}{b}=-1,\ b\neq 0,$$
$$\Leftrightarrow a+b=0.$$

In case $b=0$, [one of the lines being $x=0$, the other, *viz.*, $ax+2hy=0$ is perpendicular to it if $a=0$, so that we again have the condition $a+b=0$.

2. The lines are coincident $\Leftrightarrow \theta=0$,

$$\Leftrightarrow \tan\theta=0,$$
$$\Leftrightarrow h^2=ab.$$

10·10. COMBINED EQUATION OF THE BISECTORS OF THE ANGLE BETWEEN A PAIR OF STRAIGHT LINES REPRESENTED BY

$$\mathbf{ax^2+2hxy+by^2=0} \qquad ...(i)$$

Let the equation (i) represent the two straight lines

$$y-m_1x=0 \text{ and } y=m_2x=0, \qquad ...(ii)$$

so that

$$m_1+m_2=-2h/b, \text{ and } m_1m_2=a/b. \qquad ...(iii)$$

Recall that the locus of a point which moves so that its distances from two given intersecting straight lines are equal is the pair of straight lines bisecting the angles between them. Therefore, the point (x, y) will lie on a bisector of an angle between the lines (i) iff its distances from the lines $y-m_1x=0$ and $y-m_2x=0$ are equal, *i.e.*, iff (if and only if)

$$\frac{|y-m_1x|}{\sqrt{(1+m_1^2)}}=\frac{|y-m_2x|}{\sqrt{(1+m_2^2)}},$$

i.e., iff
$$\frac{(y-m_1x)^2}{(1+m_1^2)}=\frac{(y-m_2x)^2}{(1+m_2^2)},$$

i.e., iff
$$(1+m_2^2)(y-m_1x)^2=(1+m_1^2)(y-m_2x)^2,$$

i.e., iff
$$(m_1^2-m_2^2)(x^2-y^2)+2(m_1m_2-1)(m_1-m_2)xy=0,$$

i.e., iff
$$(m_1+m_2)(x^2-y^2)+2(m_1m_2-1)xy=0, \qquad ...(iv)$$

since
$$m_1-m_2\neq 0.$$

By (iii), we can re-write (iv) as

$$\left(\frac{-2h}{b}\right)(x^2-y^2)+2\left(\frac{a}{b}-1\right)xy=0,$$

or
$$\frac{x^2-y^2}{a-b}=\frac{xy}{h}$$

which is the desired equation.

Aliter. Let equation (i) represent the two straight lines M_1ON_1 and M_2ON_2 inclined at angles θ_1 and θ_2 to the x-axis (Fig. 10.31) so that (i) is equivalent to

$$b(y-x\tan\theta_1)(y-x\tan\theta_2)=0. \qquad ...(ii)$$

(ii) can be re-written as

$$bx^2\tan\theta_1\tan\theta_2-bxy(\tan\theta_1+\tan\theta_2)+by^2=0. \qquad ...(iii)$$

Comparing (i) and (iii) we find that

$$b\tan\theta_1\tan\theta_2=a,\ -b(\tan\theta_1+\tan\theta_2)=2h,$$

so that

$$\tan\theta_1+\tan\theta_2=-2h/b,\ \tan\theta_1\tan\theta_2=a/b. \qquad ...(iv)$$

Now
$$\angle AOM_1=\angle M_2OA,$$

therefore,
$$\angle AOX-\theta_1=\theta_2-\angle AOX,$$

i.e.,
$$2\angle AOX=\theta_1+\theta_2.$$

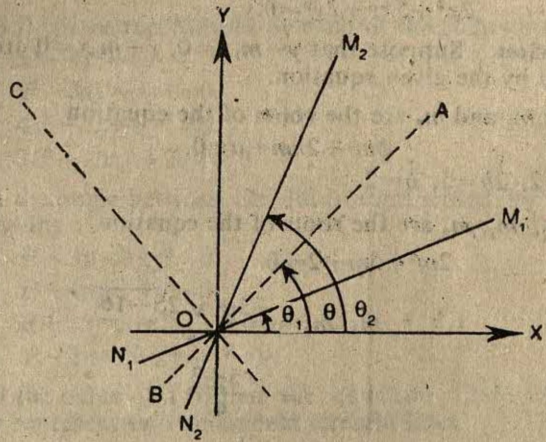


Fig. 10·31.

Also, $\angle COX = 90° + \angle AOX$,

therefore, $2 \cdot \angle COX = 180° + 2\angle AOX = 180° + \theta_1 + \theta_2$.

Hence if θ stands for *either* of the angles AOX or COX, we have

$$\tan 2\theta = \tan(\theta_1 + \theta_2),$$

$$= \frac{\tan \theta_1 + \tan \theta_2}{1 - \tan \theta_1 \tan \theta_2} = \frac{-2h}{b-a}, \text{ by } (iv). \quad \ldots(v)$$

If (x, y) be the co-ordinates of any point on either of the lines OA or OC we have

$$\tan \theta = y/x. \quad \ldots(vi)$$

Re-writing (v) as

$$\frac{-2h}{b-a} = \tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta},$$

and using (vi) we have

$$-\frac{2h}{b-a} = \frac{2(y/x)}{1-(y/x)^2},$$

i.e.,

$$\boxed{\frac{x^2 - y^2}{a-b} = \frac{xy}{h}.} \quad \ldots(vii)$$

Since (vii) is a relation between the co-ordinates (x, y) of *any* point on *either* of the two bisectors, therefore, it is the combined equation of the bisectors.

Example 30. *Find the lines represented by*

$$2x^2+5xy+2y^2=0.$$

Solution. Suppose that $y-m_1x=0$, $y-m_2x=0$ are the lines represented by the given equation.

Then m_1 and m_2 are the roots of the equation

$$bm^2+2hm+a=0,$$

where $a=2$, $2h=5$, $b=2$.

Thus, m_1, m_2 are the roots of the equation

$$2m^2+5m+2=0.$$

$$\therefore \qquad m_1, m_2=\frac{-5\pm\sqrt{25-16}}{4},$$

$$=\frac{-5\pm3}{4},$$

$$=\frac{-1}{2}, -2.$$

Hence the required equations are

$$y+\tfrac{1}{2}x=0 \text{ and } y+2x=0,$$

i.e., $\qquad 2y+x=0$ and $y+2x=0$.

Example 31. *Find the angle between the lines*

$$x^2+4xy+y^2=0.$$

Solution. If $x^2+4xy+y^2\equiv ax^2+2hxy+by^2$,

then $\qquad a=1,\ b=1,\ h=2.$

If θ is the angle between the given lines, then

$$\tan\theta=\frac{2\sqrt{(h^2-ab)}}{a+b}=\frac{2\sqrt{(2^2-1.1)}}{1+1}=\frac{2\sqrt{3}}{2}=\sqrt{3}.$$

$$\therefore \qquad \theta=\tan^{-1}\sqrt{3}=60°.$$

Hence the angle between the lines represented by the given equation is 60°.

Example 32. *Find the equation of the bisectors of the angle between the straight lines* $2x^2+4xy+y^2=0$.

Solution. If $2x^2+4xy+y^2\equiv ax^2+2hxy+by^2$,

then $\qquad a=2,\ h=2,\ b=1.$

Therefore, the equation of the bisectors of the angles between the given pair of straight lines is

$$\frac{x^2-y^2}{2-1}=\frac{xy}{2},$$

i.e., $\qquad 2x^2-xy-2y^2=0.$

EXERCISE 10 (*m*)

1. Find the lines represented by each of the following :
   (*i*) $x^2-4y^2=0$.
   (*ii*) $x^2-3xy-4y^2=0$.
   (*iii*) $4x^2-5xy+y^2=0$.
   (*iv*) $3x^2+19xy+20y^2=0$.
2. Find the angle between the lines represented by each of the following :
   (*i*) $x^2+3xy+2y^2=0$.
   (*ii*) $x^2+4xy+4y^2=0$.
   (*iii*) $x^2+2xy\cos 2\alpha-y^2=0$.
   (*iv*) $x^2-2\sec\theta\, xy+y^2=0$.
3. Find the value of λ so that the equation $12x^2+36xy+\lambda y^2=0$ may represent two coincident straight lines.
4. Show that the angle between the lines represented by
$$(x^2+y^2)\sin^2\alpha=(x\cos\theta-y\sin\theta)^2$$
is 2α.
5. Find the equation of the straight lines through the origin perpendicular to the lines represented by
$$2x^2+3xy+y^2=0.$$

10·11. CONDITION FOR THE GENERAL EQUATION OF SECOND DEGREE TO REPRESENT A PAIR OF STRAIGHT LINES

The general equation of second degree is of the form
$$ax^2+2hxy+by^2+2gx+2fy+c=0. \qquad ...(i)$$

This will represent two straight lines if we can resolve the left-hand side of (*i*) into two real linear factors.

Three different cases arise :

Case I. $a\neq 0$.

In this case, multiplying both sides of (*i*) by a, we get
$$a^2x^2+2ax(hy+g)+aby^2+2afy+ac=0,$$
i.e., $$(ax+hy+g)^2-(hy+g)^2+aby^2+2afy+ac=0,$$
i.e., $$(ax+hy+g)^2-[(h^2-ab)y^2+2(gh-af)y+g^2-ac]=0.$$

The expression on the left of the above equation can be resolved into two linear factors iff (if and only if)
$$(h^2-ab)y^2+2(gh-af)y+g^2-ac$$
is a perfect square, that is, iff
$$(gh-af)^2-(h^2-ab)(g^2-ac)=0,$$

i.e., iff $$-2afgh+a^2f^2+abg^2+ach^2-a^2bc=0,$$
i.e., iff $$a(2fgh+abc-af^2-bg^2-ch^2)=0. \qquad ...(ii)$$

Since $a\neq 0$, therefore, (*ii*) is equivalent to

$$abc+2fgh-af^2-bg^2-ch^2=0, \qquad ...(iii)$$

which is the required condition.

Case II. $a=0,\ b\neq 0$.

In this case, the equation reduces to

$$2hxy+by^2+2gx+2fy+c=0. \qquad ...(iv)$$

Multiplying both sides of (*iv*) by b, we get

$$2bhxy+b^2y^2+2bgx+2bfy+bc=0,$$

i.e., $$b^2y^2+2by(hx+f)+2bgx+bc=0,$$

i.e., $$(by+hx+f)^2-[(hx+f)^2-2bgx-bc]=0,$$

i.e., $$(by+hx+f)^2-h^2x^2+2x(hf-bg)+f^2-bc]=0.$$

The expression on the left-hand side of the above equation can be resolved into two linear factors iff

$$h^2x^2+2x(hf-bg)+f^2-bc$$

is a perfect square, that is, iff

$$(hf-bg)^2=h^2(f^2-bc),$$

i.e., iff $$b^2g^2-2bfgh+bch^2=0,$$

i.e., iff $$(2fgh-bg^2-ch^2)b=0. \qquad ...(v)$$

Since $b\neq 0$, (*v*) is equivalent to

$$2fgh-bg^2-ch^2=0,$$

which is the same condition as (*iii*) with $a=0$.

Case III. $a=0,\ b=0$.

In this case, the equation reduces to

$$2hxy+2gx+2fy+c=0. \qquad ...(vi)$$

Here $h\neq 0$, because the given equation is a second degree equation.

Now (*vi*) can be written as

$$2hx\left(y+\frac{g}{h}\right)+2f\left(y+\frac{c}{2f}\right)=0.$$

The left-hand side of the above equation can be resolved into two linear factors iff $g/h=c/2f$, that is, iff $2fg-ch=0$, which is the same condition as (*iii*) with $a=0$ and $b=0$. Thus we have the following:

Theorem 10·4. *A necessary and sufficient condition that the general equation of the second degree*

$$a^2x+2hxy+by^2+2gh+2fy+c=0$$

may represent a pair of straight lines is that

$$abc+2fgh-af^2-bg^2-ch^2=0.$$

The above condition can also be obtained by another method which we give below.

Alternative Method :

Suppose that the general equation of the second degree, *viz.*,

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

represents the two straight lines

$$lx+my+n=0,$$

and $$l'x+m'y+n'=0.$$

Then, $$ax^2+2hxy+by^2+2gx+2fy+c$$
$$\equiv(lx+my+n)(l'x+m'y+n').$$

Comparing coefficients, we get

$$ll'=a,\ mm'=b,\ nn'=c,$$
$$lm'+l'm=2h,\ ln'+l'n=2g,\ mn'+m'n=2f.$$

Multiplying the last three equations above, we get,

$$(lm'+l'm)(ln'+l'n)(mn'+m'n)=8fgh,$$

i.e., $$ll'(mn'+m'n)^2+mm'(ln'+l'n)^2$$
$$+nn'(lm'+l'm)^2-4ll'mm'nn'=8fgh,$$

i.e., $$a(2f)^2+b(2g)^2+c(2h)^2-4abc=8fgh,$$

i.e., $$abc+2fgh-af^2-bg^2-ch^2=0,$$

which is the required necessary condition.

Remark. The expression $abc+2fgh-af^2-bg^2-ch^2$ is known as the *discriminant* of equation (i), and is generally denoted by Δ.

10·12. ANGLE BETWEEN THE LINES REPRESENTED BY THE GENERAL EQUATION OF SECOND DEGREE

Consider the general equation of the second degree, *viz.*,

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

Suppose that the lines represented by this equation are

$$y=m_1x+c_1 \text{ and } y=m_2x+c_2. \quad ...(i)$$

Then, $$ax^2+2hxy+by^2+2gx+2fy+c$$
$$\equiv b(m_1x+c_1-y_1)(m_2x+c_2-y_2).$$

Comparing coefficients of xy and x^2, we get

$$m_1+m_2=-2h/b,\ m_1m_2=a/b.$$

If θ is the acute angle between the lines (i), then

$$\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|,$$

$$=\frac{2\sqrt{(h^2-ab)}}{|a+b|}.$$

$$\therefore \qquad \theta=\tan^{-1}\left\{\frac{2\sqrt{(h^2-ab)}}{|a+b|}\right\},$$

which is the required acute angle.

Remarks. 1. Note that the angle between the lines represented by

$$ax^2+2hxy+by^2+2gx+2fy+c=0, \qquad \ldots(i)$$

is the same as the angle between the lines represented by

$$ax^2+2hxy+by^2=0. \qquad \ldots(ii)$$

The reason for this is that the equation of the lines through the origin parallel to the lines represented by (i) is (ii). This can be seen as follows :

If (i) represents the lines

$$y=m_1x+c_1 \text{ and } y=m_2x+c_2,$$

then the equation of the lines through the origin parallel to the lines represented by (i) is

$$(y-m_1x)(y-m_2x)=0,$$

i.e., $$y^2-(m_1+m_2)xy+m_1m_2x^2=0,$$

i.e., $$y^2-\left(-\frac{2h}{b}\right)xy+\frac{a}{b}x^2=0,$$

i.e., $$ax^2+2hxy+by^2=0.$$

Hence the angle between the lines represented by (i) is the same as the angle between the lines represented by (ii). Note that (ii) is obtained by equating to zero the second degree terms in (i).

2. It follows from the above that the lines represented by

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

are perpendicular iff $a+b=0$ and are parallel iff $h^2-ab=0$.

10·13. CHANGE OF ORIGIN (TRANSLATION OF AXES)

Sometimes a problem can be solved more easily by changing the origin. In this section we shall show as to how we can change the origin, the new axes being parallel to the original ones.

Let Ox, Oy be a pair of rectangular co-ordinate axes, and let O′ be a point whose co-ordinates which respect to Ox, Oy are (h, k). Throught O′ draw lines O′X, O′Y parallel to Ox, Oy respectively. Let P be a point whose co-ordinates with respeøt to Ox, Oy are (x, y), and with respect to O′X, O′Y are (X, Y). To obtain the

relation between (x, y) and (X, Y), we draw PM $\perp Ox$ to meet OX in N. Also let YO′ (produced, if necessary) meet Ox in L.

Then $\quad x = OM,$

$= OL + LM,$

$= X + h$, since $LM = O'N = X$, and $OL = h$.

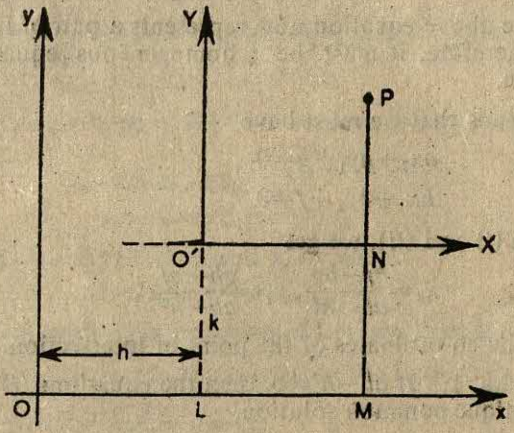


Fig. 10·32.

Similarly,

$$y = MP = MN + NP,$$

$= Y + k$, since $NP = Y$

and $\quad MN = LO' = k$.

Therefore, an equation in x and y can be transformed into an equation in X and Y by substituting

$$x = X + h \text{ and } y = Y + k.$$

10·14. THE POINT OF INTERSECTION OF THE LINES REPRESENTED BY THE GENERAL EQUATION OF SECOND DEGREE

Consider the general equation of second degree

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

such that $\triangle = 0$, so that it represents a pair of straight lines. Suppose that the lines represented by the above equation are not parallel, that is, $h^2 \neq ab$. Let (x_1, y_1) be the co-ordinates of the point of intersection of these lines. Now, transferring the origin to (x_1, y_1) by substituting

$$x = X + x_1, \; y = Y + y_1,$$

where X, Y are the new current co-ordinates, we get,

$$a(X + x_1)^2 + 2h(X + x_1)(Y + y_1) + b(Y + y_1)^2 + 2g(X + x_1) + 2f(Y + y_1) + c = 0.$$

i.e., $$(aX^2+2hXY+bY^2)+2(ax_1+hy_1+g)X+2(hx_1+by_1+f)Y +(ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c)=0.$$

Since $$ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c=0,$$

the above equation reduces to

$$(aX^2+2hXY+bY^2)+(ax_1+hy_1+g)X+(hx_1+by_1+f)Y=0.$$

Since the above equation now represents a pair of lines through the origin, therefore, it must be a homogeneous equation of the second degree.

This means that we must have

$$ax_1+hy_1+g=0, \qquad ...(i)$$

and $$hx_1+by_1+f=0. \qquad ...(ii)$$

Solving (*i*) and (*ii*), we get

$$x_1=\frac{hf-bg}{ab-h^2},\ y_1=\frac{gh-af}{ab-h^2},$$

which gives the co-ordinates of the point of intersection.

Remarks. 1. If $ab-h^2=0$, then the equations (*i*) and (*ii*) do not have a unique common solution.

If $\frac{a}{h}=\frac{b}{h}\neq\frac{g}{f}$, the equations do not have any common solution. This corresponds to the case of the two lines being parallel and distinct.

If $\frac{a}{h}=\frac{h}{b}=\frac{g}{f}$, the equations have infinitely many common solutions. This corresponds to the case when the lines are coincident (and so every point of the lines is a point of intersection).

2. If $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a pair of lines, the left-hand side of the above equation can be resolved into two linear factors. When equated to zero, these factors will give the equations of the two lines. We can then solve these two equations to obtain their point of intersection.

Example 33. *Find the equations of the lines represented by*

$$x^2+3xy+2y^2+4x+5y+3=0.$$

Also, find their point of intersection and the angle between them.

Solution. The given equation is

$$x^2+3xy+2y^2+4x+5y+3=0,$$

i.e., $$x^2+x(3y+4)+2y^2+5y+3=0,$$

i.e., $$\left[x+\left(\frac{3y}{2}+2\right)\right]^2-\left(\frac{3y}{2}+2\right)^2+2y^2+5y+3=0,$$

i.e., $\left(x+\frac{3y}{2}+2\right)^2-\frac{9y^2}{4}-6y-4+2y^2+5y+3=0,$

i.e., $\left(x+\frac{3y}{2}+2\right)^2-\left(\frac{y^2}{4}+y+1\right)=0,$

i.e., $\left(x+\frac{3y}{2}+2\right)^2-\left(\frac{y}{2}+1\right)^2=0,$

i.e., $\left[x+\frac{3y}{2}+2+\frac{y}{2}+1\right]\left[x+\frac{3y}{2}+2-\frac{y}{2}-1\right]=0,$

i.e., $(x+2y+3)(x+y+1)=0.$

Thus the equations of the lines represented by the given equation are

$$x+2y+3=0, \qquad ...(i)$$

and $$x+y+1=0. \qquad ...(ii)$$

Solving (*i*) and (*ii*) for x and y, we get

$$x=1 \text{ and } y=-2.$$

Thus the point of intersection of the two lines is $(1, -2)$.

Also, if θ is the acute angle between the lines, then

$$\tan\theta=\left|\frac{-\frac{1}{2}+1}{1+\frac{1}{2}}\right|=\frac{1}{3}$$

Hence, the angle between the lines is $\tan^{-1}\frac{1}{3}$.

Example 34. *For what value of c, does the equation*

$$2x^2+3xy+y^2+11x+6y+c=0$$

represent a pair of straight lines ?

Solution. Comparing the given equation with the general equation of the second degree

$$ax^2+2hxy+by^2+2gx+2fy+c=0, \text{ we get}$$

$$a=2, b=1, h=3/2, g=11/2, f=3.$$

A necessary and sufficient condition that the general equation of the second degree may represent a pair of lines is

$$abc+2fgh-af^2-bg^2-ch^2=0.$$

Thus the given equation will represent a pair of lines iff

$$2c+2.3 \cdot \frac{11}{2} \cdot \frac{3}{2}-2 \cdot 9-\frac{121}{4}-\frac{9}{4}c=0.$$

which on simplification yields $c=5$.

Example 35. *Find the point of intersection of the lines represented by*

$$2x^2-xy-y^2+5x+y+2=0.$$

Also, determine the angle between them.

Solution. If (x_1, y_1) is the point of intersection of the lines

$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$

then x_1, y_1 are obtained by solving the equations

$$ax_1+hy_1+g=0,$$

and $$hx_1+by_1+f=0.$$

Here, $\quad a=2, b=-1, h=-\dfrac{1}{2}, g=\dfrac{5}{2}, f=\dfrac{1}{2}, c=2.$

Thus the point of intersection of the given lines is given by

$$2x_1-\frac{1}{2}y_1+\frac{5}{2}=0,$$

and $$-\frac{1}{2}x_1-y_1+\frac{1}{2}=0.$$

These give, $x_1=-1, y_1=1.$

$\therefore$ $(-1, 1)$ is the point of intersection.

Also, if θ is the acute angle between the two lines, then

$$\tan\theta=\frac{2\sqrt{(h^2-ab)}}{|a+b|},$$

$$=\left|\frac{2\sqrt{(\frac{1}{4}+2)}}{2-1}\right|,$$

$$=3.$$

$\therefore$ $$\theta=\tan^{-1} 3.$$

Example 36. *Show that $x^2+6xy+9y^2+4x+12y-5=0$ represents two parallel straight lines. Find the distance between these lines.*

Solution. Arranged as a quadratic equation in x, the given equation is

$$x^2+2(3y+2)x+9y^2+12y-5=0,$$

or $$[x+(3y+2)]^2-(3y+2)^2+9y^2+12y-5=0,$$

or $$(x+3y+2)^2-9=0,$$

or $$(x+3y+2)^2-3^2=0,$$

or $$(x+3y+2+3)(x+3y+2-3)=0,$$

or $$(x+3y+5)(x+3y-1)=0.$$

$\therefore$ the lines represented by the given equation are

$$x+3y+5=0, \qquad \cdots(i)$$

and $$x+3y-1=0. \qquad \cdots(ii)$$

These lines are parallel since their slopes are equal.

To find the distance between these lines, suppose AB is the straight line represented by (*i*) and CD is the straight line represented by (*ii*). It can be easily seen that the relative positions of AB and CD are as shown in Fig. 10·33.

Let OM and ON be the perpendiculars drawn from the origin O to the lines AB and CD respectively.

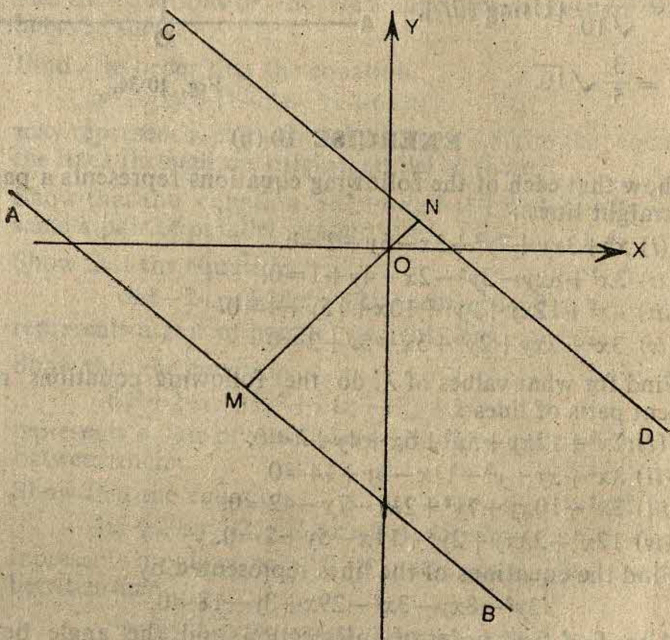


Fig. 10·33.

Clearly, the distance between AB and CD is OM+ON.

Now, $$OM=\frac{5}{\sqrt{1+9}}=\frac{5}{\sqrt{10}},$$

and $$ON=\frac{1}{\sqrt{(1+9)}}=\frac{1}{\sqrt{10}}.$$

Thus the required distance

$$=\frac{5}{\sqrt{10}}+\frac{1}{\sqrt{10}},=\frac{3\sqrt{10}}{5}.$$

Remark. The distance between the lines AB and CD in the above example can also be obtained as follows :

Let $P(x', y')$ be any point on the line CD.

Draw $PQ \perp AB$.

Since (x', y') lies on CD,

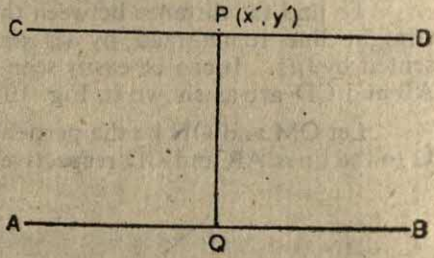
$x'+3y'-1=0, \quad ...(iii)$

Now,

$$PQ=\frac{|x'+3y'+5|}{\sqrt{1+9}},$$

$$=\frac{1+5}{\sqrt{10}} \text{ [Using } (iii)],$$

$$=\frac{3}{5}\sqrt{10}.$$



Frg, 10·34.

EXERCISE 10 (n)

1. Show that each of the following equations represents a pair of straight lines :
   (i) $x^2+3xy+2y^2+3x-5y+2=0.$
   (ii) $3x^2+8xy-3y^2-2x-4y+1=0.$
   (iii) $4x^2+12xy+9y^2+10x+15y+4=0.$
   (iv) $3x^2+7xy+2y^2+5x+5y+2=0.$

2. Find for what values of λ, do the following equations represent pairs of lines :
   (i) $7x^2+12xy+5y^2+6x+4y+\lambda=0.$
   (ii) $\lambda x^2+xy-y^2-11x-5y+14=0.$
   (iii) $3x^2-10xy+7y^2+2\lambda x-7y-42=0.$
   (iv) $12x^2+2\lambda xy+2y^2+11x-5y+2=0.$

3. Find the equations of the lines represented by
$$3y^2-8xy-3x^2-29x+3y-18=0.$$
Also, find their point of intersection and the angle between them.

4. Find the equations of the lines represented by
$$6x^2+7xy-20y^2+x+14y-2=0.$$
Also, find their point of intersection and the angle between them.

5. Show that the equation
$$2x^2-5xy+2y^2-3x+3y+1=0$$
represents two straight lines intersecting at an angle $\tan^{-1}\frac{3}{4}$.

6. Show that the equation
$$x^2-xy-6y^2-3x+14y-4=0$$
represents a pair of lines inclined at 45° to each other. Also, find their point of intersection.

7. Find the point of intersection of the lines represented by
$$x^2+3xy+y^2-7x-3y+1=0.$$
Also, find the angle between them.

8. Find the point of intersection of the lines represented by
$$3x^2+10xy+8y^2+14x+22y+15=0.$$
Also, find the angle between them.

9. If $6x^2-11xy-10y^2+19y+c=0$ represents two straight lines, find the equations of the lines and the tangent of the angle between them.

10. Find λ in order that the equation
$$x^2+\lambda xy+y^2-5x-7y+6=0$$
may represent a pair of straight lines. Write the equation of the lines through the origin parallel to these.

11. Show that the equation $9x^2+6xy+y^2+9x+3y+2=0$ represents a pair of parallel straight lines.

12. Show that the equation
$$9x^2-24xy+16y^2-12x+16y-12=0$$
represents a pair of parallel straight lines.

13. Show that the equation
$$16x^2+24xy+9y^2+24x+18y+5=0$$
represents a pair of parallel straight lines and find the distance between them.

14. Show that the equation
$$4x^2+20xy+25y^2+8x+20y-5=0$$
represents a pair of parallel straight lines and find the distance between them.

TEST YOUR UNDERSTANDING X

In each of the following, four alternatives are given. Put a tick mark ($\sqrt{}$) against the correct alternative.

1. The equation of the straight line which passes through the point $(1, -2)$ and cuts off equal intercepts from the axes is

(a) $x+y=1$ (b) $x-y=1$

(c) $x+y+1=0$ (d) $x-y-2=0$.

2. The equation of the straight line passing through $(1, 2)$ and perpendicular to $x+y+1=0$ is

(a) $y-x+1=0$ (b) $y-x-1=0$

(c) $y-x+2=0$ (d) $y-x-2=0$.

3. If the lines $3y+4x=1$, $y=x+5$ and $kx+5y-3=0$ are concurrent, then the value of k is

(a) 1 (b) 3

(c) 6 (d) 0.

4. The distance between the lines $3x+4y=9$ and $6x+8y=15$ is

(a) $\frac{3}{2}$ (b) $\frac{3}{10}$

(c) 6 (d) $\frac{3}{5}$.

5. The lines $(p+2q)x+(p-3q)y=p-q$ for different values of p and q pass through the point

(a) $\left(\frac{3}{2}, \frac{5}{2}\right)$ (b) $\left(\frac{2}{5}, \frac{2}{5}\right)$

(c) $\left(\frac{3}{5}, \frac{3}{5}\right)$ (d) $\left(\frac{2}{5}, \frac{3}{5}\right)$.

6. The equation of the straight line passing through the point (2, 1) and parallel to $2x-y+1=0$ is

(a) $2x-y=0$ (b) $x+2y-4=0$

(c) $2x-y-3=0$ (d) $2x-y+3=0$.

7. The angle between the straight lines $x-y+1=0$ and $x+y-2=0$ is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{6}$.

8. The equation of the straight line which passes through the point of intersection of $x+y=1$ and $x-y=1$ and is parallel to the line $3x+y=1$ is

(a) $x-3y+2=0$ (b) $3x+y+3=0$

(c) $3x+y-3=0$ (d) $x+3y-1=0$.

9. The straight lines represented by the equation $x^2-y^2=0$ are inclined to each other at an angle

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$

(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$.

10. The equation $x^2+4xy+4y^2+3x+6y+2=0$ represents

(a) a pair of parallel straight lines

(b) a pair of perpendicular straight lines

(c) a pair of straight lines passing through the origin

(d) a pair of straight lines passing through the point (−1, −2).

REVIEW EXERCISE X

1. The vertices of a triangle are $A(5, -2)$, $B(-9, 3)$ and $C(-3, 5)$. Find the equations of the medians.
2. Prove that the points $(4, 5)$, $(6, -1)$ and $(0, 17)$ are in a straight line and find the equation of the straight line through them.
3. Find the ratio in which the join of $(1, 2)$ and $(-9, -6)$ is divided by the join of $(3, 7)$ and $(-11, -11)$.
4. Find the equation of the straight line which bisects the join of $(2, 1)$ and $(3, 5)$ and also bisects the join of $(1, -3)$ and $(-2, 1)$.
5. Obtain the equation of the straight line through the point $(4, -1)$ and inclined at an angle of $30°$ with the x-axis.
6. Find the equation of the straight line which passes through the point $(2, -5)$ and has intercepts on the axes equal in magnitude but opposite in sign.
7. In what ratio is the line joining $(1, 3)$ and $(2, 7)$ divided by $3x+y=9$?
8. Find the equation of the straight line which passes through the point $(3, -2)$ and cuts off positive intercepts on the x and y-axes which are in the ratio $4 : 3$.
9. Find the area of the triangle formed by the lines
$$y=2x-1,\ 2y+3x=5 \text{ and } x+y+1=0.$$
10. For what value of k are the lines $4x+3y-3=0$, $kx+7y=11$, and $x+y=2$, concurrent ?
11. Find the equation of the straight line joining the origin to the point of intersection of the straight lines
$$5x+6y-1=0 \text{ and } x+2y-7=0.$$
12. Prove that the points $(3, 0)$, $(1, 3)$, $(3, 4)$ and $(5, 1)$ are the co-ordinates of the vertices of a parallelogram and find the angle between its diagonals.
13. Find the equation of the straight line perpendicular to the straight line
$$3x-5y+7=0$$
and passing through the intersection of the straight lines
$$5x-6y=1,\ 3x+2y+5=0.$$
14. Find the equations of the diagonals of the parallelogram formed by the straight lines whose equations are
$$2x-y-5=0,\ 2x-y+7=0,$$
$$3x+2y-5=0,\ 3x+2y+4=0.$$

15. The vertices of a triangle are $(4, -3)$, $(-2, 1)$ and $(2, 3)$. Find the co-ordinates of the circumcentre of the triangle.

16. The line $2x-3y-4=0$ is the perpendicular bisector of the line AB and B is the point $(5, 6)$. What are the co-ordinates of A ?

17. Find the distance between the straight lines

$$7x+24y+3=0 \text{ and } 7x+24y+28=0.$$

18. Find the length of the perpendicular from (b, a) upon the straight line

$$a(x-a)=b(y-b).$$

19. For what value of λ will the equation

$$\lambda x^2-10xy+12y^2+5x-16y-3=0$$

represent a pair of straight lines ? Also find the equations of the lines.

20. Show that the four lines given by $12x^2+7xy-12y^2=0$ and $12x^2+7xy-12y^2-x+7y-1=0$ lie along the sides of a square.

SUMMARY

1. The equation of any straight line parallel to y-axis is of the form $x=a$.
2. The equation of any straight line parallel to the x-axis is of the form $y=b$.
3. The equation of a straight line in the *slope-intercept form* is $y=mx+c$.
4. The equation of a straight line in the *point-slope form* is

$$y-y_1=m(x-x_1).$$

5. The equation of a straight line in the *two-point form* is

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}.$$

6. The equation of a straight line in the *Symmetrical form* or *distance form* is

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r.$$

7. The equation of a straight line in the *intercept form* is

$$\frac{x}{a}+\frac{y}{b}=1.$$

8. The equation of a straight line in the *perpendicular form* is

$$x\cos\alpha+y\sin\alpha=p.$$

9. The point of intersection of the straight lines

$$ax+by+c=0 \text{ and } a'x+b'y+c'=0 \text{ is}$$

$$\left(\frac{bc'-b'c}{ab'-a'b}, \frac{ca'-c'a}{ab'-a'b}\right).$$

10. The angle between the straight lines whose equations are $y=m_1x+c_1$, and $y=m_2x+c_2$, is

$$\tan^{-1}\left|\frac{m_1-m_2}{1+m_1m_2}\right|.$$

11. The angle between the straight lines whose equations are

$$A_1x+B_1y+C_1=0 \text{ and } A_2x+B_2y+C_2=0, \text{ is}$$

$$\tan^{-1}\left|\frac{A_1B_2-A_2B_1}{A_1A_2+B_1B_2}\right|.$$

12. The straight lines whose equations are

$$y=m_1x+c_1 \text{ and } y=m_2x+c_2,$$

are (*i*) parallel if $m_1=m_2$,

(*ii*) perpendicular if $m_1m_2=-1$.

13. The distance of the point (x_1, y_1) from the straight line $ax+by+c=0$ is

$$\frac{|ax_1+by_1+c|}{\sqrt{(a^2+b^2)}}.$$

14. The angle between the lines $ax^2+2hxy+by^2=0$ is

$$\tan^{-1}\frac{2\sqrt{h^2-ab}}{|a+b|}.$$

15. The equation of the bisectors of the angles between the lines represented by $ax^2+2hxy+by^2=0$ is

$$\frac{x^2-y^2}{a-b}=\frac{xy}{h}.$$

16. A necessary and sufficient condition for the equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

to represent a pair of straight lines is that

$$\Delta=abc+2fgh-af^2-bg^2-ch^2=0.$$

□□

GOTTFRIED WILHELM LEIBNIZ (1646-1716)

Gottfried Wilhelm Leibniz was born in Leipzig in 1646. When he was only eight years old, he could read Latin and Greek. Mathematics was only one of the many fields in which Leibniz showed conspicuous genius. Before he was twenty, he had mastered the ordinary text-book knowledge of mathematics, philosophy, theology and law.

Leibniz is said to have lived not one life but several. As a diplomat, historian, philosopher and mathematician, he did enough in each field to fill one ordinary working life. It was in 1672-73, while he was on a diplomatic mission, first in Paris and then in London, that he exhibited his calculating machine to the Royal Society.

The last seven years of his life were embittered by the controversy which others had brought upon him and Newton concerning whether he had discovered the Calculus independently of Newton. Leibniz made important contributions to the study of geometry as well.

CHAPTER 11

Circles

11·1. INTRODUCTION

A *circle* is the locus of a point which moves in a plane so that its distance from a fixed point is constant. The fixed point is called the *centre* of the circle and the fixed distance is called the *radius* of the circle.

11·2. THE STANDARD FORM OF THE EQUATION OF A CIRCLE

Let O (0, 0) be the centre of a circle and let r be its radius. If P(x, y) be any point on the circle, then

$$OP = r.$$

But

$$OP = \sqrt{(x-0)^2+(y-0)^2}.$$
$$= \sqrt{x^2+y^2}.$$

Hence the required equation is

$$\sqrt{x^2+y^2} = r,$$

or

$$x^2+y^2 = r^2.$$

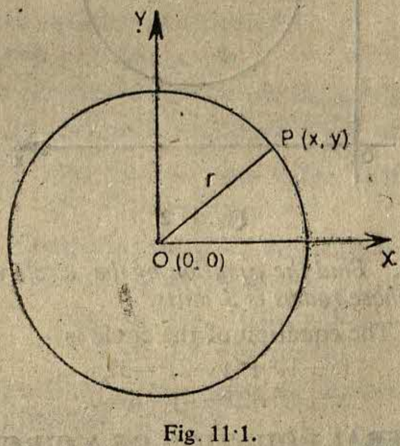


Fig. 11·1.

Example 1. *Find the equation of the circle whose centre is the origin and whose radius is 3 units.*

Solution. The equation of the circle is

$$\sqrt{(x-0)^2+(y-0)^2}=3,$$

or

$$x^2+y^2=9.$$

11·3. EQUATION OF A CIRCLE WHOSE CENTRE AND RADIUS ARE GIVEN

Let r be the radius and $C(h, k)$ the centre of a circle. If $P(x, y)$ be any point on the circle, then

$$CP=r.$$

But, by the distance formula,

$$CP=\sqrt{(x-h)^2+(y-k)^2}.$$

Hence the required equation is

$$\sqrt{(x-h)^2+(y-k)^2}=r,$$

or

$$(x-h)^2+(y-k)^2=r^2.$$

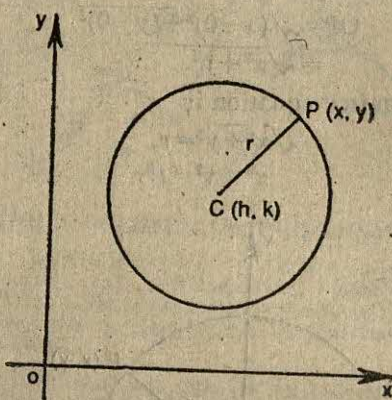


Fig. 11·2.

Example 2. *Find the equation of the circle whose centre is the point (1, 2) and whose radius is 3 units.*

Solution. The equation of the circle is

$$(x-1)^2+(y-2)^2=3^2,$$

or

$$x^2+y^2-2x-4y-4=0.$$

11·4. THE GENERAL EQUATION OF A CIRCLE

The equation of the circle whose centre is the point (h, k) and whose radius is r, is

$$(x-h)^2+(y-k)^2=r^2,$$

or $$x^2-2hx+h^2+y^2-2ky+k^2=r^2,$$

or $$x^2+y^2-2hx-2ky+h^2+k^2-r^2=0. \qquad \text{...(1)}$$

Equation (1) has the following special features :

(*i*) It is of the second degree in x, y.

(*ii*) The coefficients of x^2 and y^2 are equal.

(*iii*) The term containing xy is absent.

The equation of every circle has the above properties.

Conversely, every equation having the above properties represents a circle.

The general equation with the above properties is

$$x^2+y^2+2gx+2fy+c=0. \qquad \text{...(2)}$$

Equation (2) may be re-written as

$$(x^2+2gx+g^2)+(y^2+2fy+f^2)=g^2+f^2-c,$$

or $$(x+g)^2+(y+f)^2=g^2+f^2-c,$$

showing that the distance of the point (x, y) from the fixed point $(-g, -f)$ is constant and equal to $\sqrt{g^2+f^2-c}$. The locus of the point (x, y) is, therefore, a circle whose centre is the point $(-g, -f)$ and whose radius is $\sqrt{(g^2-f^2-c)}$.

Thus, *the equation*

$$x^2+y^2+2gx+2fy+c=0,$$

represents a circle whose centre is the point $(-g, -f)$ *and whose radius is* $\sqrt{g^2+f^2-c}$.

Rule. If the equation of a circle be

$$x^2+y^2+2gx+2fy+c=0,$$

the coordinates of the centre$=(-\frac{1}{2}$ coeff. of x, $-\frac{1}{2}$ coeff. of $y)$,

radius$=\sqrt{\{(\frac{1}{2}\text{ coeff. of } x)^2+(\frac{1}{2}\text{ coeff. of } y)^2-\text{constant term}\}}$

Example 3. *Find the centre and radius of the circle*

$$x^2+y^2+6x-8y-11=0.$$

Solution. Comparing the given equation with

$$x^2+y^2+2gx+2fy+c=0,$$

we have $$2g=6,\ 2f=-8,\ c=-11,$$

or $$g=3,\ f=-4,\ c=-11.$$

Hence the centre of the circle is the point $(-g, -f)$, *i.e.*, $(-3, 4)$ and the radius of the circle

$$=\sqrt{g^2+f^2-c}$$

$$=\sqrt{(3)^2+(-4)^2+11}=6.$$

Aliter. The given equation is

$$x^2+y^2+6x-8y-11=0,$$

or $$(x^2+6x+9)+(y^2-8y+16)=9+16+11,$$

or $$(x+3)^2+(y-4)^2=36.$$

Therefore, the centre is $(-3, 4)$ and the radius is 6.

Example 4. *Find the equation of the circle whose centre is the point $(1, -2)$ and which passes through the centre of the circle*

$$x^2+y^2+4x-4y+1=0.$$

Solution. Let A be the centre of the circle

$$x^2+y^2+4x-4y+1=0,$$

so that A is the point $(-2, 2)$.

If C be the centre of the required circle, the radius of the required circle is CA. By the distance formula,

$$CA=\sqrt{(-2-1)^2+(2+2)^2}=5.$$

Thus the centre of the required circle is the point $(1, -2)$ and its radius is 5.

Hence the equation of the required circle is

$$(x-1)^2+(y+2)^2=5^2,$$

or $$x^2+y^2-2x+4y-20=0.$$

EXERCISE 11 (*a*)

1. Find the equation of the circle whose
   (*i*) centre is $(3, 2)$ and radius is 5 units ;
   (*ii*) centre is $(5, -12)$ and radius is 13 units ;
   (*iii*) centre is (a, b) and radius is $\sqrt{a^2+b^2}$;
   (*iv*) centre is $\left(\frac{3}{2}, -\frac{5}{2}\right)$ and radius is 1 unit.
2. Find the centre and the radius of each of the following circles :
   (*i*) $x^2+y^2=16$; (*ii*) $x^2+y^2-6y=0$;
   (*iii*) $x^2+y^2-4x=0$; (*iv*) $x^2+y^2-8x-6y=0$;
   (*v*) $x^2+y^2+2x+2y+1=0$; (*vi*) $7x^2+7y^2-4x-y=3$.
3. Find the centre and the radius of the circle whose equation is $x^2+y^2-6x+4y-36=0.$
4. Find the centre and the radius of the circle whose equation is $ax^2+ay^2=bx+cy.$
5. What is the locus of the equation
   $$x^2+y^2+6x-6y+9=0 ?$$

6. Find the equation of the circle through $(4, -2)$ and having the same centre as the circle

$$x^2+y^2-6x+4y-1=0.$$

7. Find the equation of the circle whose centre is $(-1, -2)$ and which has the same radius as the circle

$$x^2+y^2+3x-7y-2=0.$$

8. Find the equation of the circle whose centre is $(-5, 7)$ and which passes through the centre of the circle

$$x^2+y^2+8x-6y+1=0.$$

9. Find the equation of the circle which is concentric with the circle

$$x^2+y^2+7x-5y+1=0$$

and whose radius is 4 units.

10. Find the equation of the circle which is concentric with the circle

$$x^2+y^2-3x+4y-2=0$$

and which passes through the point $(-1, -2)$.

11·5. EQUATION OF THE CIRCLE THROUGH THREE GIVEN POINTS

Since the general equation of a circle contains three independent constants *viz.*, g, f and c, therefore, in general one circle can be drawn so as to pass through three points. For, on substituting the co-ordinates of the points in turn in the equation

$$x^2+y^2+2gx+2fy+c=0,$$

we obtain three equations in g, f and c, which have one solution **in general**.

Remark. The words *in general* have been used here to indicate that there is an exceptional case, *viz*, when the three points are distinct and collinear.

Example 5. *Find the equation of the circle that passes through the points $(1, 3)$, $(-1, 1)$ and $(2, -1)$.*

Solution. Let $x^2+y^2+2gx+2fy+c=0$, ...(*i*)
be the equation of the circle. By substituting the co-ordinates of the given points in turn in equation (*i*), we get

$$10+2g+6f+c=0, \qquad ...(ii)$$

$$2-2g+2f+c=0, \qquad ...(iii)$$

and

$$5+4g-2f+c=0. \qquad ...(iv)$$

Subtracting (*iv*) from (*iii*), we get

$$-3-6g+4f=0. \qquad ...(v)$$

Subtracting (*iii*) from (*ii*), we get

$$8+4g+4f=0. \qquad ...(vi)$$

Subtracting (*vi*) from (*v*), we get

$$-11-10g=0,$$

or $$g=-\frac{11}{10}.$$

Substituting the value of g in (*vi*), we get

$$8-\frac{22}{5}+4f=0,$$

or $$f=-\frac{9}{10}.$$

Substituting the values of g and f in (*ii*), we get

$$10-\frac{11}{5}-\frac{27}{5}+c=0,$$

or $$c=-\frac{12}{5}.$$

Substituting the value of g, f and c in (*i*), the required equation becomes

$$x^2+y^2-\frac{11}{5}x-\frac{9}{5}y-\frac{12}{5}=0.$$

or $$5x^2+5y^2-11x-9y-12=0.$$

Example 6. *Find the equation of the circle that passes through the points (1, −2) and (−2, 2) and whose centre is on the straight line*

$$8x-4y+9=0.$$

Solution. Let the equation of the circle be

$$x^2+y^2+2gx+2fy+c=0. \qquad ...(i)$$

Since the point (1, −2) lies on the circle, we have

$$5+2g-4f+c=0. \qquad ...(ii)$$

Since the point (−2, 2) lies on the circle, we have

$$8-4g+4f+c=0. \qquad ...(iii)$$

The centre of the circle (*i*) is $(-g, -f)$. Since it lies on the straight line

$$8x-4y+9=0,$$

we have $$-8g+4f+9=0. \qquad ...(iv)$$

Subtracting (*iii*) from (*ii*), we have

$$6g-8f-3=0. \qquad ...(v)$$

Solving (*iv*) and (*v*), we have

$$\frac{g}{60}=\frac{f}{30}=\frac{1}{40},$$

or $$g=\frac{3}{2},\ f=\frac{3}{4}.$$

Substituting the values of g, f in (*iii*), we have

$$c = -5.$$

Substituting the values of g, f and c in (*i*), the required equation becomes

$$x^2+y^2+3x+\frac{3}{2}y-5=0,$$

or $$2x^2+2y^2+6x+3y-10=0.$$

EXERCISE 11 (*b*)

1. Find the equation of the circle that passes through the points
   (*i*) (2, 0), (0, 3), (2, 3) ;
   (*ii*) (0, 1), (2, 3), (−2, 5) ;
   (*iii*) (2, 3), (3, 2), (5, 1) ;
   (*iv*) (1, 2), (3, −1), (5, −6).
2. Find the equation of the circle that passes through the points $(2a, 0)$, $(0, 2b)$ and $(a+b, a+b)$.
3. Find the equation of the circle that passes through the points (4, 1) and (6, 5) and has its centre on the line $4x+y=16$.
4. A circle has its centre on the line $x=2y$ and passes through the points (−1, 2), (3, −2). Find the co-ordinates of the centre and the equation of the circle.
5. Show that the points (2, 2), (5, 3), (6, 0), (3, −1) lie on a circle. Find the centre and the radius of this circle.
6. Show that the four points (3, 0), (0, 3), (3, 3) and (0, 0) are concyclic.
7. Show that the points (4, 3), (8, −3), (0, 9) cannot lie on a circle.
8. Find the equation of the circle circumscribing the triangle formed by the lines

   $$x-3y+5=0,\ x+2y=0,\ 3x+y-5=0.$$
9. Find the equations of the circles which pass through the points (0, 3) and (0, −3) and have radius equal to 5.
10. Find the equations of the circles which pass through the points (7, 10) and (−7, −4) and have radius equal to 10.

16·6. THE EQUATION OF THE CIRCLE WHEN THE CO-ORDINATES OF THE ENDS OF A DIAMETER ARE GIVEN

Let (x_1, y_1) and (x_2, y_2) be the co-ordinates of the ends A, B of a diameter.

Let (x, y) be the co-ordinates of any point P on the circle.

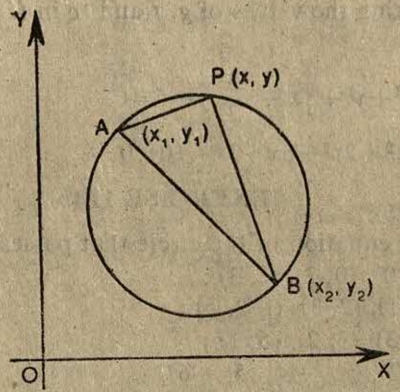


Fig. 11·3.

Since angle in a semi-circle is a right angle, $\angle$APB is a right angle. Therefore, the lines AP, BP are perpendicular to each other. Now

Slope of AP$=\dfrac{y-y_1}{x-x_1}$,

slope of BP$=\dfrac{y-y_2}{x-x_2}$.

$$\therefore \quad \frac{y-y_1}{x-x_1}\cdot\frac{y-y_2}{x-x_2}=-1,$$

or $$(y-y_1)(y-y_2)=-(x-x_1)(x-x_2).$$

Hence $$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0,$$

is the required equation.

Aliter. Since $\angle \text{APB}=90°$,

therefore, $\text{PA}^2+\text{PB}^2=\text{AB}^2$,

or $$\{(x-x_1)^2+(y-y_1)^2\}+\{(x-x_2)^2+(y-y_2)^2\}$$
$$=(x_2-x_1)^2+(y_2-y_1)^2,$$

or $$x^2+y^2-x(x_1+x_2)-y(y_1+y_2)+x_1x_2+y_1y_2=0,$$

or $$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0.$$

Example 7. *Find the equation of the circle when the co-ordinates of the extremities of a diameter are (4, 1) and (6, 5). Find the radius and the co-ordinates of the centre of this circle.*

Solution. Since the equation of the circle described on the join of the points (x_1, y_1) and (x_2, y_2) as diameter is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0,$$

therefore, the equation of the circle described on the join of the points (4, 1), (6, 5) as diameter is

$$(x-4)(x-6)+(y-1)(y-5)=0,$$

or $$x^2+y^2-10x-6y+29=0.$$

Co-ordinates of the centre are (5, 3).

$$\text{Radius}=\sqrt{5^2+3^2-29}=\sqrt{5}.$$

Aliter. The centre of the circle is the mid-point of join of (4, 1) and (6, 5). Therefore, the centre of the circle is the point

$$\left(\frac{4+6}{2}, \frac{1+5}{2}\right), \text{ i.e., } (5, 3).$$

$$\text{Diameter of the circle}=\sqrt{(6-4)^2+(5-1)^2},$$
$$=2\sqrt{5}.$$

$$\therefore \qquad \text{Radius}=\sqrt{5}.$$

Hence the equation of the circle is

$$(x-5)^2+(y-3)^2=(\sqrt{5})^2,$$

or $$x^2+y^2-10x-6y+29=0.$$

Parametric equations of a circle

The equation of a circle can be expressed in parametric form.

Let C be a circle with centre at the origin and radius r. Let $P(x, y)$ be any point on the circle and let M be the foot of the perpendicular drawn from P on OX. If $\angle POX=\theta$, then from the right-angled triangle POM, we have

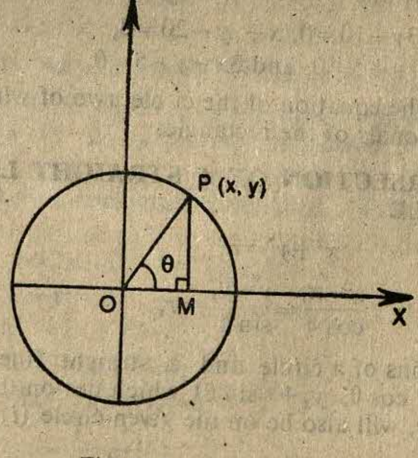


Fig. 11.4.

$x = \text{OM} = \text{OP} \cos \theta = r \cos \theta,$

$y = \text{MP} = \text{OP} \sin \theta = r \sin \theta.$

We find that the co-ordinates of any point (x, y) on C can be expressed in terms of the parameter θ in the form

$$x = r \cos \theta, \quad y = r \sin \theta. \qquad \text{...(A)}$$

Conversely, if the co-ordinates of a point in the plane be given by (A), then we have $x^2+y^2=r^2$, so that the point lies on the circle. It follows that a point (x, y) lies on a circle C if and only if its co-ordinates are of the form (A). The equations (A) are known as *parametric equations* of the circle. In a similar fashion we can easily see that the parametric equations of a circle whose centre is (h, k) and radius is r, are given by

$$x = h + r \cos \theta, \ y = k + r \sin \theta,$$

θ being the parameter.

EXERCISE 11 (*c*)

1. Find the equation of the circle having for a diameter the line joining the points (0, 1) and (1, 1).
2. Find the equation of the circle described on the join of the points (3, 4) and (2, -7) as a diameter.
3. Find the equation of the circle drawn on the line joining (-1, 2) and (3, -4) as a diameter. Find the radius and co-ordinates of the centre also.
4. Find the equation of the circle which passes through the origin and cuts off intercespts equal to 2 and 3 units respeectively from the axes.
5. The equations of the sides of a rectangle are

   $x+3y-10=0, \ x+3y-20=0,$

   $3x-y+5=0, \text{ and } 3x-y-5=0.$

   Obtain the equation of the circle, two of whose diameters are the diagonals of the rectangle.

11·8. INTERSECTION OF A STRAIGHT LINE AND A CIRCLE

Let $$x^2+y^2=a^2 \qquad \text{...}(i)$$

and $$\frac{x-x_1}{\cos \theta} = \frac{y+y_1}{\sin \theta} = r, \qquad \text{...}(ii)$$

be the equations of a circle and a straight line respectively. The point $(x_1+r \cos \theta, \ y_1+r \sin \theta)$ which lies on the given line (*ii*) for all values of r, will also be on the given circle (*i*), if r satisfies the equation

$$(x_1+r \cos \theta)^2+(y_1+r \sin \theta)^2=a^2,$$

i.e., $$r^2+2r(x_1 \cos \theta+y_1 \sin \theta)+x_1^2+y_1^2-a^2=0, \quad ...(iii)$$

and this being a quadratic equation in r, gives two values, say r_1 and r_2, of r which may be real and distinct, equal, or complex. Then

$(x_1+r_1 \cos \theta, y_1+r_1 \sin \theta)$ and $(x_1+r_2 \cos \theta, y_1+r_2 \sin \theta)$ are the two points of intersection, which may be real (either distinct or coincident) or imaginary.

Thus, *every straight line meets a circle in two points which may be real (either distinct or coincident) or imaginary.*

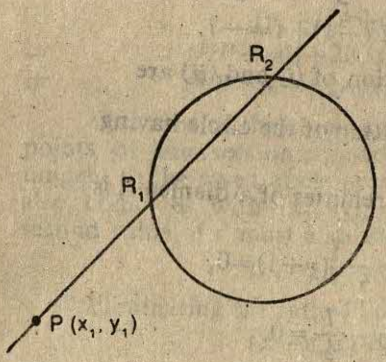


Fig. 11·5.

Example 8. *Find the points of intersection of the straight line $x+y=2$ and the circle $x^2+y^2=4$.*

Solution. For any point (x, y) on the straight line

$$x+y=2, \text{ we have } y=2-x. \quad ...(i)$$

Substituting the above value of y in the equation

$$x^2+y^2=4,$$

we have $$x^2+(2-x)^2=4, \quad ...(ii)$$

i.e., $$2x^2-4x=0,$$

i.e., $$x(x-2)=0,$$

so that $$x=0 \text{ or } x=2.$$

From (i) we find that

when $$x=0, y=2-0=2;$$

when $$x=2, y=2-2=0.$$

Therefore, the points of intersection are (0, 2) and (2, 0).

Example 9. *Find the equation of the circle described on the intersection of the circle $x^2+y^2=2$ and the straight line $y=2x+1$ as a diameter.*

Solution. The x-co-ordinates of the points of intersection of the straight line

$$y=2x+1 \quad ...(i)$$

and the circle $$x^2+y^2=2$$

are given by $$x^2+(2x+1)^2=2,$$

i.e., $$5x^2+4x-1=0, \quad ...(ii)$$

i.e., $$(5x-1)(x+1)=0,$$

i.e., $$x=\frac{1}{5} \text{ or } -1. \quad ...(iii)$$

When $$x=\frac{1}{5},\ y=2.\frac{1}{5}+1=\frac{7}{5}.$$

When $$x=-1,\ y=2.(-1)+1=-1.$$

Therefore, the points of intersection of (*i*) and (*ii*) are $\left(\frac{1}{5}, \frac{7}{5}\right)$ and $(-1, -1)$. The equation of the circle having $\left(\frac{1}{5}, \frac{7}{5}\right)$ and $(-1, -1)$ as the extremities of a diameter, is

$$\left(x-\frac{1}{5}\right)(x+1)+\left(y-\frac{7}{5}\right)(y+1)=0,$$

or $$x^2+y^2+\frac{4}{5}x+y^2-\frac{2}{5}y-\frac{7}{5}=0,$$

or $$5x^2+5y^2+4x-2y-8=0.$$

EXERCISE 11 (*d*)

1. Find the points of intersection of the straight line
$$3x+4y+7=0$$
and the circle $x^2+y^2-4x-6y-12=0$.
2. Find the points of intersection of the straight line
$$2x+y-2=0$$
and the circle $x^2+y^2-x-2y=0$.
3. Find the middle point of the intercept of the circle
$$x^2+y^2-4x-6y-12=0$$
on the straight line $x+y+1=0$.
4. Find the equation of the circle which passes through the point (1, 2) and also through the points of intersection of the straight line $x+y=0$ and the circle
$$x^2+y^2-4x-4y-2=0.$$
5. Find the equation of the circle having as the ends of a diameter the points of intersection of the straight line
$$x+y-2=0$$
and the circle $x^2+y^2+2x-2y-1=0$.

11·9. EQUATION OF THE TANGENT TO THE CIRCLE $x^2+y^2=a^2$ AT THE POINT (x_1, y_1)

Let (x_1, y_1) be a point on the circle, so that we have

$$x_1^2+y_1^2=a^2. \qquad \cdots(i)$$

The points of intersection of any line

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=r \qquad \cdots(ii)$$

through (x_1, y_1) with the given circle are (x_1+lr, y_1+mr), where the values of r are the roots of the equation

$$(x_1+lr)^2+(y_1+mr)^2=a^2,$$

or $$r^2(l^2+m^2)+2r(lx_1+my_1)+x_1^2+y_1^2-a^2=0,$$

or $$r^2(l^2+m^2)+2r(lx_1+my_1)=0, \text{ by } (i).$$

One root of the above equation in r is zero, so that one of the points of intersection coincides with (x_1, y_1). The line (ii) will be a tangent to the given circle provided the second point of intersection also coincides with (x_1, y_1), the condition for which is that the second value of r must also vanish. This requires

$$lx_1+my_1=0. \qquad \ldots(iii)$$

Eliminating the ratio $l:m$ between (ii) and (iii), we have

$$(x-x_1)x_1+(y-y_1)y_1=0,$$

or $$xx_1+yy_1-(x_1^2+y_1^2)=0,$$

or $$xx_1+yy_1-a^2=0, \text{ by } (i),$$

or $$xx_1+yy_1=a^2.$$

We shall now state and prove two basic facts about tangents to a circle. You must be already familiar with these facts. We are proving them here to illustrate the use of the methods of co-ordinate geometry.

Theorem 11·1. *The length of the perpendicular from the centre of a circle on any tangent to it is equal to the radius.*

Proof. Let the equation of any circle be $x^2+y^2=a^2$. The equation of the tangent to the circle at a point (x_1, y_1) on it is

$$xx_1+yy_1=a^2. \qquad \ldots(i)$$

The length of the perpendicular from the centre $(0, 0)$ on (i) is

$$a^2/\sqrt{(x_1^2+y_1^2)}. \qquad \ldots(ii)$$

Since (x_1, y_1) lies on the circle, therefore,

$$x_1^2+y_1^2=a^2. \qquad \ldots(iii)$$

From (ii) and (iii) we find that the required length is $a^2/a=a=$ radius of the circle.

Theorem 11·2. *The tangent to a circle at any point is perpendicular to the radius through that point (i.e., the line joining the point to the centre of the circle).*

Proof. Let the equation of any circle be $x^2+y^2=a^2$, and let (x_1, y_1) be any point on it.

The equation of the tangent to the circle at (x_1, y_1) is $xx_1+yy_1=a^2$, so that the slope of the tangent is $-x_1/y_1$. Also, the slope of the line joining the centre $(0, 0)$ to the point (x_1, y_1) is y_1/x_1. Since the product of the slopes of this line and the

tangent is $(y_1/x_1) \cdot (-x_1/y_1) = -1$, therefore, the two lines are perpendicular to each other.

Example 10. *Find the condition that the straight line $y=mx+c$ may meet the circle $x^2+y^2=a^2$ in coincident points.*

Solution. To find the points of intersection of the straight line

$$y=mx+c, \qquad \ldots(i)$$

and the circle $\quad x^2+y^2=a^2, \qquad \ldots(ii)$

we have to obtain the common solutions of equations (i) and (ii). Substituting the value of y from (i) in (ii), we have

$$x^2+(mx+c)^2=a^2,$$

i.e., $\quad (1+m^2)x^2+2mcx+c^2-a^2=0. \qquad \ldots(iii)$

The points of intersection coincide if and only if the roots of (iii) are equal, the condition for which is

$$4m^2c^2=4(1+m^2)(c^2-a^2),$$

i.e., $\quad c^2=a^2(1+m^2).$

Hence the points of intersection coincide, if and only if

$$c^2=a^2(1+m^2).$$

Example 11. *Find the equation of the tangent to the circle $x^2+y^2=25$ at the point $(3, 4)$.*

Solution. The equation of any straight line through the point (3, 4) is

$$y-4=m(x-3). \qquad \ldots(i)$$

The length of the perpendicular from (0, 0) on (i) is

$$\frac{|3m-4|}{\sqrt{(1+m^2)}}. \qquad \ldots(ii)$$

If the straight line (i) touches the circle $x^2+y^2=25$, the length (ii) must be equal to 5 units, the radius of the circle.

i.e., $\quad \dfrac{|3m-4|}{\sqrt{(1+m^2)}}=5,$

i.e., $\quad (3m-4)^2=25(1+m^2),$

i.e., $\quad 16m^2+24m+9=0,$

i.e., $\quad m=-\dfrac{3}{4}.$

Substituting this value of m in (i), we find that the required equation is

$$y-4=-\frac{3}{4}(x-3),$$

or $\quad 3x+4y-25=0.$

Example 12. *Find the equation of the tangent to the circle*

$$x^2+y^2+2x+4y-20=0$$

through the point $(2, 2)$.

Solution. The given equation can be put in the form

$$(x+1)^2+(y+2)^2=25,$$

so that the centre of the circle is the point $(-1, -2)$. The slope of the line joining the point $(2, 2)$ to the centre $(-1, -2)$ of the circle is $\dfrac{-2-2}{-1-2}=\dfrac{4}{3}$.

Since the tangent at $(2, 2)$ is perpendicular to the radius through $(2, 2)$, therefore, the slope of the tangent at $(2, 2)$ must be $-\left(\dfrac{4}{3}\right)^{-1}$, *i.e.*, $-\dfrac{3}{4}$.

The equation of the straight line through $(2, 2)$ having $-\dfrac{3}{4}$ as its slope is

$$y-2=-\frac{3}{4}(x-2),$$

i.e., $$3x+4y-14=0,$$

which is the required equation.

Aliter. The equation of any straight line through the point $(2, 2)$ is

$$y-2=m(x-2). \qquad \ldots(i)$$

If (i) touches the given circle, the length of the perpendicular from the point $(-1, -2)$ must be equal to the radius of the circle,

i.e., $$\frac{|(-2-2)-m(-1-2)|}{\sqrt{(1+m^2)}}=5,$$

i.e., $$(3m-4)^2=25(1+m^2),$$

i.e., $$16m^2+24m+9=0,$$

i.e., $$(4m+3)^2=0,$$

i.e., $$m=-\frac{3}{4}.$$

Substituting $m=-\dfrac{3}{4}$ in (i), we find that the equation of the required tangent is

$$(y-2)=-\frac{3}{4}(x-2),$$

i.e., $$3x+4y=14.$$

Example 13. *Find the condition that the straight line $y=mx+c$ may touch the circle $x^2+y^2=a^2$.*

Solution. The length of the perpendicular from the centre (0, 0) of the given circle on the straight line $y=mx+c$ is $\dfrac{|c|}{\sqrt{(1+m^2)}}$.
If the given straight line touches the circle, this length must be equal to a, the radius of the circle. Therefore, the required condition is

$$\frac{|c|}{\sqrt{1+m^2}}=a,$$

i.e.,
$$c^2=a^2(1+m^2).$$

Aliter. The straight line

$$y=mx+c \qquad \ldots(i)$$

touches the circle

$$x^2+y^2=a^2, \qquad \ldots(ii)$$

provided the points of intersection of (*i*) and (*ii*) are coincident. As in Example 10, the condition for this is

$$c^2=a^2(1+m^2).$$

Remark. Since $c^2=a^2(1+m^2) \Leftrightarrow c=\pm a\sqrt{1+m^2}$, therefore, the lines $y=mx\pm a\sqrt{1+m^2}$ touch the circle $x^2+y^2=a^2$ for all values of m. These are, in fact, the two tangents to the given circle which are parallel to the straight line $y=mx$.

Example 14. *Find the equations of the tangents to the circle*

$$x^2+y^2-4x-6y+12=0,$$

which are parallel to the straight line

$$3x-4y+7=0.$$

Solution. The equation of any straight line parallel to the line

$$3x-4y+7=0 \text{ is}$$
$$3x-4y+k=0. \qquad \ldots(i)$$

If (*i*) is a tangent to the circle

$$x^2+y^2-4x-6y+12=0,$$

the length of the perpendicular from the centre (2, 3) of the circle on (*i*) must be equal to the radius $\sqrt{(2^2+3^2-12)}=1$.

Therefore, we must have

$$\frac{|3.2-4.3+k|}{\sqrt{3^2+4^2}}=1,$$

i.e.,
$$(k-6)^2=25,$$

i.e.,
$$k=1 \text{ or } 11.$$

Substituting these values of k in (i), we have

$$3x-4y+1=0,$$

and $$3x-4y+11=0,$$

as the required equations.

11·10. LENGTH OF A TANGENT TO THE CIRCLE $x^2+y^2=a^2$ FROM A GIVEN POINT (x_1, y_1)

Let PT be a tangent to the circle $x^2+y^2=a^2$ from an external point P (x_1, y_1).

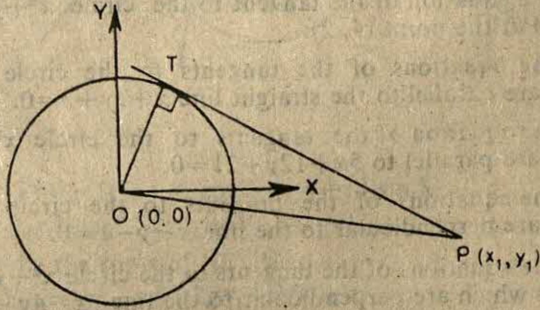


Fig. 11·6.

If O be the centre of the circle, then PT must be perpendicular to OT. Therefore, in the right-angled triangle OTP, we must have

$$PT^2=PO^2-OT^2. \qquad ...(i)$$

Also, $$PO^2=(x_1-0)^2+(y_1-0)^2=x_1^2+y_1^2, \qquad ...(ii)$$

and $$OT^2=a^2. \qquad ...(iii)$$

From (i), (ii) and (iii), we have

$$PT^2=x_1^2+y_1^2-a^2,$$

so that $$PT=\sqrt{(x_1^2+y_2^2-a^2)}.$$

Remark. *The length of a tangent from an external point (x_1, y_1) to the circle $x^2+y^2+2gx+2fy+c=0$ is*

$$\sqrt{(x_1^2+y_1^2+2gx_1+2fy_1+c)}.$$

With the same notation as above, taking O to be the point $(-g, -f)$,

$$PO^2=(x_1+g)^2+(y_1+f)^2,$$

$$OT^2=g^2+f^2-c,$$

$$PT^2=PO^2-OT^2,$$

$$=(x_1+g)^2+(y_1+f)^2-(g^2+f^2-c),$$

$$=x_1^2+y_1^2+2gx_1+2fy_1+c.$$

Hence $PT=\sqrt{(x_1^2+y_1^2+2gx_1+2fy_1+c)}$.

EXERCISE 11(e)

1. Find the equation of the tangent to the circle $x^2+y^2=169$ at the point $(5, -12)$.
2. Find the equation of the tangent to the circle $x^2+y^2=a^2$ at the point $(a \cos \theta, a \sin \theta)$
3. Find the equations of the tangents to the circle $x^2+y^2=169$ at the points $(5, 12)$ and $(12, -5)$ Show that the tangents intersect at right angle at the point $(17, 7)$.
4. Find the equation of the tangent to the circle $x^2+y^2-4x-4y+4=0$ at the point $(4, 2)$.
5. Find the equations of the tangents to the circle $x^2+y^2=4$ which are parallel to the straight line $x+2y+3=0$.
6. Find the equation of the tangents to the circle $x^2+y^2=169$ which are parallel to $5x+12y+21=0$.
7. Find the equations of the tangents to the circle $x^2+y^2=9$ which are perpendicular to the line $x-y-1=0$.
8. Find the equations of the tangents to the circle $x^2+y^2-2x-4y-4=0$ which are perpendicular to the line $3x-4y-1=0$.
9. Find the equations of the tangents to the circle $x^2+y^2=25$ which are inclined at an angle of 30° to the axis of x.
10. Find the condition that the line $x \cos \alpha+y \sin \alpha=p$ may touch the circle $x^2+y^2=a^2$.
11. Find the condition that the line $lx+my+n=0$ may touch the circle $x^2+y^2=a^2$.
12. Find the length of the tangent from the point $(2, 3)$ to the circle $x^2+y^2=4$.
13. Find the length of the tangent from the point $(4, 2)$ to the circle $x^2+y^2+2x+4y-11=0$.

11·11. POWER OF A POINT WITH RESPECT TO A CIRCLE

Let
$$x^2+y^2+2gx+2fy+c=0 \qquad \cdots(i)$$
be the equation of a circle and let P (x_1, y_1) be any point.

The equation of any straight line through P is

$$\frac{x-x_1}{\cos \theta}=\frac{y-y_1}{\sin \theta}=r. \qquad \cdots(ii)$$

The points of intersection of the straight line (ii) and the circle (i) are given by

$$(x_1+r \cos \theta)^2+(y_1+r \sin \theta)^2+2g (x_1+r \cos \theta)$$
$$+2f (y_1+r \sin \theta)+c=0,$$

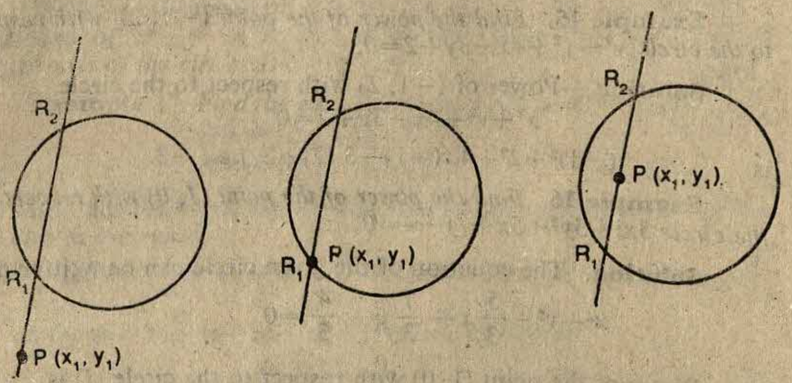


Fig. 11·7.

or $$r^2+2r\,[(x_1+g)\cos\theta+(y_1+f)\sin\theta]$$
$$+x_1^2+y_1^2+2gx_1+2fy_1+c=0. \qquad ...(iii)$$

If r_1, r_2 be the roots of (*iii*), then

$$r_1r_2=x_1^2+y_1^2+2gx_1+2fy_1+c. \qquad ...(iv)$$

From (*iv*) we find that if the straight line (*ii*) meets the circle (*i*) in the points R_1, R_2 then

$$PR_1 . PR_2=x_1^2+y_1^2+2gx_1+2fy_1+c. \qquad ...(v)$$

Since the right-hand side of (*v*) does not depend upon θ, therefore, we find that $PR_1.PR_2$ is the same for all straight lines through P. Thus we find that *if P be a given point and R_1, R_2 be the points in which any straight line through P meets a given circle, then $PR_1.PR_2$ is constant.* This constant is called the **power** *of P with respect to the circle.* Thus *the power of the point* (x_1, y_1) *with respect to the circle* $x^2+y^2+2gx+2fy+c=0$ *is*

$$x_1^2+y_1^2+2gx_1+2fy_1+c.$$

Remark. It is obvious that *the power of a point with respect to a circle is positive if the point lies outside the circle, zero if the point lies on the circle, and negative if the point lies within the circle.*

Comparing (*iv*) with the expression for the length of a tangent to a circle from a point lying outside the circle, we find that *if a point P lies outside a circle, then the power of P with respect to the circle is equal to the square of the length of tangent from P to the circle.*

The above statement is simply the following theorem which you must have already read in earlier classes :

If through a point outside a circle, a chord and a tangent be drawn to the circle, then the square of the length of the tangent is equal to the rectangle contained by the segments of the chord.

Example 15. *Find the power of the point $(-1, 2)$ with respect to the circle $x^2+y^2+4x-3y+2=0$.*

Solution. Power of $(-1, 2)$ with respect to the circle

$$x^2+y^2+4x-3y+2=0$$

is $$(-1)^2+2^2+4.(-1)-3\,(2)+2, \text{ i.e., } -3.$$

Example 16. *Find the power of the point $(1, 0)$ with respect to the circle $3x^2+3y^2+5x+7y-4=0$.*

Solution. The equation of the given circle can be written as

$$x^2+y^2+\frac{5}{3}x+\frac{7}{3}y-\frac{4}{3}=0. \qquad ...(i)$$

Power of the point $(1, 0)$ with respect to the circle (i) is

$$1^2+0^2+\frac{5}{3}.1+\frac{7}{3}.0-\frac{4}{3}=\frac{4}{3}.$$

We now state and prove an important fact about the power of a point with respect to two circles.

Theorem 11·3. *The locus of the point, the powers of which with respect to two given circles are equal, is a straight line.*

Proof. Let the equations of a pair of circles be

$$x^2+y^2+2gx+2fy+c=0, \qquad ...(i)$$
$$x^2+y^2+2g'x+2f'y+c'=0. \qquad ...(ii)$$

The powers of a point $P\,(x_1, y_1)$ with respect to the circles (i) and (ii) are

$$x_1^2+y_1^2+2gx_1+2fy_1+c, \qquad ...(iii)$$
$$x_1^2+y_1^2+2g'x_1+2f'y_1+c' \qquad ...(iv)$$

respectively. If (iii) and (iv) are equal, we must have

$$x_1^2+y_1^2+2gx_1+2fy_1+c=x_1^2+y_1^2+2g'x_1+2f'y_1+c',$$

i.e., $$2(g-g')\,x_1+2\,(f-f')\,y_1+c-c'=0. \qquad ...(v)$$

From (v) we find that the locus of (x_1, y_1) is the straight line

$$2(g-g')\,x+2\,(f-f')\,y+c-c'=0.$$

Definition 11·1. *The locus of the point whose powers with respect to two given circles are equal, is called the* **radical axis** *of the circles.*

In view of the above theorem, we have the following result :

The radical axis of the circles

$$x^2+y^2+2gx+2fy+c=0,$$

and $$x^2+y^2+2g'x+2f'y+c'=0,$$

is the straight line

$$2\,(g-g')\,x+2\,(f-f')\,y+c-c'=0.$$

Rule. *If the equation of two circles be $S=0$, $S'=0$, and the coefficient of x^2+y^2 in S and S' be unity, then the equation of the radical axis of the circles is $S-S'=0$.*

Example 17. *Find the equation of the radical axis of the circles*

$$x^2+y^2+3x+4y+5=0,$$

and

$$x^2+y^2+2x+3y+4=0,$$

and show that the lengths of the tangents from any point on it to the two circles are equal.

Solution. The equation of the radical axis of the given circles is

$$(x^2+y^2+3x+4y+5)-(x^2+y^2+2x+3y+4)=0,$$

i.e.,

$$x+y+1=0. \quad ...(i)$$

Let (x_1, y_1) be any point on the radical axis. The lengths l_1, l_2 of tangents from (x_1, y_1) to the circles

$$x^2+y^2+3x+4y+5=0,$$

and

$$x^2+y^2+2x+3y+4=0,$$

are given by

$$l_1^2=x_1^2+y_1^2+3x_1+4y_1+5,$$

and

$$l_2^2=x_1^2+y_1^2+2x_1+3y_1+4,$$

respectively.

$$l_1^2-l_2^2=x_1+y_1+1. \quad ...(ii)$$

Also, since (x_1, y_1) is a point on the radical axis of the given circles, therefore, $x_1+y_1+1=0$. ...(*iii*)

From (*ii*) and (*iii*), we find that

$$l_1^2-l_2^2=0,$$

i.e.,

$$l_1=l_2.$$

Hence the lengths of the tangents to the given circles from any point on the radical axis are equal.

Example 18. *Find the radical axis of the circles*

$$x^2+y^2+2x+3y-7=0,$$

and

$$x^2+y^2-2x-y+1=0.$$

Also, find the points of intersection of the circles and show that they lie on the radical axis.

Solution. The equation of the radical axis of the circles

$$x^2+y^2+2x+3y-7=0, \quad ...(i)$$

and

$$x^2+y^2-2x-y+1=0 \quad ...(ii)$$

is

$$x^2+y^2+2x+3y-7=x^2+y^2-2x-y+1,$$

i.e.,

$$4x+4y-8=0,$$

i.e.,

$$x+y-2=0. \quad ...(iii)$$

To find the points of intersection of (*i*) and (*ii*), we have to find the common solutions of (*i*) and (*ii*). Subtracting (*ii*) from (*i*), we have

$$4x+4y-8=0,$$

i.e., $$x+y-2=0,$$

i.e., $$x=2-y. \qquad ...(iv)$$

Substituting the above value of x in (*i*), we have

$$(2-y)^2+y^2+2(2-y)+3y-7=0,$$

i.e., $$2y^2-3y+1=0,$$

i.e., $$y=1, \text{ or } \tfrac{1}{2}.$$

When $y=1, x=2-1=1$.

When $y=\frac{1}{2}, x=2-\frac{1}{2}=\frac{3}{2}$.

Hence the points of intersection are $(1, 1)$ and $(\frac{3}{2}, \frac{1}{2})$. By actual substitution we find that both these points lie on (*iii*).

Example 19. *Find the equation of the radical axis of the circles*

$$x^2+y^2-4x-2y+4=0,$$
$$x^2+y^2-5x-2y+7=0,$$

and show that it touches both the circles at the point (3, 1).

Solution. The equation of the radical axis of the circles

$$x^2+y^2-4x-2y+4=0, \qquad ...(i)$$

and $$x^2+y^2-5x-2y+7=0, \qquad ...(ii)$$

is $$x^2+y^2-4x-2y+4=x^2+y^2-5x-2y+7,$$

i.e., $$x=3.$$

The line $x=3$ intersects (*i*) in the points given by

$$9+y^2-4.3-2y+4=0,$$

i.e., $$y^2-2y+1=0,$$

i.e., $$(y-1)^2=0.$$

The line $x=3$, therefore, meets (*i*) in two coincident points *i.e.*, it touches (*i*), the point of contact being (3, 1). Similarly it also touches (*ii*) at (3, 1).

Remark. It can be shown that given any three circles, the radical axes of the circles taken in pairs all pass through a point. This point is called the **radical centre** of the circles.

Theorem 11·4. *The radical axis of two given circles is perpendicular to the line joining the centres of the circles.*

Proof. Let the equations of two circles be

$$x^2+y^2+2g_1x+2f_1y+c_1=0, \qquad ...(i)$$

and $$x^2+y^2+2g_2x+2f_2y+c_2=0. \qquad ...(ii)$$

The radical axis of the circles (*i*) and (*ii*) is the line

$$2(g_1-g_2)x+2(f_1-f_2)y+c_1-c_2=0,$$

so that the slope of the radical axis is

$$-\frac{g_1-g_2}{f_1-f_2}.$$

The co-ordinates of the centres of the two circles are $(-g_1, -f_1)$ and $(-g_2, -f_2)$, and therefore, the slope of the line joining the centres is

$$\frac{f_1-f_2}{g_1-g_2}.$$

The product of the slopes

$$=-\frac{g_1-g_2}{f_1-f_2}\cdot\frac{f_1-f_2}{g_1-g_2}=-1.$$

Since the product of the slopes is -1, therefore, the two lines are at right angles to each other.

EXERCISE 11 (f)

1. Find the power of the point $(1, -2)$ with respect to the circle $x^2+y^2+4x-3y+1=0$.

2. Find the power of the point $(-3, 1)$ with respect to the circle $x^2+y^2+6x+y-8=0$.

3. Find the locus of the point whose power with respect to the circle $x^2+y^2=4$ is equal to twice its power with respect to the circle $x^2+y^2-3x+6y+4=0$.

4. Find the radical axis of the circles $x^2+y^2-6x+8y+3=0$ and $x^2+y^2+2x-5y+1=0$.

5. Find the radical axis of the circles $x^2+y^2+4x-3y-7=0$ and $x^2+y^2-3x+2y-5=0$, and show that it is perpendicular to the line joining the centres of the circles.

6. Find the radical axis of the circles $x^2+y^2+2x-3y+1=0$ and $x^2+y^2+3x-4y-2=0$, and show that it is perpendicular to the line joining the centres of the circles.

7. Find the radical axis of the circles

$$x^2+y^2=4,\ x^2+y^2+4x-10y+4=0,$$

and show that it passes through the points of intersection of the circles.

8. Find the radical axis of the circles $x^2+y^2-2x=0$ and $x^2+y^2-4x=0$ and show that it touches the circles at the origin.

9. Show that the radical axis of the circles $x^2+y^2+4x+7=0$, $2x^2+2y^2+3x+5y+9=0$ and $x^2+y^2-7x-8y-9=0$ taken in pairs all pass through the point $\left(-\frac{24}{19}, -\frac{5}{19}\right)$.

10. Show that the radical axes of the circles $x^2+y^2+x+2y+3=0$, $x^2+y^2+2x+4y+5=0$ and $x^2+y^2-7x-8y-9=0$ taken in pairs all pass through the point $(-\frac{2}{3}, -\frac{2}{3})$.

11·12. FAMILY OF CIRCLES PASSING THROUGH THE INTERSECTION OF TWO CIRCLES

Let $\quad S=x^2+y^2+2gx+2fy+c=0,$

and $\quad S'=x^2+y^2+2g'x+2f'y+c'=0,$

be the equations of two circles.

Consider the equation $S+kS'=0$.

Now

$$S+kS'=(1+k)(x^2+y^2)+2(g+kg')x+2(f+kf')y +(c+kc'). \qquad ...(i)$$

If $k \neq -1$, then $S+kS'=0$ represents a circle. Also, for each value of k other than -1, $S+kS'=0$ gives one circle. Thus $S+kS'=0$ represents in general (except when $k=-1$) a family of circles. If we impose one condition on $S+kS'=0$, we can determine the value of k, *i.e.*, we can find that member of the family which satisfies the prescribed condition.

Example 20. *Find the equation of the circle which passes through the points of intersection of the circles*

$$x^2+y^2-6x-8y+1=0,$$

and $$x^2+y^2+9x+12y-4=0,$$

and also passes through the point $(1, 1)$.

Solution. The general equation of the family of circles passing through the points of intersection of the given circles is

$$(x^2+y^2-6x-8y+1)+k(x^2+y^2+9x+12y-4)=0, \qquad ...(i)$$

provided $k \neq -1$.

The point $(1, 1)$ lies on (i) provided

$$(1^2+1^2-6.1-8.1+1)+k(1^2+1^2+9.1+12.1-4)=0,$$

i.e., provided $\quad -11+19k=0,$

which gives $\quad k=\frac{11}{19}.$

Substituting $k=\frac{11}{19}$ in (*i*) and simplifying we find that the equation of the desired circle is

$$30x^2+30y^2-15x-20y-25=0.$$

EXERCISE 11 (*g*)

1. Find the equation of the circle which passes through the points of intersection of the circles $x^2+y^2=1$ and $x^2+y^2+6x+8y-4=0$ and also passes through the origin.

2. Find the equation of the circle which passes through the points of intersection of the circles
$$x^2+y^2+8x+6y-4=0$$
and
$$x^2+y^2+6x+8y-2=0$$
and whose centre lies on the line $x-y=3$.

3. Find the equation of the circle which passes through the point of intersection of the circles $x^2+y^2-3x=0$ and $x^2+y^2+6y=0$, and also passes through the point (1, 1).

11·13. ORTHOGONAL CIRCLES

Definition 11·2. *Two circles are said to cut each other orthogonally if the tangents to them at any of their points of intersection are at right angles. If two circles cut each other orthogonally, they are said to be orthogonal.*

Let C, C′ be the centres of two orthogonal circles S and S′ (Fig. 11·8) and let P be a point of intersection of these circles. The tangent to S at P is perpendicular to PC, and the tangent to S′ at P is perpendicular to PC′. Since the circles are orthogonal, therefore, the tangents at P to the two circles are at right angles. This means that PC and PC′ must be at right angles. Thus we find that *if two circles intersect each other orthogonally, the radii to the two circles through a point of intersection are at right angles.*

11.13.1. Condition for the Circles $x^2+y^2+2gx+2fy+c=0$ and $x^2+y^2+2g'x+2f'y+c'=0$ to Intersect Each Other Orthogonally

Let C, C′ be the centres of the two circles and let P be a point of intersection of the circles.

If the circles cut each other orthogonally, then $\angle CPC'$ must be a right angle. By applying Pythagoras theorem to the right-angled triangle CPC′, we then have

$$PC^2+PC'^2=CC'^2, \qquad \ldots(i)$$

Now
$$PC^2=g^2+f^2-c,\ PC'^2=g'^2+f'^2-c',$$

and
$$CC'^2=(g'-g)^2+(f'-f)^2.$$

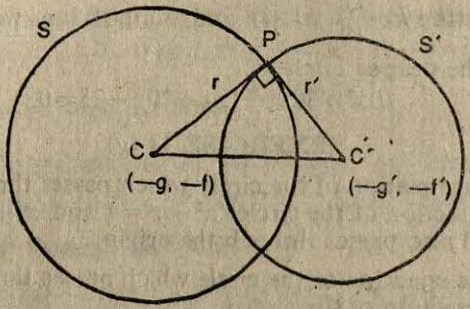


Fig. 11·8.

Substituting the above values in (*i*), we have

$$(g^2+f^2-c)+(g'^2+f'^2-c')=(g'-g)^2+(f'-f)^2, \quad ...(ii)$$

or
$$2gg'+2ff'=c+c',$$

as the required condition.

Remark. The condition (*ii*) can be stated as

$$r^2+r'^2=d^2,$$

where r, r' are the radii of the circles and d is the distance between their centres.

Example 21. *Show that the circles*

$$x^2+y^2-2x+4y-9=0,\ x^2+y^2+6x-4y-5=0$$

cut each other orthogonally.

Solution. The equations of the given circles can be written as

$$(x-1)^2+(y+2)^2=14,$$

and
$$(x+3)^2+(y-2)^2=18.$$

The centres of the circles are, therefore, $(1, -2)$, $(-3, 2)$, and their radii are $\sqrt{14}$, $\sqrt{18}$.

The distance between the centres is)

$$d=\sqrt{(1+3)^2+(-2-2)^2}=\sqrt{32}.$$

Now
$$r=\sqrt{14},\ r'=\sqrt{18},\ d=\sqrt{32},$$

so that
$$r^2+r'^2=d^2.$$

Hence the circles cut each other orthogonally.

Aliter.

$$g=-1,\ f=2,\ c=-9,\ g'=3,\ f'=-2,\ c'=-5.$$

$$\therefore \quad 2gg'+2ff'=2(-1).3+2.2.(-2)=-14.$$

$$c+c'=(-9)+(-5)=-14.$$

Hence $2gg'+2ff'=c+c'$, showing that the circles cut each other orthogonally.

Example 22. *Find the equation of the circle which passes through the points (1, 3) and (−2, 4), and cuts orthogonally the circle*

$$x^2+y^2+4x-6y+1=0.$$

Solution. Let the equation of the desired circle be

$$x^2+y^2+2gx+2fy+c=0. \qquad ...(i)$$

The points (−2, 4) and (1, 3) lie on (*i*), provided

$$20-4g+8f+c=0, \qquad ...(ii)$$

and

$$10+2g+6f+c=0. \qquad ...(iii)$$

The circle (*i*) cuts the circle

$$x^2+y^2+4x-6y+1=0$$

orthogonally if $\quad 2g.2+2f(-3)=1+c,$

i.e., if $\quad 1-4g+6f+c=0. \qquad ...(iv)$

Solving equations (*ii*), (*iii*) and (*iv*), we have

$$g=-\frac{3}{2},\ f=-\frac{19}{2},\ c=50.$$

Substituting the above values of g, f, c in (*i*), we find that the required equation is

$$x^2+y^2-3x-19y+50=0.$$

Example 23. *Find the equation of the circle which cuts orthogonally the three circles*

$$x^2+y^2-6x+4y-1=0,$$
$$x^2+y^2+8x-6y+3=0,$$

and

$$x^2+y^2-2x+8y-7=0.$$

Solution. Let the equation of the desired circle be

$$x^2+y^2+2gx+2fy+c=0. \qquad ...(i)$$

Since the circle (*i*) cuts the given circles orthogonally, therefore,

$$2g(-3)+2f.2=c-1,$$
$$2g.4+2f(-3)=c+3,$$

and

$$2g.(-1)+2f.4=c-7.$$

Solving the above equations we have

$$g=-\frac{11}{24},\ f=-\frac{25}{24},\ c=-\frac{5}{12}$$

Substituting the above values of g, f and c in (*i*), we have

$$x^2+y^2-\frac{11}{12}x-\frac{25}{12}y-\frac{5}{12}=0,$$

i.e., $$12x^2+12y^2-11x-25y-5=0,$$

as the desired equation.

EXERCISE 11 (*h*)

1. Prove that the circles $x^2+y^2-3x=0$ and $x^2+y^2+6y=0$ cut each other orthogonally.

2. Prove that the circles $x^2+y^2-2ax=0$ and $x^2+y^2-2by=0$ cut each other orthogonally.

3. Find the equation of the circle which passes through the points $(2, -1)$ and $(1, -2)$, and cuts orthogonally the circle
$$x^2+y^2-2x+3y-5=0.$$

4. Find the equation of the circle which passes through the origin and cuts orthogonally the circles $x^2+y^2-8y+12=0$ and $x^2+y^2-4x-6y-3=0$.

5. Find the equation of the circle which passes through the points of intersection of the straight line $x+y-1=0$ and the circle $x^2+y^2-6x-8y=0$, and cuts the circle $x^2+y^2=1$ orthogonally.

6. Find the equation of the circle which passes through the point $(-3, 2)$ and cuts orthogonally the circles
$$x^2+y^2-2x-2y+1=0,$$
and $$x^2+y^2-3x+6y-2=0.$$

7. Find the equation of the circle which cuts orthogonally the three circles $x^2+y^2+4x-5y+6=0$, $x^2+y^2+5x-6y+7=0$, and $x^2+y^2-x-y-1=0$.

8. Find the equation of the circle which cuts orthogonally the circles
$$x^2+y^2-2x+3y-7=0,$$
$$x^2+y^2+5x-5y+9=0,$$
and $$x^2+y^2+7x-9y+29=0.$$

9. Find the equation of the circle which cuts orthogonally the circles $x^2+y^2+3x+5y+7=0$ and $x^2+y^2+x-y-1=0$, and has its centre on the line $3x+2y+5=0$.

10. Find the equation of the circle which cuts orthogonally the circles $x^2+y^2-6x+4y-3=0$, passes through $(3, 0)$, and touches the axis of y.

TEST YOUR UNDERSTANDING XI

In each of the following problems, four alternatives are given. Put a tick-mark (√) against the correct alternative :

1. The centre of the circle $x^2+y^2-4x+6y-8=0$ is at

   (*a*) $(-2, 3)$ (*b*) $(2, -3)$ (*c*) $(-4, 6)$ (*d*) $(4, -6)$.

2. The radius of the circle $x^2+y^2-8x-6y=0$ is

   (*a*) 8 (*b*) 6 (*c*) 5 (*d*) 4.

3. The equation of the tangent to the circle $x^2+y^2+6x-4y-4=0$ at the point $(1, 1)$ is

   (*a*) $x+y-4=0$ (*b*) $4x-y-3=0$
   (*c*) $x-4y-3=0$ (*d*) $4x+y+3=0$.

4. The line $y=2x+c$ touches the circle $x^2+y^2=16$ if c equals

   (*a*) 4 (*b*) $\sqrt{5}$ (*c*) $2\sqrt{5}$ (*d*) $4\sqrt{5}$.

5. The power of the point $(1, 2)$ with respect to the circle

   $$x^2+y^2-3x-4y-6=0 \text{ is}$$

   (*a*) 12 (*b*) -12 (*c*) 6 (*d*) $\sqrt{12}$.

6. The circle $x^2+y^2-6x-8y=0$ cuts the circle

   $$x^2+y^2+8x+6y+c=0$$

   orthogonally provided c equals

   (*a*) -48 (*b*) 48 (*c*) 24 (*d*) -24.

7. The length of the tangent to the circle $x^2+y^2+2x+4y-6=0$ from the point $(1, 2)$ is

   (*a*) 4 (*b*) 3 (*c*) 9 (*d*) 8.

8. The equation of the normal to the circle $x^2+y^2-4x-8y+3=0$ at the point $(1, 0)$ is

   (*a*) $4x+y-4=0$ (*b*) $x-4y+1=0$
   (*c*) $x+4y+1=0$ (*d*) $4x-y-4=0$.

9. The equation of the common chord of the circles

   $x^2+y^2+2x+3y+1=0$ and $x^2+y^2+4x+3y+2=0$ is

   (*a*) $6x+6y+3=0$ (*b*) $2x+1=0$
   (*c*) $2y-1=0$ (*d*) $x-y+1=0$.

10. The normal to the circle $x^2+y^2+4x+6y-39=0$ at the point $(2, 3)$ meets the circle again at the point

    (*a*) $(2, 6)$ (*b*) $(-6, -9)$ (*c*) $(-2, 3)$ (*d*) $(3, -2)$.

REVIEW EXERCISE XI

1. Find the equation of the circle which passes through $(1, -2)$ and $(4, -3)$, and which has its centre on the straight line $3x+4y=7$. *(A.I.S.S.E., 1984)*

2. The length of the tangent from (f, g) to the circle $x^2+y^2=6$ is twice the length of the tangent to the circle
$$x^2+y^2+3x+3y=0.$$
Show that $f^2+g^2+4f+4g+2=0$. *(A.I.S.S.E., 1984)*

3. Show that the four points $(0, 0)$, $(1, 1)$, $(-5, 5)$ and $(-5, 1)$ are concyclic. *(A.I.S.S.E., 1985)*

4. Find the equation of the tangent to the circle $x^2+y^2-4x-4y+4=0$ at the point $(4, 2)$. *(A.I.S.S.E., 1985)*

5. Find the equation of the circle which is concentric with the circle $x^2+y^2-6x-8y+1=0$ and whose radius is 5 units.

6. Find the equation of the circle circumscribing the triangle formed by the lines $x+y=6$, $2x+y=4$, and $x+2y=5$.

7. Find the equation of the chord of the circle $x^2+y^2=81$ which is bisected at the point $(-2, 3)$.

8. Find the equation of the circle whose diameter is the common chord of the circles $x^2+y^2+2x+3y+1=0$ and
$$x^2+y^2+4x+3y+2=0.$$

9. Show that the circles $x^2+y^2+5x-5y+9=0$ and $x^2+y^2-16x-18y=4$ cut each other orthogonally.

10. For what values of p and q will powers of the point (p, q) with respect to the circles
$$x^2+y^2+4x+7=0,\ 2x^2+2y^2+3x+5y+9=0$$
and $x^2+y^2+y=0$ be equal ?

SUMMARY

1. The equation of the circle with centre (h, k) and radius r is
$$(x-h)^2+(y-k)^2=r^2.$$
2. The equation $x^2+y^2+2gx+2fy+c=0$ represents a circle whose centre is $(-g, -f)$ and radius is
$$\sqrt{(g^2+f^2-c)}.$$
3. The equation of the circle having (x_1, y_1) and (x_2, y_2) as the extremities of a diameter is $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$.
4. The equation of the tangent to the circle $x^2+y^2=a^2$ at the point (x_1, y_1) is $xx_1+yy_1=a^2$.
5. The straight line $y=mx+c$ touches the circle $x^2+y^2=a^2$ provided
$$c^2=a^2(1+m^2).$$

6. The length of the tangent drawn to the circle $x^2+y^2+2gx+2fy+c=0$ from an external point (x_1, y_1) is

$$\sqrt{(x_1^2+y_1^2+2gx_1+2fy_1+c)}.$$

7. The power of the point (x_1, y_1) with respect to the circle

$$x^2+y^2+2gx+2fy+c=0$$

is
$$x_1^2+y_1^2+2gx_1+2fy_1+c.$$

8. The equation of the radical axis of the circles

$$x^2+y^2+2gx+2fy+c=0$$

and
$$x^2+y^2+2g_1x+2f_1y+c_1=0$$

is
$$2x(g-g_1)+2y(f-f_1)+c-c_1=0.$$

9. The circles
$$x^2+y^2+2gx+2fy+c=0$$

and
$$x^2+y^2+2g_1x+2f_1y+c_1=0$$

intersect each other orthogonally provided

$$2gg_1+2ff_1=c+c_1.$$

□ □

JOHANN KEPLER (1571-1630)

Johann Kepler, a contemporary of Galileo and a forerunner of Newton devoted himself to the study of conics because he needed them for applications to astronomy. The three laws of planetary motion discovered by him are well-known. His discovery that the planets describe ellipses with the sun at one of foci has given him a permanent place in the history of science. He also found a method for calculating volumes of solids of revolution.

CHAPTER 12

Conic Sections

12·1. CONIC SECTIONS

A **conic section** or a **conic** is the locus of a point which moves so that its distance from a fixed point is in a constant ratio to its distance from a fixed straight line.

The fixed point is called a **focus**, the fixed straight line is called a **directrix**, and the constant ratio is called the **eccentricity** of the conic. The straight line passing through the focus and perpendicular to the directrix is called the **axis** of the conic. A point of intersection of a conic with its axis is called a **vertex.**

The eccentricity of a conic is denoted by e. When $e=1$, the conic is called a **parabola**; when $e<1$, the conic is called an **ellipse,** and when $e>1$, the conic is called **a hyperbola.**

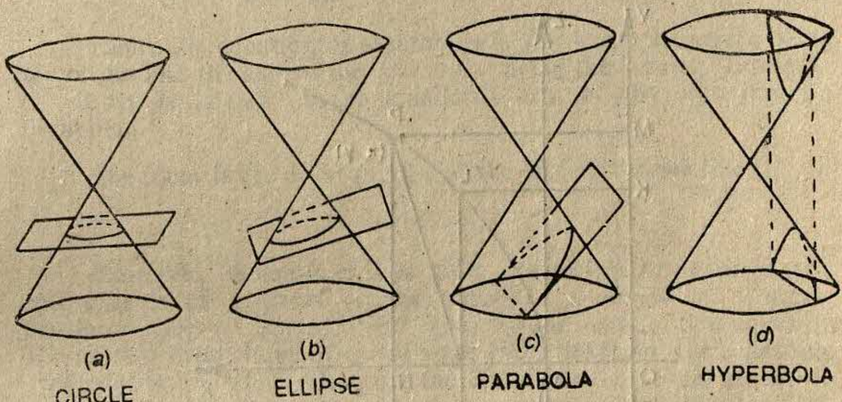


Fig. 12·1.

The name conic section is derived from the fact that if a right circular cone (extending infinitely in both directions ; the figure is incomplete !) be cut by any plane, the section will be in all cases a conic as defined above. In fact, if the plane is perpendicular to the axis of the cone, the section obtained is a circle as shown in Fig. 12·1 (a), (except when the plane passes through the vertex of the cone) ; if the plane cuts the cone obliquely in such a way that it cuts all the generators of the cone of points lying on one sheet of the cone, then the section obtained is an ellipse as shown in Fig. 12·1 (b) ; if the plane cutting the cone is paralled to

a generator of the cone, then the section obtained is a parabola as shown in Fig. 12·1 (*c*); and if the plane cuts both the sheets of the cone, then the section obtained is a hyperbola consisting of two infinite branches as shown in Fig. 12.1 (*d*). Here, we exclude the case when the plane passes through the vertex of the cone in which case the section obtained reduces either to a pair of straight lines or to a point.

We have already studied pairs of straight lines and circles (often referred to as degenerate conics). In the present chapter we propose to study the remaining three conic sections, *viz.*, the parabola, the ellipse and the hyperbola.

12·2. PARABOLA

Definition 12·1. *A* **parabola** *is the locus of a point which moves in such a way that its distance from a fixed point (called the* **focus**) *is equal to its distance from a fixed straight line (called the* **directrix**) *not containing the point.*

12·3. THE STANDARD EQUATION OF A PARABOLA

Let S be the focus and let Y′Y be the directrix of a parabola.

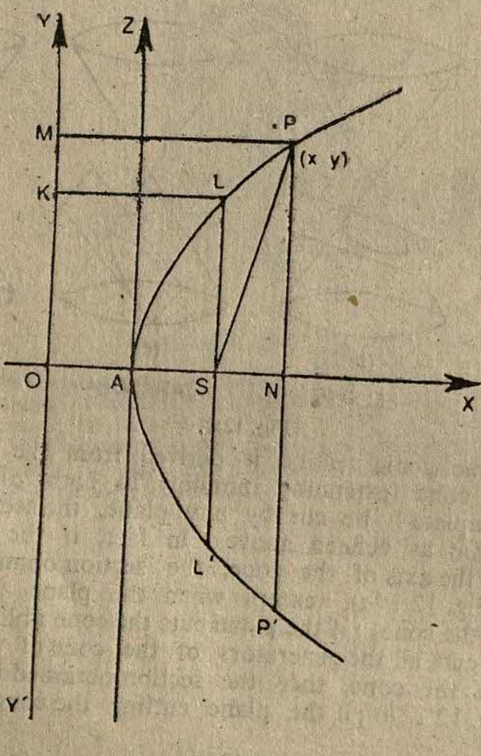


Fig. 12·2.

Draw SO perpendicular to Y′Y and let OS$=2a$. Then by definition, OS is the axis of the parabola. Let A be the middle point of OS. Take the origin of co-ordinates at A, the x-axis along AS, the y-axis along the perpendicular AZ to AS at A.

Let P(x, y) be any point on the parabola. Draw PM and PN perpendiculars to OY and OX respectively as shown in Fig. 12·2. Then

$$\text{MP}=\text{ON}=\text{OA}+\text{AN}=a+x.$$

But by definition, MP$=$PS,

or $$\text{MP}^2=\text{PS}^2,$$

or $$(a+x)^2=(x-a)^2+y^2,$$

or $$(x+a)^2-(x-a)^2=y^2,$$

or $$4ax=y^2,$$

or $$y^2=4ax. \qquad \ldots(i)$$

Hence the equation of a parabola is $y^2=4ax$, the vertex being the origin and the axis of the parabola being the x-axis. Equation (i) above is known as the standard form of the equation of a parabola.

The focus is the point $(a, 0)$ and the directrix is the line

$$x+a=0.$$

Remark. Sometimes it is quite convenient to express the co-ordinates of a point on the parabola in terms of a single parameter, say t. Since $x=at^2$ and $y=2at$ satisfy the equation $y^2=4ax$ for every value of t, and since every point on the parabola corresponds to some value of t, therefore, $x=at^2$, $y=2at$ are called the parametric co-ordinates of a point on the parabola $y^2=4ax$. We shall frequently refer to this point as the point 't'.

12·4. TRACING A PARABOLA

We shall now trace the parabola $y^2=4ax$, where a is positive.

(i) Since y^2 is a positive quantity, x must also be positive. Therefore, the curve lies wholly on the right side of the axis of y.

(ii) For every positive value of x, there are two equal and opposite values of y, that is, if (x, y) be any point on the curve, then $(x, -y)$ also lies on the curve. Therefore, *the curve is symmetrical about the x-axis.*

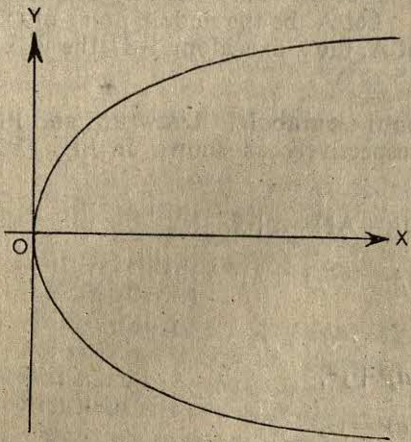


Fig. 12·3.

(*iii*) If $y=0$, then $x=0$, that is the curve meets the x-axis only at the origin.

(*iv*) As x increases, y also increases and there is no limit to this increase of x and y. There fore, there is no limit to the curve on the right side of the axis of y.

Taking all the above facts into consideration, the parabola can be traced as shown in Fig. 12·3.

Remark. *A point (x', y') lies outside, on or inside the parabola $y^2=4ax$ according as y'^2-4ax' is positive, zero or negative.*

Let P be the point with co-ordinates (x', y'). Draw the ordinate PM meeting the curve in N. Then P will lie outside the parabola if PM > NM, that is if $PM^2-NM^2>0$. Now, $PM=y'$ and since N lies on the porabola $y^2=4ax$, $NM^2=4ax'$. Thus, $PM^2-NM^2=y'^2-4ax'$. Therefore, if P lies outside the parabola, then $y'^2-4ax'>0$. Similarly, P lies inside the parabola if $y'^2-4ax'<0$. Obviously, if $y'^2-4ax'=0$, then the point P (x', y') lies on the parabola.

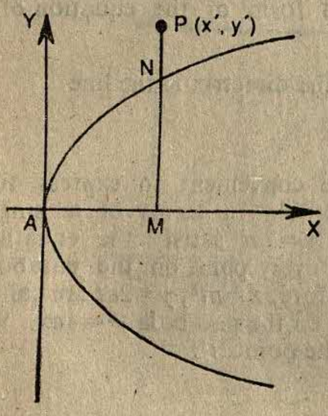


Fig. 12·4.

Example 1. *Find the co-ordinates of the focus and the equation of the directrix of the parabola $y^2=8x$.*

Solution. We know that focus of the parabola $y^2=4ax$ is $(a, 0)$ and the equation of its directrix is $x+a=0$. Now, comparing the equation $y^2=8x$ with the equation $y^2=4ax$, we have $a=2$.

Thus, focus is the point (2, 0). Also the equation of the directrix is $x+2=0$.

Example 2. *Find the focus and directrix of the parabola*

$$(y+3)^2=2(x+2).$$

Solution. To change the origin of co-ordinates to the point $(-2, -3)$,

let $$x+2=X \text{ and } y+3=Y.$$

Then the equation of the given parabola becomes

$$Y^2=2X.$$

Comparing it with $y^2=4ax$, we get

$$a=\tfrac{1}{2}.$$

Therefore, the co-ordinates (X, Y) of the focus (referred to the new origin) of the parabola

$$Y^2=2X,$$

are given by $$X=\tfrac{1}{2},\ Y=0.$$

$$\therefore \quad x+2=\tfrac{1}{2},\ y+3=0,$$

or $$x=-\tfrac{3}{2},\ y=-3.$$

Thus, $(-\frac{3}{2}, -3)$ are the desired co-ordinates of the focus of the given parabola.

Also, the equation of the directrix of the parabola

$$Y^2=2X \text{ is } X+\tfrac{1}{2}=0.$$

Therefore, $x+2+\frac{1}{2}=0$, *i.e.*, $2x+5=0$

is the equation of the directrix of the given parabola.

Example 3. *Find the focus and directrix of the parabola*

$$x^2=-8y.$$

Solution. The given equation can be written in the form

$$x^2=(y-2)^2-(y+2)^2, \qquad \dots(i)$$

or $$x^2+(y+2)^2=(y-2)^2, \qquad \dots(ii)$$

or $$\sqrt{x^2+(y+2)^2}=|y-2|. \qquad \dots(iii)$$

If P (x, y) is any point on the given parabola, then (*iii*) shows that the distance of P from the point $(0, -2)$ is equal to its perpendicular distance from the straight line

$$y=2.$$

Therefore, $x^2=-8y$ is the equation of the parabola whose focus is $(0, -2)$ and whose directrix is the straight line $y=2$.

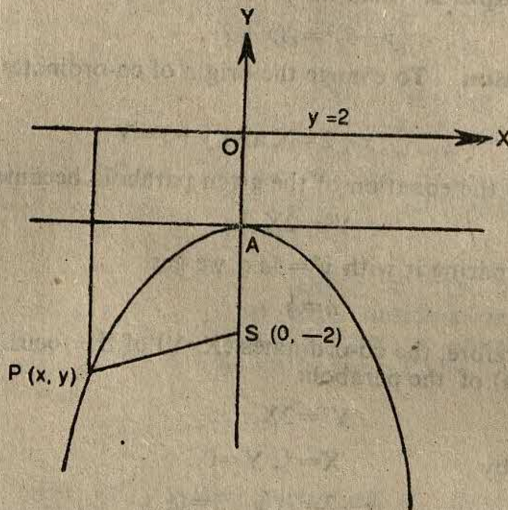


Fig. 12·5.

Remarks. 1. The given equation has been put in form (*i*) by writing $-8y$ as $4y.(-2)$ and then using the formula

$$4pq=(p+q)^2-(p-q)^2,$$

with $$p=y,\ q=-2.$$

This enables us to get (*ii*).

2. It can be shown, as in the above example, that the focus of the parabola $x^2=-4ay$ $(a>0)$ is $(0, -a)$ and the equation of its directrix is $y=a$.

12·5. THE LATUS RECTUM OF A PARABOLA

The chord of a parabola through the focus and perpendicular to the axis is called the **latus rectum**. Let the chord of the parabola $y^2=4ax$ through the focus S perpendicular to the axis AX meet the parabola in L and L′. Then L′SL is the latus rectum. (Fig. 12·6).

Now, the co-ordinates of L, L' are easy to determine. Since L, L′ are the points of intersection of the line $x=a$ with the parabola $y^2=4ax$, therefore by solving these two equations together we get $x=a$ and $y=\pm 2a$. Hence L, L′ are the points $(a, 2a)$ and $(a, -2a)$ respectively.

Thus the length of the latus rectum$=$L′L$=4a$.

Hence the length of the latus rectum of the parabola $y^2=4ax$ is equal to $4a$, which is also twice the distance between the focus and the directrix.

Since the equation of a parabola contains only one arbitrary

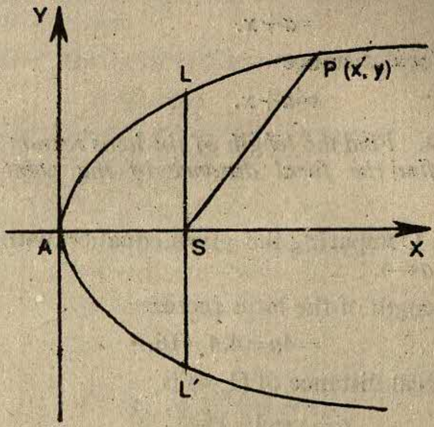


Fig. 12·6.

constant, it follows that the lengh of the latus rectum determines the size of a parabola.

12·6. FOCAL DISTANCES

If $P(x, y)$ is any point on the parabola $y^2=4ax$, then the distance of P from the focus S is called the **focal distance** of P. If B′B is the directrix and PM and PN the perpendiculars drawn from P to the directrix and the axis respectively, then the focal distance of P is

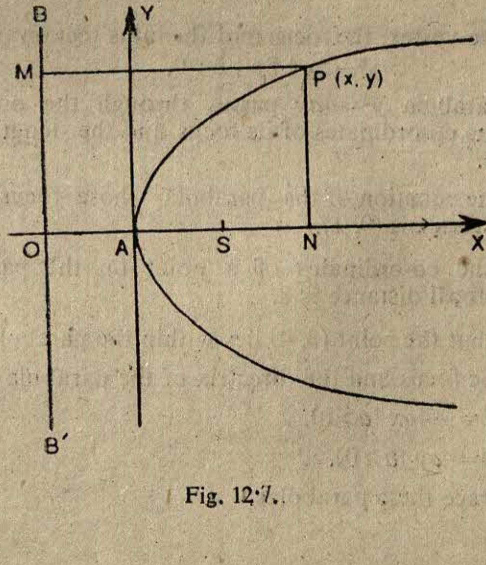


Fig. 12·7.

$$PS=PM=OA+AN,$$
$$=a+x.$$

Thus, the focal distance

$$=a+x.$$

Example 4. *Find the length of the latus rectum of the parabola $y^2=16x$. Also, find the focal distance of the point (1, 4) on the parabola.*

Solution. Comparing the given equation with the equation $y^2=4ax$, we get $a=4$.

Thus, the length of the latus rectum

$$=4a=4.4=16.$$

Also, the focal distance of (1, 4) is

$$a+x=4+1=5.$$

EXERCISE 12 (*a*)

1. Find the equation of the parabola whose focus is $(-3, 0)$ and whose directrix is $x+5=0$.
2. Find the co-ordinates of the focus and the equation of the directrix of the parabola $y^2=-6x$.
3. Find the focus and the directrix of the parabola
$$(y+5)^2=4(x+3).$$
4. Find the length of the latus rectum of the parabola $y^2=9x$. Also, find the focal distance of the point (1, 3) on the parabola.
5. Find the vertex, the focus and the latus rectum of the parabola
$$y^2-5x+8y+6=0.$$
6. The parabola $y^2=4ax$ passes through the point $(3, -2)$. Find the co-ordinates of its focus and the length of its latus rectum.
7. Find the equation of the parabola whose focus is (0, 0) and whose vertex is (0, 1).
8. Find the co-ordinates of a point on the parabola $y^2=8x$ whose focal distance is 8.
9. Show that the point (3, 4) lies within the parabola $y^2=6x$.
10. Find the focus and the directrix of the parabola :

    (*i*) $y^2=-4ax$ $(a>0)$.

    (*ii*) $x^2=4ay$ $(a>0)$.

    Also, trace these parabolas.

12·7. CONDITION THAT $y=mx+c$ MAY TOUCH THE PARABOLA $y^2=4ax$

The abscissae of the points of intersection of the line

$$y=mx+c$$

with the parabola $y^2=4ax$ are given by the equation

$$(mx+c)^2=4ax,$$

or
$$m^2x^2+(2mc-4a)x+c^2=0. \qquad \text{...}(i)$$

If $y=mx+c$ is to be a tangent to the parabola, the two roots of (i) must be equal. The condition for this is

$$(2mc-4a)^2=4m^2c^2,$$

or
$$16a^2-16amc=0,$$

or
$$c=\frac{a}{m}, \text{ provided } m\neq 0.$$

For this value of c, the equation (i) now becomes

$$m^2x^2+2(a-2a)x+\frac{a^2}{m^2}=0,$$

or
$$mx-\frac{a}{m}=0.$$

Thus,
$$x=\frac{a}{m^2}.$$

The corresponding value of y is given by

$$y=mx+c=m\frac{a}{m^2}+\frac{a}{m}=\frac{2a}{m}.$$

Thus, $y=mx+c$, $m\neq 0$, touches the parabola $y^2=4ax$ if $c=\dfrac{a}{m}$; the point of contact is $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$.

Remarks. 1. The straight line $y=mx+\dfrac{a}{m}$ touches the parabola $y^2=4ax$ for all values of m $(\neq 0)$. This equation is generally called the *slope form* of the equation of the tangent.

2. The tangent to the parabola $y^2=4ax$ parallel to the line $y=mx+c$ is

$$y=mx+\frac{a}{m}.$$

It follows, therefore, that only one tangent can be drawn to a parabola parallel to a given straight line.

Example 5. *Find the equation of the tangent to the parabola $y^2=8x$ parallel to the straight line $2x-3y+1=0$. Also, find the co-ordinates of the point of contact.*

Solution. The slope of the given line is

$$m=\tfrac{2}{3}.$$

Comparing the equation $y^2=8x$ with $y^2=4ax$, we get

$$a=2.$$

Hence the equation of the tangent with slope $\tfrac{2}{3}$ is

$$y=\tfrac{2}{3}x+\frac{2}{\frac{2}{3}},$$

or $$2x-3y+9=0.$$

The co-ordinates of the point of contact are

$$\left(\frac{a}{m^2}, \frac{2a}{m}\right), \text{ where } m=\tfrac{2}{3}, a=2.$$

Thus the point of contact is $(\tfrac{9}{2}, 6)$.

Example 6. *Find the equation of the tangent to the parabola $y^2=4ax$ which makes an angle of $60°$ with the axis of x.*

Solution. Suppose m is the slope of the required tangent. Then the equation of the tangent is

$$y=mx+\frac{a}{m}.$$

Since the tangent makes an angle of $60°$ with the axis of x,

$$\therefore \quad m=\tan 60°=\sqrt{3}.$$

$\therefore$ the equation of the tangent is

$$y=\sqrt{3}x+\frac{a}{\sqrt{3}},$$

or $$3x-\sqrt{3}y+a=0$$

EXERCISE 12 (*b*)

1. Find the equation of the tangent to the parabola $y^2=3x$ which is parallel to the line $2x-y=1$. Also, find the point of contact.
2. Find the equation of the tangent to the parabola $y^2=6x$ which is perpendicular to the line $5x-7y+6=0$. Also, find the point of contact.
3. Find the equations of the tangents to the parabola $y^2=3x$ passing through the point $(1, -2)$.
4. Find the equations of the tangents to the parabola $y^2=16x$ which are parallel and perpendicular respectively to the line $2x-y+5=0$. Also, find the co-ordinates of their points of contact.

5. Show that the line $3y=6x+2$ touches the parabola $3y^2=16x$ and find the co-ordinates of the point of contact.

6. Show that the line $7x+6y=13$ is a tangent to the parabola $y^2-7x-8y+14=0$ and find the co-ordinates of its point of contact.

12·8. ELLIPSE

The ellipse was studied by the ancient Greeks, notably by Menaechmus (fourth century BC), a pupil of Plato and Eudoxus. He called it a section of the acute-angled cone and obtained it as a section of a right circular cone of semi-vertical angle less than 45° by a plane perpendicular to a generating line of the cone. Apollonius, the great Greek geometer, later on (3rd century B.C.) obtained it as a section of a right circular cone of arbitrary semi-vertical angle by taking a section at a suitable inclination. The focus-directrix property was first treated, much later, by Pappus of Alexandria (about 300 A.D.). Kepler, working at Prague with the Danish astronomer Tycho Brahe discovered that the planets move in elliptical orbits with the sun at one focus. In 1680 A.D. Newton proved that the elliptical orbit was a consequence of the 'inverse square law of gravitation.' The ellipse has thus turned out to be of great physical significance to us.

As discussed earlier, an ellipse is a conic section for which the eccentricity is less than unity.

Thus, we have the following :

Definition 12·2. *An* **ellipse** *is the locus of a point which moves so that its distance from a fixed point bears to its distance from a fixed straight line (not containing the point) a constant ratio which is less than unity. The fixed point is called a* **focus** *and the fixed straight line is called a* **directrix.** *The constant ratio is called the* **eccentricity** *and is denoted by e.*

12·9. EQUATION OF AN ELLIPSE IN THE STANDARD FORM

Let S be a focus and XM a directrix of the ellipse whose equation is required. Draw SX perpendicular to XM. Since the eccentricity e is less than one, we can divide SX internally and externally in the ratio $e : 1$ at A and A′, so that

$$SA=eAX, \qquad \cdots(i)$$

and

$$SA'=eA'X. \qquad \cdots(ii)$$

By the definition of an ellipse, A and A′ lie on the ellipse. Let C be the middle point of AA′ and let $AA'=2a$. Then

$$AC=CA'=a.$$

Produce XA′ to X′ making $A'X'=XA$. Cut off $A'S'=SA$, so that A′ lies between S′ and X′.

Now, adding (*i*) and (*ii*), we get

$$AA' = eXX'.$$

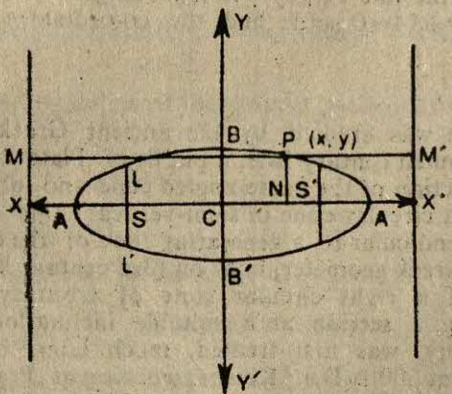


Fig. 12·8.

Again, subtracting (*i*) from (*ii*), we get

$$SS' = eAA'.$$

Now, $\quad AA' = eXX' \Rightarrow 2a = eXX',$

$$\Rightarrow 2a = 2eCX,$$

$$\Rightarrow CX = a/e.$$

Similarly $\quad SS' = eAA'$ gives $CS = ae$.

Take C as the origin, CA′ as the axis of x and a line through C perpendicular to CA′ as the axis of y. Let P be any point on the ellipse with co-ordinates (x, y) and let PM be perpendicular to the directrix. From the definition, we have,

$$PS = ePM,$$

or $\quad \sqrt{(x+ae)^2+y^2} = eXN,$

$$= e(XC+CN),$$

$$= e\left(\frac{a}{e} + x\right).$$

Squaring, we get

$$x^2+2aex+a^2e^2+y^2 = a^2+2aex+e^2x^2,$$

or $\quad x^2(1-e^2)+y^2 = a^2(1-e^2),$

or $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{a^2(1-e^2)} = 1.$

Since $e<1$, $a^2(1-e^2)>0$. Put $a^2(1-e^2)=b^2$.

Then,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

is the required equation of the ellipse.

12·10. TRACING OF ELLIPSE

Consider the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \qquad ...(i)$$

Putting $x=0$ in (i), we get

$$y^2=b^2,$$

or $$y=\pm b.$$

Hence the y-axis cuts the curve in two points B$(0, b)$ and B$'(0, -b)$.

Putting $y=0$ in (i), we get

$$x=\pm a.$$

Hence the x-axis cuts the curve in two points A$'(a, 0)$ and A$(-a, 0)$.

Also, equation (i) gives

$$x=\pm\frac{a}{b}\sqrt{b^2-y^2}.$$

Thus for any value of y, there are two values of x which are equal in magnitude but opposite in sign. This shows that the curve is symmetrical about the axis of y. Also, x is real if $|y|<b$ and imaginary if $|y|>b$. Therefore, the curve lies within the region bounded by the lines $y=\pm b$.

Similarly, we have

$$y=\pm\frac{b}{a}\sqrt{a^2-x^2},$$

which shows that the curve is symmetrical about the axis of x and lies within the region bounded by the lines $x=\pm a$.

It follows that the curve lies within the rectangle whose sides are $x=\pm a$, and $y=\pm b$.

If (x', y') satisfies the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

then obviously $(-x', -y')$ also satisfies it. Therefore, every chord drawn through the origin is bisected at it and so origin is the centre of the curve.

Also, as x increases, y decreases and as y increases, x decreases. Therefore, the curve is a closed curve.

The shape of the curve is shown in Fig. 12·8.

Remark. Since $b^2=a^2(1-e^2)$ and $e<1$,

$\therefore\ b^2<a^2$ and hence $AA'>BB'$.

The line segment $AA'=2a$ is called the **major axis** and the line segment $BB'=2b$ is called the **minor axis** of the ellipse.

The lengths a and b are called the **major semi-axis** and **minor semi-axis** respectively.

Remarks. 1. Sometimes the axes of co-ordinates are so chosen that the minor axis lies along the x-axis and the major axis lies along the y-axis. The equation of the ellipse then takes the form

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1.$$

2. A point $P(x_1, y_1)$ lies outside, on, or inside the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

according as the expression

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1>, =, \text{ or } <0.$$

Proof. Draw PN perpendicular to the major axis and let PN or PN produced meet the ellipse in Q. If $QN=y_2$, then Q is the point (x_1, y_2).

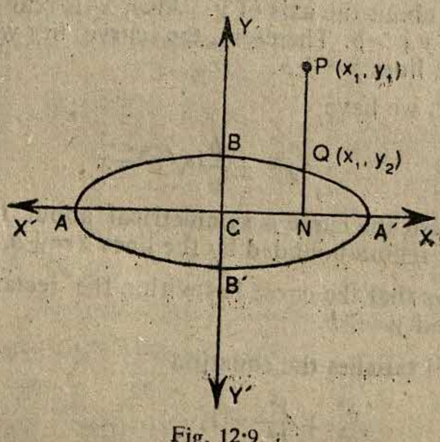


Fig. 12·9.

Since Q lies on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

$\therefore$ we have

$$\frac{x_1^2}{a^2}+\frac{y_2^2}{b^2}=1.$$

Now, the point P will be outside, on, or inside the ellipse according as

$$PN >, =, \text{ or } < QN,$$

i.e., according as $y_1 >, =, \text{ or } < y_2$,

i.e., according as $\dfrac{y_1^2}{b^2} >, =, \text{ or } < \dfrac{y_2^2}{b^2}$,

i.e., according as $\dfrac{y_1^2}{b^2} >, =, \text{ or } < 1-\dfrac{x_1^2}{a^2}$,

i.e., according as $\dfrac{x_1^2}{a^2}+\dfrac{y_1^2}{b^2}-1 >, =, \text{ or } < 0$.

This proves the assertion.

12·11. THE SECOND FOCUS AND DIRECTRIX

The equation of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1$$

can be re-written as

$$x^2-2aex+a^2e^2+y^2=a^2-2aex+e^2x^2,$$

or

$$(x-ae)^2+y^2=e^2\left(\frac{a}{e}-x\right)^2.$$

The above form of the equation shows that the given ellipse is also the locus of a point which moves so that its distance from the point S′$(ae, 0)$ is e times its distance from the line $x=\dfrac{a}{e}$.

Hence there is a second focus S′$(ae, 0)$ and a second directrix whose equation is $x=\dfrac{a}{e}$.

Remark. In Fig. 12·8, since CS$=ae$ and CX$=\dfrac{a}{e}$, therefore, the co-ordinates of the focus S are $(-ae, 0)$ and the equation of the directrix XM is $x=-\dfrac{a}{e}$. Also, the second directrix $x=\dfrac{a}{e}$ is the line parallel to the y-axis and at a distance $\dfrac{a}{e}$ from the origin.

12·12. THE LATUS RECTUM

The chord LL′ through the focus S perpendicular to the major axis is called the **latus rectum** (see Fig. 12·8).

Since CS$=ae$, the equation of LL′ is $x=-ae$. Therefore, the abscissae of L, L′ are given by

$$\frac{a^2e^2}{a^2}+\frac{y^2}{b^2}=1,$$

or
$$b^2e^2+y^2=b^2,$$

or
$$1-e^2=\frac{y^2}{b^2},$$

or
$$\frac{b^2}{a^2}=\frac{y^2}{b^2},$$

or
$$y=\pm\frac{b^2}{a}.$$

Hence L is the point $(-ae,\ b^2/a)$ and L′ is the point

$$(-ae,\ -b^2/a).$$

Similarly, the co-ordinates of the extremities of the latus rectum through the other focus S′ will have co-ordinates

$$(ae,\ b^2/a) \text{ and } (ae,\ -b^2/a).$$

Also, $\text{LL}'=2\text{SL}=\dfrac{2b^2}{a}=2a(1-e^2).$

SL is called the **semi-latus rectum** and its length is b^2/a.

12·13. FOCAL DISTANCES

As in case of the parabola, the distance of a point on the ellipse from a focus is called a focal distance of the point. Since an ellipse has two foci, there are two focal distances for each point. The following theorem gives a relationship between the two focal distances of a point.

Theorem 12·1. *The sum of the focal distances of any point on the ellipse is constant and is equal to the length of the major axis.*

Proof. Let P(x_1, y_1) be any point on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

As in Fig. 12·8,

$$\begin{aligned} PS &= ePM, \\ &= e\left(\frac{a}{e}+x_1\right), \\ &= a+ex_1. \end{aligned}$$

Similarly, $$\begin{aligned} PS' &= ePM'. \\ &= a-ex_1. \end{aligned}$$

$\therefore$ $$\begin{aligned} PS+PS' &= (a+ex_1)+(a-ex_1), \\ &= 2a. \end{aligned}$$

This proves the theorem.

Remark. Because of the above theorem, a popular way of drawing an ellipse is as follows: Take a piece of string. Fix the ends of the string to two points S and S′ on a piece of paper. Now take a pencil and with the head P of the pencil, tighten the string. Now move the pencil keeping the string tight. The closed curve traced by the pencil is an ellipse whose foci are S, S′ and whose major axis is equal to the length of the string.

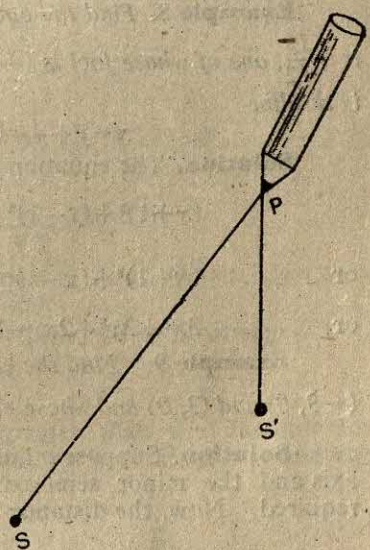


Fig. 12·10.

12·14. THE EQUATION OF AN ELLIPSE WITH GIVEN ECCENTRICITY, A GIVEN FOCUS AND A GIVEN DIRECTRIX

Suppose we want to find out the equation of an ellipse with eccentricity e, one of whose foci is the point (h, k) and the corresponding directrix being the line

$$ax+by+c=0.$$

Let $P(x, y)$ be any point on the ellipse. Then by definition, the distance of P from the point (h, k) is equal to e times its perpendicular distance from the line

$$ax+by+c=0.$$

This gives

$$(x-h)^2+(y-k)^2=e^2\left(\frac{ax+by+c}{\sqrt{a^2+b^2}}\right)^2,$$

or $$(a^2+b^2)[(x-h)^2+(y-k)^2]=e^2(ax+by+c)^2,$$
which is the required equation.

Example 7. *Find the equation of the ellipse whose major axis lies along the x-axis and is of length 10 units, and whose minor axis lies along the y-axis and is of length 8 units.*

Solution. The equation of the ellipse is
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$
where $2a=10$ and $2b=8$. Thus the equation of the ellipse is
$$\frac{x^2}{5^2}+\frac{y^2}{4^2}=1,$$
i.e., $$\frac{x^2}{25}+\frac{y^2}{16}=1,$$
i.e., $$16x^2+25y^2=400.$$

Example 8. *Find the equation of the ellipse whose eccentricity is $\frac{1}{\sqrt{2}}$, one of whose foci is $(-1, 1)$, and the corresponding directrix is the line*
$$x-y+4=0.$$

Solution. The equation of the required ellipse is
$$(x+1)^2+(y-1)^2=\left(\frac{1}{\sqrt{2}}\cdot\frac{x-y+4}{\sqrt{1+1}}\right)^2,$$
or $$(x+1)^2+(y-1)^2=\frac{1}{4}(x-y+4)^2,$$
or $$3x^2+3y^2+2xy-8=0.$$

Example 9. *Find the equation of the ellipse whose foci are $(-3, 0)$ and $(3, 0)$ and whose eccentricity is $\frac{1}{4}$.*

Solution. Suppose a and b are respectively the major semi-axis and the minor semi-axis of the ellipse whose equation is required. Now, the distance between the foci of the ellipse is $2ae$.

$\therefore$ $$2ae=6, \text{ where } e=\frac{1}{4}.$$

This gives $a=12$.

Now, $$b^2=a^2(1-e^2),$$
$$=144\left(1-\frac{1}{16}\right),$$
$$=135.$$

$\therefore$ the equation of the ellipse is
$$\frac{x^2}{144}+\frac{y^2}{135}=1,$$
or $$15x^2+16y^2=2160.$$

Example 10. *Find the foci, the equations of the directrices and the eccentricity of the ellipse* $4x^2+9y^2=36$.

Solution. The equation of the given ellipse can be re-written as

$$\frac{x^2}{9}+\frac{y^2}{4}=1.$$

If a and b are the major semi-axis and minor semi-axis of the ellipse, then

$$a^2=9,\ b^2=4.$$

Also, the eccentricity e is given by

$$b^2=a^2(1-e^2),$$

or

$$e^2=1-\frac{b^2}{a^2},$$

$$=1-\frac{4}{9}=\frac{5}{9}.$$

$\therefore$

$$e=\frac{\sqrt{5}}{3}.$$

The foci being the points $(\pm ae, 0)$, are $(\pm\sqrt{5}, 0)$.

The equations of the directrices are

$$x=\pm\frac{a}{e}=\pm\frac{3}{\sqrt{5}/3}=\pm\frac{9}{\sqrt{5}}.$$

Example 11. *Find the eccentricity, latus rectum, foci and directrices of the ellipse* $25x^2+16y^2=400$.

Solution. The equation of the ellipse can be re-written as

$$\frac{x^2}{16}+\frac{y^2}{25}=1.$$

Here, the denominator of y^2 is greater than the denominator of x^2. Therefore, the major axis lies along the axis of y and the minor axis lies along the axis of x. (See remark on page 408.)

Comparing the given equation with

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1,$$

we find that the lengths a and b of the major semi-axis and minor semi-axis are given by

$$a^2=25,$$
$$b^2=16.$$

The eccentricity e is given by

$$b^2=a^2(1-e^2),$$

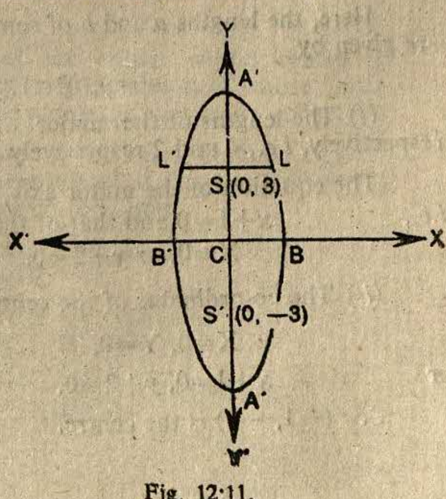


Fig. 12·11.

or $$e^2=1-\frac{b^2}{a^2}=1-\frac{16}{25}=\frac{9}{25},$$

or $$e=\frac{3}{5}.$$

The length of the latus rectum is

$$\frac{2b^2}{a}=\frac{2\cdot16}{5}\left(=\frac{32}{5}\right)=6\frac{2}{5}.$$

The co-ordinates of the foci are

$$(0,\ \pm ae)=\left(0,\ \pm\frac{5\cdot3}{5}\right)=(0,\ \pm3).$$

The equations of the directrices are

$$y=\pm a/e=\pm\frac{5\cdot5}{3}=\pm\frac{25}{3}.$$

Example 12. *Find the centre, the lengths and equations of the major and minor axes, the length of the latus rectum and the equations of the latera recta, the foci and the directrices of the ellipse*

$$x^2+4y^2+2x+16y+13=0.$$

Solution. The given equation can be re-written as

$$(x^2+2x+1)+4(y^2+4y+4)-4=0,$$

or $$\frac{(x+1)^2}{4}+(y+2)^2=1.$$

Transferring the origin to the point $(-1, -2)$, we get the equation of the ellipse as

$$\frac{X^2}{4}+\frac{Y^2}{1}=1.$$

Here, the lengths a and b of semi-major and semi-minor axes are given by

$$a^2=4,\ b^2=1.$$

(*i*) The lengths of the major and minor axes are $2a$ and $2b$ respectively, *i.e.*, 4 and 2 respectively.

The equation of the minor axis is $X=0$,

i.e., $x+1=0$ and that of the major axis is

$$X=0, \text{ or } y+2=0.$$

(*ii*) The co-ordinates of the centre are given by

$$X=0,\ Y=0,$$

or $$x+1=0,\ y+2=0.$$

$\therefore$ $(-1, -2)$ is the centre.

(*iii*) The length of the latus rectum is

$$\frac{2b^2}{a}=1.$$

Now, the eccentricity e is given by

$$e^2=1-\frac{b^2}{a^2}=1-\frac{1}{4}=\frac{3}{4}.$$

$\therefore$ The equations of the latera recta are

$$X=\pm ae=\pm 2\,.\,\sqrt{3}/2=\pm\sqrt{3},$$

or $$x+1=\pm\sqrt{3}, \text{ or } x=\pm\sqrt{3}-1.$$

(*iv*) The foci are the points given by

$$X=\pm ae=\pm\sqrt{3}, \text{ and } Y=0,$$

or $$x+1=\pm\sqrt{3} \text{ and } y+2=0.$$

Thus, the foci are the points $(\sqrt{3}-1, -2)$ and $(-\sqrt{3}-1, -2)$.

(*v*) The equations of the directrices are given by

$$X=\pm\frac{a}{e}=\pm\frac{4}{\sqrt{3}},$$

or $$x+1=\pm\frac{4}{\sqrt{3}},$$

or $$x=\frac{4}{\sqrt{3}}-1 \text{ and } x=\frac{-4}{\sqrt{3}}-1.$$

EXERCISE 12 (*c*)

1. Find the standard equation of the ellipse whose major axis lies along the x-axis and is of length 5/2, and whose minor axis lies along the y-axis and is of length 1/2.
2. Find the equation of the ellipse whose eccentricity is $\frac{2}{3}$, a focus is (3, 4), and the corresponding directrix is the line
   $3x+4y=5.$
3. Find the standard equation of the ellipse whose centre is the origin, whose eccentricity is $\frac{1}{2}$ the length of whose latus rectum is 8, and whose axes lie along the axes of co-ordinates.
4. Find the equation of the ellipse the distance between whose foci is 8 and the distance between whose directrices is 18, and whose axes lie along the axes of co-ordinates.
5. Find the foci, directrices, eccentricity and the length of the latus rectum of the ellipse $5x^2+4y^2=1$.
6. Find the eccentricity, the foci, the directrices and the length of the latus rectum of the ellipse $2x^2+3y^2-1=0$.
7. Find the co-ordinates of the extremities of the latera recta of the ellipse given in questions 5 and 6 above.

8. Find the centre, the eccentricity, the foci, the latus rectum and the directrices of the ellipse

$$x^2+4y^2-10x-24y+45=0.$$

9. Find the eccentricity, the foci, and the length of the latus rectum of the ellipse

$$8(x-1)^2+6(y+1)^2-1=0.$$

10. Find the distance between the foci of the ellipses

$$3x^2+4y^2=1.$$

11. Find the lengths of the axes of the ellipse whose foci are (2, 0) and (−2, 0) and whose eccentricity is $\frac{1}{3}$.

12. Is the point (2, 3) outside, on, or inside the ellipse :

(*i*) $2x^2+3y^2=1$, (*ii*) $\dfrac{x^2}{9}+\dfrac{y^2}{31}=1$,

(*iii*) $\dfrac{x^2}{16}+\dfrac{y^2}{12}=1$?

12·15. CONDITION THAT $y=mx+c$ MAY TOUCH THE ELLIPSE $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

Theorem 12·2. *With a given slope m, there are two tangents to the ellipse* $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, *given by* $y=mx\pm\sqrt{a^2m^2+b^2}$.

Proof. Let $y=mx+c$ be any straight line with slope m,

The abscissae of the points of intersection of the line $y=mx+c$ with the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ are given by

$$\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1,$$

or $$b^2x^2+a^2(mx+c)^2=a^2b^2,$$

or $$(a^2m^2+b^2)x^2+2mca^2x+a^2(c^2-b^2)=0.$$

If the line is a tangent to the ellipse, the two roots of the above equation must be equal. The condition for this is

$$(2mca^2)-4(b^2+a^2m^2)a^2(c^2-b^2)=0,$$

or $$c^2=a^2m^2+b^2,$$

or $$c=\pm\sqrt{a^2m^2+b^2}\,.$$

Thus the line $y=mx+c$ touches the ellipse $x^2/a^2+y^2/b^2=1$ provided $c=\pm\sqrt{a^2m^2+b^2}$.

Consequently, the equations of the tangents to the ellipse $x^2/a^2+y^2/b^2=1$ with a given slope m are

$$y=mx+\sqrt{a^2m^2+b^2},$$

and
$$y=mx-\sqrt{a^2m^2+b^2}$$

Remarks. 1. The lines

$$y=mx\pm\sqrt{a^2m^2+b^2}$$

will always touch the ellipse whatever be the value of m.

2. There are two tangents to an ellipse parallel to a given straight line. These two parallel tangents are equidistant from the centre of the ellipse, each being at a distance $\sqrt{\left(\dfrac{a^2m^2+b^2}{m^2+1}\right)}$ from the centre.

Example 13. *Find the equations of the tangents to the ellipse $4x^2+3y^2=6$ which are parallel to the line $y=2x+3$.*

Solution. We know that the equations of the tangents to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, parallel to the line with slope m are given by

$$y=mx\pm\sqrt{a^2m^2+b^2}$$

Here the equation of the ellipse is

$$4x^2+3y^2=6,$$

or
$$\frac{x^2}{3/2}+\frac{y^2}{2}=1.$$

$$\therefore \qquad a^2=\frac{3}{2},\ b^2=2.$$

Also, m, the slope of the given line is 2. Therefore, the required equations of the tangents are

$$y=2x\pm\sqrt{\left(\frac{3}{2}\right).4+2}=2x\pm2\sqrt{2}.$$

$$\therefore \qquad y=2x+2\sqrt{2}, \text{ and } y=2x-2\sqrt{2}$$

are the two tangents parallel to the line $y=2x+3$.

Example 14. *Find the value of a so that the straight line $y=x+a$ may touch the ellipse $2x^2+3y^2=6$.*

Solution. The equation of the ellipse can be written as

$$\frac{x^2}{3}+\frac{y^2}{2}=1.$$

Therefore, the equations of the tangents to the given ellipse parallel to the given line are

$$y=x\pm\sqrt{3+2}=x\pm\sqrt{5}.$$

If $y=x+a$ is to be one of these tangents, then we must have

$$a=\pm\sqrt{5}.$$

EXERCISE 12 (*d*)

1. Find the equations of the tangents to the ellipse
$$9x^2+16y^2=144$$
parallel to the line $y=x+4$.

2. Find the equations of the tangents to the ellipse
$$x^2+16y^2=16$$
which make an angle of 60° with the major axis.

3. Find the equations of the tangents to the ellipse $x^2+3y^2=3$ which are perpendicular to the line $x=3y+5$.

4. Show that the straight line $4x+3y=12\sqrt{2}$ touches the ellipse
$$4x^2+9y^2=36.$$
Also, find the co-ordinates of the point of contact.

5. Find the values of c so that the line $y=x+c$ may touch the ellipse $2x^2+3y^2=1$.

6. Show that the line $x \cos \alpha+y \sin \alpha=p$ touches the ellipse
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ if } a^2 \cos^2 \alpha+b^2 \sin^2 \alpha=p^2.$$

12·16. HYPERBOLA

The hyperbola was studied like the other conic sections, by the ancient Greek geometer Menaechmus, Apollonius, Pappus, Aristaeus and Euclid. The lost works of Aristaeus and Euclid probably dealt with the general hyperbola, but only a single branch of it. Apollonius was the first one to treat the double branched curve obtaining it as a section of a double cone.

A branch of a hyperbola may often be seen as the edge of the shadow cast on a wall by a circular lampshade.

The property that the difference of the focal distances of a hyperbola is constant has important applications in theory and practice of radar navigation.

As pointed out earlier, a hyperbola is a conic section whose eccentricity is greater than unity. Thus, we have the following :

Definition 12·3. *A* **hyperbola** *is the locus of a point which moves so that its distance from a fixed point bears to its distance from a fixed straight line (not containing the point) a constant ratio which is greater than unity. The fixed point is called the focus, the fixed straight line is called the directrix and the constant ratio is called the eccentricity.*

12·17. THE STANDARD EQUATION OF A HYPERBOLA

Theorem 12·3. *The equation of a hyperbola in the standard form is*

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

Proof. Let S be the focus and ZM the directrix. Draw SZ perpendicular to the directrix. Since $e>1$, we can divide SZ internally and externally in the ratio $e:1$. Let the points of division be A and A′ as shown in Fig. 12·12.

Let C be the middle point of A′A and let $A'A=2a$.

Then $\quad SA=e.AZ$

and $\quad SA'=e.ZA'.$

$\therefore \quad SA+SA'=e(AZ+ZA'),$

or $\quad SA+(SA+AA')=eAA',$

or $\quad 2(SA+AC)=eAA',$

or $\quad 2SC=2e.AC,$

or $\quad CS=ae. \qquad \ldots(i)$

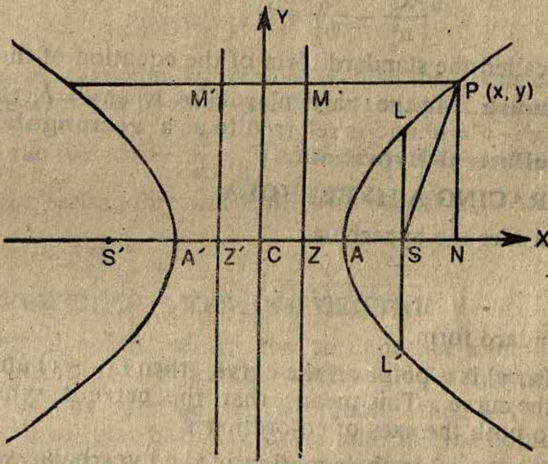


Fig. 12·12.

Also, $\quad SA'-SA=e(ZA'-AZ),$

i.e., $\quad AA'=e(AA'-2AZ),$

i.e., $\quad 2AC=eZZ',$

i.e., $\quad AC=e.ZC,$

i.e., $\quad CZ=\dfrac{a}{e}. \qquad \ldots(ii)$

Now, let C be the origin of co-ordinates, CA the axis of x, and a line through C perpendicular to CA the axis of y. Let $P(x, y)$ be any point on the curve.

Then from Fig. 12·12, by the definition of hyperbola,

$$SP^2 = e^2 PM^2,$$

or $$SN^2 + NP^2 = e^2 ZN^2.$$

Now, $$SN = CN - CS = x - ae,$$

and $$ZN = CN - CZ = x - \frac{a}{e}.$$

$$\therefore \quad (x-ae)^2 + y^2 = e^2\left(x - \frac{a}{e}\right)^2,$$

or $$y^2 + x^2(1-e^2) = a^2(1-e^2),$$

or $$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1. \qquad \ldots(iii)$$

Since $e > 1$, $\therefore$ $a^2(1-e^2)$ is negative.

Let. $a^2(1-e^2) = -b^2$.

Then the equation of the hyperbola becomes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

which is called the standard form of the equation of the hyperbola.

Remark. In the particular case when $a = b$, the hyperbola becomes $x^2 - y^2 = a^2$ and is referred to as a **rectangular hyperbola** or an **equilateral hyperbola**.

12·18. TRACING A HYPERBOLA

Consider the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

in its standard form.

If (x, y) is a point on the curve, then $(x, -y)$ and $(-x, y)$ are also on the curve. This means that the curve is symmetrical with respect to both the axes of co-ordinates.

Putting $y = 0$ in the equation of the hyperbola, we get, $x = \pm a$. Thus, the curve cuts the x-axis in two points $A(a, 0)$ and $A'(-a, 0)$. If $x = 0$, then the equation gives $y^2 = -b^2$, which shows that y-axis does not meet the curve in real points.

Now, writing the equation in the form

$$x^2 = \frac{a^2}{b^2}(y^2 + b^2),$$

we see that for every real value of y, x is real.

Also, x increases as y increases.

Again, writing the equation in the form

$$y^2=\frac{b^2}{a^2}(x^2-a^2),$$

we see that for all values of x such that $-a<x<a$, y^2 is negative. This means that no part of the curve lies between the lines $x=a$ and $x=-a$.

Also, when x tends to $+\infty$, $|y|$ also tends to $+\infty$. Thus, taking all the above facts into consideration, the shape of the curve is as shown in Fig 12·12, and consists of two infinite branches, each branch roughly resembling the portion of a parabola near its vertex.

12·19. THE SECOND FOCUS AND THE SECOND DIRECTRIX

In Fig. 12·12, S is a focus with co-ordinates $(ae, 0)$. Also, ZM is a directrix with equation $x=a/e$. Now, the equation of the hyperbola as deduced earlier, is

$$\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1,$$

or $$(1-e^2)x^2+y^2=a^2(1-e^2),$$

or $$x^2+y^2+a^2e^2+2aex=e^2\cdot\left(x^2+\frac{2ax}{e}+\frac{a^2}{e^2}\right),$$

or $$(x+ae)^2+y^2=e^2\left(x+\frac{a}{e}\right)^2.$$

This shows that the hyperbola is also the locus of a point whose distance from the point $(-ae, 0)$ is e times its perpendicular distance from the line $x=-\dfrac{a}{e}$. Therefore, there is a second focus $S'(-ae, 0)$ and a second directrix Z′M′ with equation

$$x=-\frac{a}{e}.$$

12·20. THE VERTICES, AXES AND CENTRE

The points A and A′ where the straight line joining the two foci cuts the hyperbola are called the **vertices** of the hyperbola.

If (x', y') is any point on the hyperbola, then obviously, the point $(-x', -y')$ will also be on the hyperbola. But the points (x', y') and $(-x', -y')$ are on a straight line through the origin and are equidistant from the origin. Hence, the origin bisects every chord which passes through it and is, therefore, called the **centre** of the hyperbola.

AA′ is called the **transverse axies** of the hyperbola. The line through the centre C perpendicular to AA′ does not meet the hyperbola in real points, because if we put $x=0$ in the equation $x^2/a^2-y^2/b^2=1$, we get $y^2=-b^2$, which does not give any real value of y. But if B and B′ be thepoints on this line such that $B'C=CB=b$, the line B′B is called the **conjugate axis.**

12·21. RELATION BETWEEN FOCAL DISTANCES

Theorem 12·4. *The difference of the focal distances of a point on the hyperbola is constant and is equal to the length of the transverse axis.*

Proof. In Fig. 12·12, we have

$$\begin{aligned} PS'-PS &= ePM'-ePM, \\ &= e(PM'-PM), \\ &= eMM', \\ &= eZZ' = e\,.\left(\frac{2a}{e}\right) = 2a, \end{aligned}$$

= the length of the transverse axis.

This proves the theorem.

12·22. THE LATUS RECTUM

The chord L′L through either focus perpendicular to the transverse axis is called the **latus rectum.**

In Fig. 12·12, if SL $=l$, the co-ordinates of L are (ae, l). Since L lies on the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

we must have

$$\frac{a^2e^2}{a^2}-\frac{l^2}{b^2}=1.$$

$$\therefore \qquad l^2=b^2(e^2-1),$$

$$=\left(\frac{b^2}{a^2}\right)a^2(e^2-1)=\frac{b^4}{a^2}.$$

$$\therefore \qquad l=\pm\frac{b^2}{a},$$

the negative sign corresponding to L′.

Thus, the latus rectum

$$L'L=2SL=\frac{2b^2}{a}.$$

It follows that the co-ordinates of the extremities of the latera recta are $\left(\pm ae, \pm\dfrac{b^2}{a}\right)$.

12·23. THE EQUATION OF A HYPERBOLA WITH A GIVEN ECCENTRICITY, A GIVEN FOCUS AND A GIVEN DIRECTRIX

Suppose e is the eccentricity, (h, k) is a focus and

$$ax+by+c=0$$

the corresponding directrix of a hyperbola. If $P(x, y)$ be any point on the hyperbola, then, by definition,

$$(x-h)^2+(y-k)^2=e^2 \cdot \frac{(ax+by+c^2)}{a^2+b^2},$$

which is the required equation.

Example 15. *Find the equation of the hyperbola with one focus (1, 1), the correspondi ngdirectrix being $2x+y=1$, and with eccentricity $\sqrt{3}$.*

Solution. The required equation is

$$(x-1)^2+(y-1)^2=(\sqrt{3})^2\left(\frac{2x+y-1}{\sqrt{5}}\right)^2,$$

or $$x^2-2x+1+y^2-2y+1=3\left(\frac{4x^2+y^2+1+4xy-4x-2y}{5}\right),$$

or $$7x^2+12xy-2y^2-2x+4y-7=0.$$

Example 16. *Find the equation of the hyperbola the distance between whose foci is 32 and whose eccentricity is $2\sqrt{2}$.*

Solution. Here, $e=2\sqrt{2}$

and $$2ae=32.$$

$\therefore$ $$2a.2\sqrt{2}=32,$$

or $$a^2=32.$$

Now, $$b^2=a^2(e^2-1),$$
$$=32\ (8-1)=224.$$

Therefore, the equation of the required hyperbola is

$$\frac{x^2}{32}-\frac{y^2}{224}=1.$$

or $$7x^2-y^2=224.$$

Example 17. *Find the centre, the foci, the eccentricity, the directrices and the lengths of the axes of the hyperbola*

$$\frac{(x-1)^2}{9}-\frac{(y-2)^2}{16}=1.$$

Solution. Transfer the origin to the point (1, 2). The equation of the hyperbola then becomes

$$\frac{X^2}{9}-\frac{Y^2}{16}=1.$$

Here, $a^2=9, b^2=16$, so that $a=3, b=4$. The length of the transverse axis is, therefore, equal to $2a=2.3=6$. Also, the transverse axis lies along the line $Y=0$ or $y-2=0$.

Again, the length of the conjugate axis $=2b=2.4=8$. The conjugate axis lies along the line $X=0$ or $x-1=0$.

The centre is given by $X=0$, $Y=0$ or $x-1=0$, $y-2=0$. Therefore, the co-ordinates of the centre are (1, 2).

Now, $b^2=a^2(e^2-1)$ gives

$$16=9(e^2-1),$$

or $$e=\frac{5}{3}.$$

$\therefore$ the co-ordinates of the foci are given by

$$X=\pm ae=\pm 3.5/3=\pm 5,$$

and $$Y=0,$$

or $$x-1=\pm 5,$$

and $$y-2=0.$$

Hence the co-ordinates of the foci are $(6, 2)$ and $(-4, 2)$.

The equations of the directrices are

$$X=\pm\frac{a}{e}=\pm\frac{3.3}{5}=\pm\frac{9}{5},$$

or $$x-1=\pm\frac{9}{5},$$

or $$x=\frac{14}{5},$$

and $$x=-\frac{4}{5}.$$

EXERCISE 12 (*e*)

1. Find the equation of the hyperbola one of whose foci is $(2, 3)$, the corresponding directrix is $x+2y+1=0$ and whose eccentricity is $\sqrt{2}$.
2. Find the equation of a hyperbola such that the distance between its foci is 16 and its eccentricity is $\sqrt{2}$.
3. Find the equation of a hyperbola with a vertex $(4, 0)$ and a focus $(0, 0)$.
4. Find the equation of the hyperbola with foci $(\pm 2, 0)$ and eccentricity $\frac{3}{2}$.
5. Find the equation of a hyperbola with latus rectum 4 and eccentricity 3.
6. Find the eccentricity, foci, directrices and the length of the latus rectum of the hyperbola $16x^2-9y^2=144$.
7. Find the eccentricity, the foci, the directrices and the length of the latus rectum of the hyperbola
$$16x^2-25y^2=400.$$
Also, find the co-ordinates of the extremities of the latera recta.
8. Find the axes, eccentricity, foci and the length of the latus rectum of the hyperbola $4x^2-9y^2=36$.

9. Find the lengths of the axes, the centre, the foci, the length of the latus rectum, the eccentricity and the directrices of the hyperbola

$$\frac{(x+4)^2}{16}-\frac{(y+1)^2}{9}=1.$$

10. Find the lengths of the axes, the centre, the foci, the length of the latus rectum, the eccentricity and the directrices of the hyperbola

$$\frac{(x-3)^2}{9}-\frac{(y-4)^2}{16}=1.$$

12·24. CONDITION THAT $y=mx+c$ MAY TOUCH THE HYPERBOLA $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

The abscissae of the points of intersection of the straight line $y=mx+c$ with the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are given by

$$\frac{x^2}{a^2}-\frac{(mx+c)^2}{b^2}=1,$$

or $$b^2x^2-a^2(mx+c)^2-a^2b^2=0,$$

or $$(-a^2m^2+b^2)x^2-2mca^2x-a^2(c^2+b^2)=0.$$

If the line is a tangent to the hyperbola, both the roots of the above equation must be equal. The condition for this is

$$(2mca^2)^2+4(-a^2m^2+b^2)a^2(c^2+b^2)=0,$$

or $$c^2=a^2m^2-b^2.$$

Corollary. The condition $c^2=a^2m^2-b^2$ gives two values of c, namely, $c=\pm\sqrt{a^2m^2-b^2}$.

Hence there are two tangents to the hyperbola with a given slope m and the equations of these tangents are

$$y=mx+\sqrt{a^2m^2-b^2},$$

and $$y=mx-\sqrt{a^2m^2-b^2}.$$

Remarks. 1. The lines $y=mx\pm\sqrt{a^2m^2-b^2}$ will always touch the hyperbola whatever be the value of m.

2. There are two tangents to a hyperbola parallel to a given straight line. The two parallel tangents are equidistant from the centre of the hyperbola, the common distance being $\sqrt{\left(\frac{a^2m^2-b^2}{m^2+1}\right)}$.

Example 18. *Show that the straight line $3x-y=5$ touches the hyperbola $\frac{x^2}{5}-\frac{y^2}{20}=1$ and find the point of contact.*

Solution. The abscissa of the points of intersection of the straight line

$$3x-y=5 \qquad \ldots(i)$$

and the hyperbola

$$\frac{x^2}{5}-\frac{y^2}{20}=1 \qquad \ldots(ii)$$

are given by

$$\frac{x^2}{5}-\frac{(3x-5)^2}{20}=1.$$

or $$4x^2-(3x-5)^2=20,$$

or $$-5x^2+30x-45=0,$$

or $$x^2-6x+9=0,$$

or $$(x-3)^2=0.$$

Since the values of x are equal, the straight line touches the hyperbola. The point of contact is given by $x=3$, $y=3x-5=4$. Hence the point of contact is $(3, 4)$.

Example 19. *Find the equations of the tangent to the hyperbola*

$$\frac{x^2}{36}-\frac{y^2}{25}=1$$

which make an angle of 45° with the axis of x.

Solution. The slope m of the tangent $=\tan 45°=1$. Therefore, the required equations are

$$y=1.x\pm\sqrt{36.1-25},$$
$$=x\pm\sqrt{11},$$

or $$x-y=\pm\sqrt{11}.$$

EXERCISE 12 (*f*)

1. Show that the straight line $8x-5y=20$ touches the hyperbola $32x^2-25y^2=400$ and find the point of contact.
2. Find the equations of the tangents to the hyperbola $4x^2-9y^2=1$ which are parallel to the straight line $5x-4y+7=0$.
3. Find the equations of the tangents to the hyperbola $\frac{x^2}{36}-\frac{y^2}{25}=1$ which are perpendicular to the straight line $3x+4y=5$.
4. Show that the straight line $y=x+\sqrt{2}$ touches the hyperbola $3x^2-5y^2=15$. Also, find the co-ordinates of the point of contact.
5. Find the co-ordinates of the point at which the line $4x-3y=1$ is a tangent to the hyperbola $4x^2-3y^2=1$.

TEST YOUR UNDERSTANDING XII

In each of the following problems four alternatives are given. Put a tick (√) mark against the correct alternative :

1. The latus rectum of the parabola $y^2=-8x$ is
   (*a*) 2 (*b*) -2 (*c*) -8 (*d*) 8.
2. The focus of the parabola $y^2=16x$ is the point
   (*a*) (0, 4) (*b*) (16, 0) (*c*) (4, 0) (*d*) (−4, 0).
3. The axis of the parabola $y^2=ax$ is the line
   (*a*) $x=0$ (*b*) $x=-3$ (*c*) $y=3$ (*d*) $y=0$.
4. The straight line $x+y-c=0$ touches the parabola $y^2=12x$ provided c equals
   (*a*) 3 (*b*) -3 (*c*) 12 (*d*) -12.
5. The eccentricity of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ is
   (*a*) $\frac{7}{9}$ (*b*) $\frac{4}{3}$ (*c*) $\sqrt{7}/3$ (*d*) $\frac{3}{4}$.
6. The latus rectum of the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ is
   (*a*) $\frac{5}{2}$ (*b*) $16/5$ (*c*) $\frac{5}{4}$ (*d*) $32/5$.
7. The equation of a directrix of the ellipse $4x^2+8y^2=1$ is
   (*a*) $x+\sqrt{2}=0$ (*b*) $y-\sqrt{2}=0$ (*c*) $y+\sqrt{2}=0$ (*d*) $x\sqrt{2}+1=0$.
8. The straight line $y=1$ touches the ellipse $x^2+16y^2=16$ at the point
   (*a*) (−1, 1) (*b*) (1, 1) (*c*) (2, 1) (*d*) (0, 1).
9. The equation $4x^2-y^2+6=0$ represents a
   (*a*) circle (*b*) parabola (*c*) hyperbola (*d*) ellipse.
10. The length of the transverse axis of the hyperbola $9x^2-16y^2=144$ is
    (*a*) 16 (*b*) 8 (*c*) 4 (*d*) 9.

REVIEW EXERCISE XII

1. The parabola $y^2=4ax$ passes through the point (5, −10). Find the co-ordinates of its focus and the length of its latus rectum.
2. Find the equation of the parabola whose focus is (4, 0) and whose directrix is the line $x+4=0$.
3. Find the equation of the tangent to the parabola $y^2=6x$ which is parallel to the line $3x-y+4=0$. Also, find the point of contact.
4. For what value of c does the line $y=4x+c$ touch the parabola $y^2=-8x$?
5. What is the distance between the foci of the ellipse $4x^2+9y^2=1$?
6. Find the eccentricity of the ellipse $16x^2+25y^2=1$.
7. For what values of c does the straight line $y=x+c$ touch the ellipse $6x^2+8y^2=1$?

8. Find the equations of the tangents to the ellipse $4x^2+3y^2=1$ which are parallel to $2x+y+3=0$.

9. Find the eccentricity of the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$.

10. Show that the straight line $4x-3y=1$ touches the hyperbola $4x^2-3y^2=1$ at the point $(1, 1)$.

SUMMARY

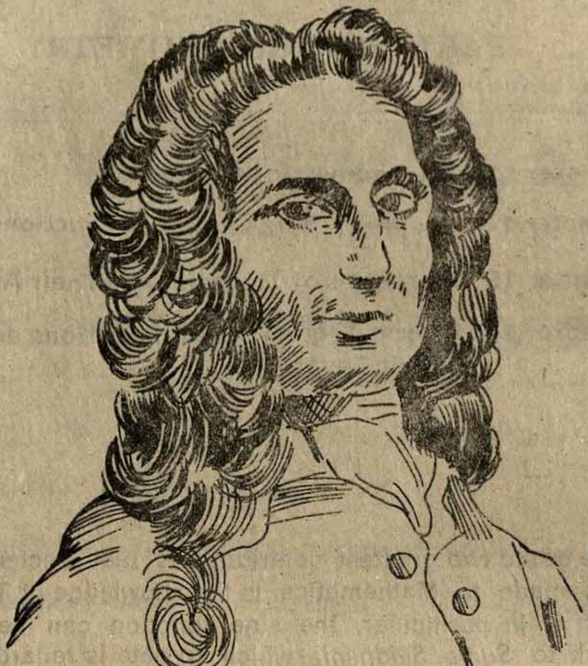
1. The equation of the parabola with vertex at the origin, axis along the x-axis, tangent at the vertex along the y-axis, and latus rectum $4a$, is $y^2=4ax$.
2. The equation of the ellipse with centre at the origin, major axis along the x-axis and of length $2a$, minor axis along the y-axis and of length $2b$, is $x^2/a^2+y^2/b^2=1$. The eccentricity e is given by $b^2=a^2(1-e^2)$, the co-ordinates of the foci are $(\pm ae, 0)$, the directrices are the lines $x=\pm\frac{a}{e}$, and the length of the latus rectum is $\frac{2b^2}{a}$.
3. The equation of the hyperbola with centre at the origin, transverse axis along the x-axis and of length $2a$, conjugate axis along the y-axis and of length $2b$, is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. The eccentricity e is given by $b^2=a^2(e^2-1)$. The co-ordinates of the foci are $(\pm ae, 0)$, the directrices are the lines $x=\pm\frac{a}{e}$, and the length of the latus rectum is $\frac{2b^2}{a}$.
4. The straight line $y=mx+c$ touches the parabola $y^2=4ax$ provided $c=\frac{a}{m}$.
5. The straight line $y=mx+c$ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ provided $c^2=a^2m^2+b^2$.
6. The straight line $y=mx+c$ touches the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ provided $c^2=a^2m^2-b^2$.

□ □

PART III : TRIGONOMETRY

□ □ □ □

One of the two greatest contributions that ancient India has made to Mathematics, is the knowledge of Trigonometry. In pariticular, the sine function can be traced back to *Surya Siddhanta* which to date is regarded as a marvellous work on Astronomy.



ABRAHAM DE MOIVRE (1667-1754)

Abraham de Moivre was born at Vitry in Champagne, France on May 26, 1667. At the age of twenty-one he left France for England, where he spent the rest of his life as a tutor in Mathematics to sons of rich noblemen.

De Moivre was elected a Fellow of the Royal Society in 1697. His friends included Halley and Newton. Newton held him in high esteem who (as the story goes !), when approached by students with problems often used to say, "Go to Mr. de Moivre, he knows these things better than I do". He is remembered most for 'De Moivre's Theorem' in Trigonometry. He was among the founders of the mathematical theory of probability. His *Doctrine of Chances*, a masterpiece, ran into three editions—in 1718, 1738 and a posthumous one in 1756. He died on November 27, 1754 at the age of 87, poverty stricken, with failing eye sight, and without friends, all of whom (with the exception of James Stirling) had already passed away by then.

CHAPTER 13

Trigonometric Functions

13·1. INTRODUCTION

The literal meaning of the word 'Trigonometry' is the 'science of triangle measurement'. It had its beginning more than two thousand years ago as a tool for astronomers. The Babylonians, Egyptians, Greeks and the Hindus studied trigonometry only because it helped them in unraveling the mysteries of the universe. In modern times trigonometry has tremendous applications to physics and engineering.

We shall devote Chapters 13-16 to the study of trigonometry. In the present chapter we shall introduce trigonometric functions and obtain identities and some basic formulae connecting various trigonometric functions. We shall also try to learn the use of tables of values of trigonometric functions. In Chapter 14 we shall study the graphs of trigonometric functions. In Chapter 15 we shall study properties of triangles and their applications to solution of triangles and problems on heights and distances. In Chapter 16 we shall study trigonometric equations and inverse trigonometric functions.

13·2. EVEN AND ODD FUNCTIONS

In the first chapter we had reviewed various concepts which included functions, their domain and range etc. It would be worthwhile if you refresh yourself about those concepts because we shall be using them in this chapter. In this section, we shall introduce the important concept of odd and even functions which will be useful to us in the study of trigonometric functions.

Consider the function f given by $f(x)=x^2$. Since $(-x)^2=x^2$, for all x, therefore, $f(-x)=(-x)^2=x^2=f(x)$, for all x. Similarly, for each of the functions g and h defined by $g(x)=|x|$, $h(x)=x^4$, we have $g(-x)=g(x)$ and $h(-x)=h(x)$, for all x. Such functions are called *even functions.*

Definition 13·1. *A function f is said to be* **even** *if*

$$f(-x)=f(x)$$

for all x in the domain of f.

We give below graphs of some even functions.

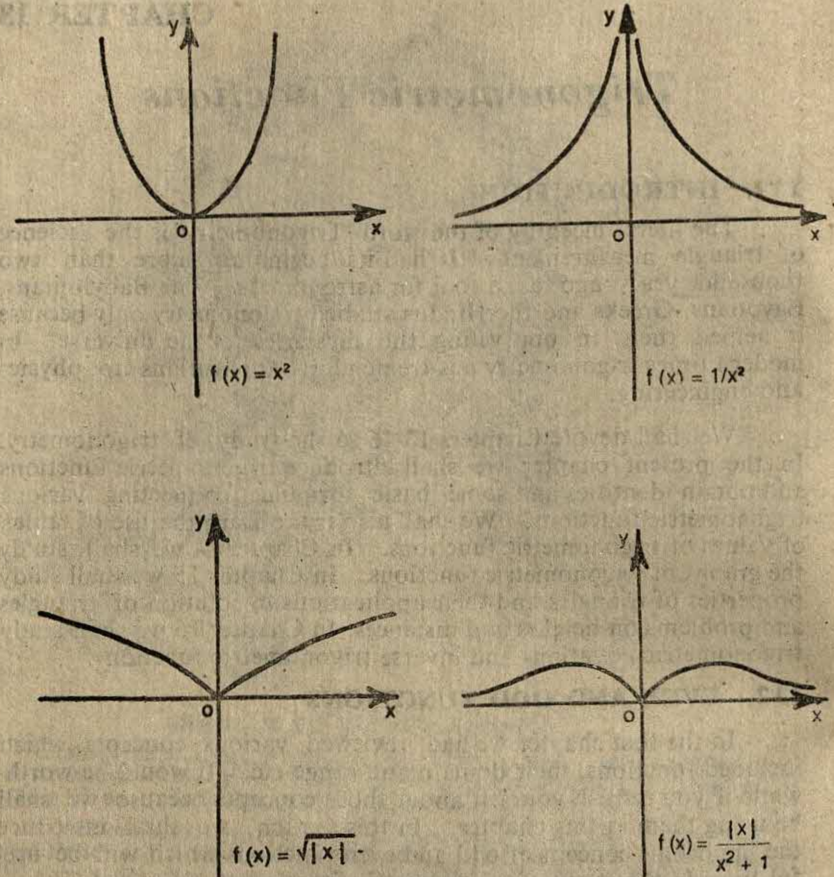


Fig. 13·1. Graphs of some even functions

Next let us consider the function F given by $F(x)=x^3$. Since $(-x)^3=-x^3$ for all x, therefore, $F(-x)=(-x)^3=-x^3=-F(x)$, for all x. Similarly for the functions G and H given by $G(x)=x^5$ and $H(x)=x^7$, we have $G(-x)=-G(x)$, $H(-x)=-H(x)$ for all x. Such functions are called *odd functions.*

Definition 13·2. *A function f is said to be* **odd** *if*

$$f(x)=-f(-x)$$

for all x in the domain of f.

We give below graphs of two odd functions.

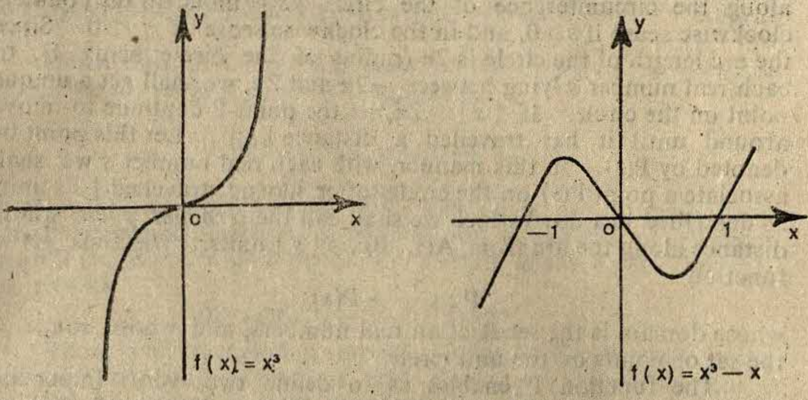


Fig. 13·2. Graphs of two odd functions

Finally, let us consider the function

$$f : f(x)=x^2+x.$$

Here $f(-x)=x^2-x$ which is neither equal to (fx) nor to $-f(x)$ for any x other than $x=0$. This function is, therefore, neither even nor odd. Similarly the functions,

$$g : g(x)=x^2-3x+2,$$
$$h : h(x)=x^3-2x+1,$$

are neither odd nor even.

13·3. SINE AND COSINE FUNCTIONS

Consider the unit circle (*i.e.*, a circle of radius 1) having its centre at the origin of a cartesian co-ordinate system (see Fig. 1·3·3). The equation of the circle is $x^2+y^2=1$.

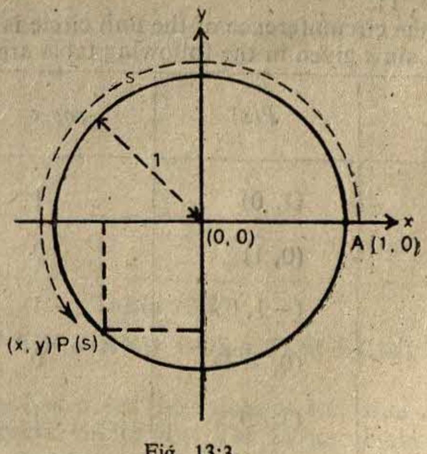


Fig. 13·3.

Suppose a variable point P starts at the point A(1, 0) and moves along the circumference of the circle $|s|$ units, in the counter-clockwise sense if $s>0$, and in the clockwise sense if $s<0$. Since the arc length of the circle is 2π (radius of the circle being 1), to each real number s lying between -2π and 2π, we shall get a unique point on the circle. If $|s|>2\pi$, let the point P continue to move around until it has travelled a distance $|s|$. Let this point be denoted by P(s). In this manner. with each real number s we shall associate a point P(s) on the circle (after having travelled $|s|$ units as described just now) which we shall call the *terminal point* whose distance along the arc from A(1, 0) is $|s|$ units. We thus get a function

$$\text{P} : s \longrightarrow \text{P}(s)$$

whose domain is the set **R** of all real numbers, and whose range is the set of points on the unit circle.

The function P enables us to define two very important functions in a very natural manner. If the co-ordinates of P(s) be (x, y), then

$$s \longrightarrow \text{abscissa of P}(s), \text{ and } s \longrightarrow \text{ordinate of P}(s)$$

are two functions. These are the cosine and the sine functions, commonly abbreviated as cos and sin.

Definition 13·3. *If the terminal point P(s) has abscissa x and ordinate y, then*

$$\boxed{\begin{aligned} &\cos s = x, \text{ for all } s \in \mathbf{R}, \\ &\sin s = y, \text{ for all } s \in \mathbf{R}. \end{aligned}}$$

The domain of each of the above functions is **R**.

Since for every point (x, y) on the unit circle $-1 \leqslant x \leqslant 1$ and $-1 \leqslant y \leqslant 1$, therefore, the range of both cos and sin is the closed interval $[-1, 1]$.

Since the circumference of the unit circle is 2π units, the values of cos s and sin s given in the following table are obvious :

s	$P(s)$	$\cos s$	$\sin s$
0	(1, 0)	1	0
$\pi/2$	(0, 1)	0	1
π	(−1, 0)	−1	0
$3\pi/2$	(0, −1)	0	−1
2π	(1, 0)	1	0

Table 13·1.

We now proceed to define another trigonometric function, namely the tangent function (abbreviated as tan) which is equally important.

Definition 13.4. *If the terminal point $P(s)$ has co-ordinates (x, y), then the tangent function is defined by*

$$\boxed{\tan s = y/x, \text{ whenever } x \neq 0.}$$

Since $x=0$ if and only if $s=k\pi+\pi/2$, where k is an integer, therefore, the tangent function is defined for all values of s except $k\pi+\pi/2$ ($k \in \mathbf{Z}$). We shall denote the domain of tan by $\mathbf{R}^*$ throughout, so that

$$\mathbf{R}^* = \mathbf{R} \sim \{k\pi+\pi/2 : k \in \mathbf{Z}\}.$$

There is an obvious but important relation between the three trigonometric functions sin, cos, tan. Since $\tan s = y/x = \sin s/\cos s$, provided $\cos s \neq 0$, *i.e.*, $x \neq 0$, therefore, we have

$$\boxed{\tan s = \frac{\sin s}{\cos s}}$$

for all values of $s \in \mathbf{R}^*$.

Let us recall the signs of the co-ordinates of the points in various quadrants. By noting the quadrant in which $P(s)$ lies for a given value of s, we can find the sign of the value of a trigonometric function for a given value of s. The entries in the following table are easy to verify :

Quadrant in which P(s) lies	*Sign of sin s*	*Sign of cos s*	*Sign of tan s*
I	+	+	+
II	+	−	−
III	−	−	+
IV	−	+	−

Table 13·2

13·4. VALUES OF SIN s AND COS s FOR SOME SPECIAL VALUES OF s

In Table 13·1 we had noted the values of the sine and cosine functions for some special values of s. The values of sin s and cos s

cannot be easily found for all values of s. However, there are many values of s for which the values of $\sin s$ and $\cos s$ can be found without much difficulty. Two such values are $\pi/6$ and $\pi/3$. We shall now find the values of $\cos s$ and $\sin s$ when s has these values.

Consider the unit circle as shown in Fig. 13·4.

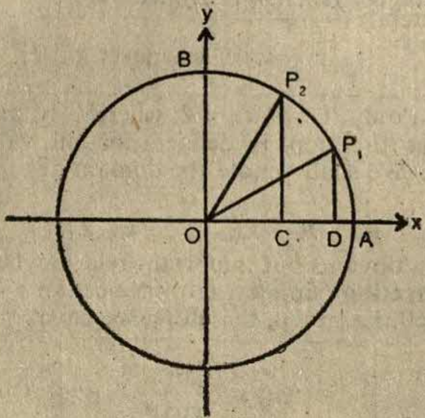


Fig. 13.4.

Let P_1, P_2 be the points of trisection of the arc of the unit circle from (1, 0) to (0, 1). This arc is one-fourth of the circumference of the circle and is, therefore, of length $\pi/2$. The arc-length of each of the trisected portions is, therefore, $\pi/6$. Let $P_1=P(\pi/6)$, and $P_2=P(\pi/3)$. Since in a circle equal arcs subtend equal angles at the centre, therefore, $\angle P_1Ox=30°$ and $\angle P_2Ox=60°$.

Consider the right-angled triangle P_1OD. Its acute angles are 30° and 60°. From your knowledge of plane geometry you know (or you can easily prove if you do not remember) that in such a triangle, the length of the hypotenuse is twice that of the shorter leg. Using this fact we find $P_1=P(\pi/6)$ is the point $(\sqrt{3}/2, 1/2)$. The triangle P_2OC is also of the same type and therefore, $P_2=P(\pi/3)$ is the point $(1/2, \sqrt{3}/2)$. We immediately have the following table of values :

s	$\sin s$	$\cos s$	$\tan s$
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

Table 13·3.

Remark. By using the fact that in an isosceles right-angled triangle the length of the hypotenuse is $\sqrt{2}$ times the length of either leg and proceeding in exactly the same manner as above, it is easy to show that $\cos \pi/4=\sin \pi/4=1/\sqrt{2}$ and $\tan \pi/4=1$.

You should be able to prove it yourself.

EXERCISE 13 (*a*)

1. Find the value of
   (*a*) $\sin^2 0+\cos^2 0$.
   (*b*) $\sin^2(\pi/6)+\cos^2(\pi/6)$.
   (*c*) $\sin^2(\pi/3)+\cos^2(\pi/3)$.
   (*d*) $\sin^2(\pi/2)+\cos^2(\pi/2)$.
   (*e*) $\sin(\pi/4)\cos(\pi/6)+\cos(\pi/4)\sin(\pi/6)$.
   (*f*) $\cos(\pi/3)\cos(\pi/4)-\sin(\pi/3)\sin(\pi/4)$.
2. Substitute the values of the trigonometric functions and verify that each of the following statements is true :
   (*a*) $\sin(\pi/3)=\cos(\pi/6)$.
   (*b*) $\sin(\pi/6)=\cos(\pi/3)$.
   (*c*) $\sin(\pi/3)=2\sin(\pi/6)\cos(\pi/6)$.
   (*d*) $\sin(\pi/2)=2\sin(\pi/4)\cos(\pi/4)$.
   (*e*) $\cos(\pi/3)=\cos^2(\pi/6)-\sin^2(\pi/6)=2\cos^2(\pi/6)-1$.

13·5. TRIGONOMETRIC IDENTITIES

Recall that an equation that holds true for all those values of the variables for which both sides of the equation are meaningful is called an **identity**. An identity which holds when certain conditions are imposed on the variables is called a **conditional identity**.

An identity involving trigonometric functions is called a **trigonometric identity.** For example, $\tan\theta\cos\theta-\sin\theta=0$ is a trigonometric identity. In the present section we shall study some basic trigonometric identities.

Theorem 13·1. *Far all* $s\in\mathbf{R}$,

$$\boxed{\cos^2 s+\sin^2 s=1.}$$

Proof. With the same notation as in section 13·3,

$\cos^2 s=\sin^2 s=x^2+y^2=1$, since (x, y) lies on the unit circle.

Remark. As we have seen above, the proof of the above identity follows directly from the definitions of the sine and cosine functions. It is a fundamental identity which we shall be using again and again.

Theorem 13·2. *If k be any integer, then*
$\sin(2k\pi+s)=s\sin s$, $\cos(2k\pi+s)=\cos s$, *for all* $s\in\mathbf{R}$.

Proof. Let us consider the notation of section 13·3 and Figure 13·3. If s_1 and s_2 be two real numbers differing from each other by an integral multiple of 2π. Then $P(s_1)=P(s_2)$. In particular, if $s \in \mathbf{R}$, then $P(2k\pi+s)=P(s)$.

Now $\quad \sin(2k\pi+s) = \text{ordinate of } P(2k\pi+s),$

$= \text{ordinate of } P(s),$

$= \sin s.$

Again, $\quad \cos(2k\pi+s) = \text{abscissa of } P(2k\pi+s),$

$= \text{abscissa of } P(s),$

$= \cos s.$

Thus $\boxed{\sin(2k\pi+s)=\sin s,\ \cos(2k\pi+s)=\cos s.}$

Remark. The above identities enable us to write, by using Table 13·1 and 13·3, the value of cos s and sin s for infinitely many values of s. For example, if $k \in \mathbf{Z}$, then

$\cos 2k\pi = \cos 0 = 1$, $\sin 2k\pi = \sin 0 = 0$,

$\cos(2k\pi+\pi/2)=\cos(\pi/2)=0$, $\sin(2k\pi+\pi/2)=\sin(\pi/2)=1$,

$\cos(2k\pi+\pi/6)=\cos(\pi/6)=\sqrt{3}/2$, $\sin(2k\pi+\pi/6)=\sin(\pi/6)$

$=\frac{1}{2}.$

The two identities proved above involved only one variable. We shall now prove a general identity involving two variables. As we shall see in the following, it is a source for many other identities.

Theorem 13·3. *For all* $s, t \in \mathbf{R}$,

$$\cos(s-t)=\cos s \cos t+\sin s \sin t.$$

Proof. Let s and t be any two real numbers and let $P(s)$, $P(t)$ be their corresponding terminal points on the unit circle. See Fig. 13.5. Since the co-ordinates of $P(s)$ are $(\cos s, \sin s)$, and those of $P(t)$ are $(\cos t, \sin t)$, the distance d between $P(s)$ and $P(t)$ is given by

$$\begin{aligned} d^2 &= (\cos s-\cot t)^2+(\sin s-\sin t)^2, \\ &= (\cos^2 s-2\cos s\cos t+\cos^2 t) \\ &\qquad +(\sin^2 s-2\sin s\sin t+\sin^2 t), \\ &= 2-2(\cos s\cos t+\sin s\sin t), \qquad \dots(i) \end{aligned}$$

since $\quad \cos^2 s+\sin^2 s=1$, $\cos^2 t+\sin^2 t=1$.

We shall now find the value of d^2 by another method. Since equal chords of a circle cut off equal arcs, and *vice-versa*, therefore, the length d depends only on the arc-length (let us call it u) between $P(s)$ and $P(t)$, and not on the actual positions of these points. Clearly $u=(s-t)\pm$ an integral multiple of 2π.

In Fig. 13·6, the chord AB cuts off an arc u from the unit circle. Therefore, by the distance formula

$$d^2=(\cos u-1)^2+(\sin u-0)^2,$$
$$=1-2\cos u+\cos^2 u+\sin^2 u,$$
$$=2-2\cos u. \qquad \text{...}(ii)$$

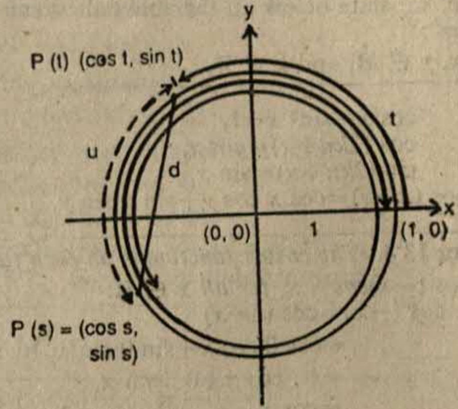


Fig. 13·5.

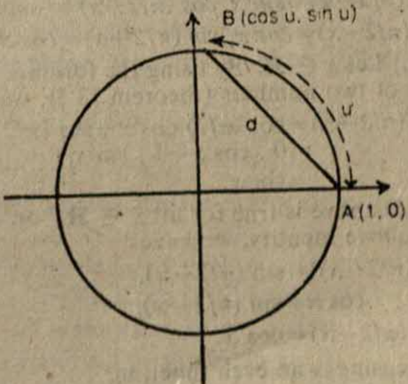


Fig. 13·6.

Since $u=(s-t)+2k\pi$, for some integer k, therefore, by theorem 13·2, $\cos u=\text{cost}\ (s-t)$. We may, therefore, write (*ii*) as

$$d^2=2-2\cos(s-t). \qquad \text{...}(iii)$$

On equating the two expressions for d^2 obtained in (*i*) and (*iii*), and simplifying, we have

$$\cos(s-t)=\cos s\cos t+\sin s\sin t,$$

which proves the theorem.

Remark. In all the three theorems proved above we have used s and t as the variables instead of x and y as is the usual

practice. This was only to avoid confusion since we had to use them for abscissa and ordinate of the point P. We shall not have many occasions now to use the unit circle, and therefore, henceforth we shall use the variables x and y instead of s and t. For the sake of convenience, we state below all the above theorems using x and y as the variables :

For all $x, y \in \mathbf{R}$, and $k \in \mathbf{Z}$,

$$\boxed{\begin{aligned} &\cos^2 x+\sin^2 x=1, \\ &\cos (2k\pi+x)=\cos x, \\ &\sin (2k\pi+x)=\sin x, \\ &\cos (x-y)=\cos x \cos y+\sin x \sin y. \end{aligned}}$$

Theorem 13·4. *The cosine function is an even function, that is,* $\cos(-x)\cos = x$, *for all* $x \in \mathbf{R}$.

Proof. $\cos(-x)=\cos(0-x)$

$=\cos 0 \cos x+\sin 0 \sin x$, by theorem 13·3,

$=1 \cdot \cos x+0 \cdot \sin x$,

$=\cos x$.

Theorem 13·5. *For all* $x \in \mathbf{R}$,

$\cos(\pi/2-x)=\sin x$, $\cos(\pi/2+x)=-\sin x$,

$\sin(\pi/2-x)=\cos x$, $\sin(\pi/2+x)=\cos x$.

Proof. (*a*) Let $x \in \mathbf{R}$. By using the formula for the cosine of the difference of two numbers (theorem 13·3), we have

$$\begin{aligned} \cos(\pi/2-x)&=\cos(\pi/2)\cos x+\sin(\pi/2)\sin x, \\ &=0 \cdot \cos x+1 \cdot \sin x, \\ &=\sin x. \end{aligned}$$

(*b*) Since (*a*) above is true for all $x \in \mathbf{R}$, on replacing x by $(\pi/2)-x$ in the above identity, we have

$\cos[\pi/2-(\pi/2-x)]=\sin(\pi/2-x)$,

or $\cos x=\sin(\pi/2-x)$,

i.e., $\sin(\pi/2-x)=\cos x$.

(*c*) Since cosine is an even function,

$$\begin{aligned} \cos(\pi/2+x)&=\cos[-(\pi/2+x)], \\ &=\cos(-\pi/2-x), \\ &=\cos(-\pi/2)\cos x+\sin(-\pi/2)\sin x, \\ &=0 \cdot \cos x+(-1)\sin x, \qquad \text{(Why ?)} \\ &=-\sin x. \end{aligned}$$

(*d*) $\sin(\pi/2+x)=\sin[\pi/2-(-x)]$,

$=\cos(-x)$, by (*b*) above,

$=\cos x$,

since cosine is an even function.

Theorem 13·6. *The sine function is an odd function, i.e.,*

$$\boxed{\sin(-x) = -\sin x, \text{ for all } x \in \mathbf{R}.}$$

Proof. We know that for all $y \in \mathbf{R}$,

$$\cos(\pi/2 - y) = \sin y.$$

Replacing y by $-x$ throughout, we have

$$\cos(\pi/2 + x) = \sin(-x). \qquad \text{...}(i)$$

Also, as already proved in Theorem 13.5,

$$\cos(\pi/2 + x) = -\sin x. \qquad \text{...}(ii)$$

From (*i*) and (*ii*) above, we have

$$\sin(-x) = -\sin x, \text{ for all } x \in \mathbf{R}.$$

Theorem 13·7. *For all real numbers x and y,*

$$\boxed{\cos(x+y) = \cos x \cos y - \sin x \sin y.}$$

Proof. $\cos(x+y) = \cos\{x-(-y)\}$,

$$= \cos x \cos(-y) + \sin x \sin(-y),$$

$$= \cos x \cos y - \sin x \sin y,$$

since $\cos(-y) = \cos y$ and $\sin(-y) = -\sin y$.

Remark. The above theorem is often called the *addition theorem for cosine.*

Theorem 13·8. (*Addition theorem for sine*). *For all real numbers x and y,*

$$\boxed{\sin(x+y) = \sin x \cos y + \cos x \sin y.}$$

Proof. $\sin(x-y) = \sin[x-(-y)]$,

$$= \sin x \cos(-x) + \cos x \sin(-y),$$

$$= \sin x \cos y - \cos x \sin y,$$

since $\cos(-y) = \cos y$ and $\sin(-y) = -\sin y$.

Theorem 13·9. *For all* $x \in \mathbf{R}$

$$\sin(\pi - x) = \sin x, \ \sin(\pi + x) = -\sin x,$$

$$\cos(\pi - x) = -\cos x, \ \cos(\pi + x) = -\cos x.$$

Proof. (*a*) $\sin(\pi - x) = \sin\pi \cos x - \cos\pi \sin x$,

$$= 0.\cos x - (-1)\sin x,$$

$$= \sin x.$$

(*b*) $\sin(\pi + x) = \sin\pi \cos x + \cos\pi \sin x$,

$$= 0 \,.\, \cos x + (-1)\sin x,$$

$$= -\sin x.$$

(c) $\cos(\pi-x)=\cos\pi\cos x+\sin\pi\sin x,$
$=(-1)\cos x+0\,.\,\sin x,$
$=-\cos x.$

(d) $\cos(\pi+x)=\cos\pi\cos x-\sin\pi\sin x,$
$=(-1)\cos x-0\,.\,\sin x,$
$=-\cos x.$

13·5·1. Trigonometric Functions of $2x$ and $3x$ in Terms of Trigonometric Functions of x

We shall now express trigonometric functions of $2x$ and $3x$ in terms of trigonometric functions of x.

Theorem 13·10. *(Trigonometric functions af 2x). For all* $x \in \mathbf{R}$,

(a) $\sin 2x=2\sin x\cos x.$

(b) $\cos 2x=\cos^2 x-\sin^2 x.$

(c) $\cos 2x=2\cos^2 x-1.$

(d) $\cos 2x=1-2\sin^2 x.$

Proof. (a) By the addition theorem for sine, we have

$$\begin{aligned}\sin 2x&=\sin(x+x),\\&=\sin x\cos x+\cos x\sin x,\\&=2\sin x\cos x.\end{aligned}$$

(b) By the addition theorem for cosine, we have

$$\begin{aligned}\cos 2x&=\cos(x+x),\\&=\cos x\cos x-\sin x\sin x,\\&=\cos^2 x-\sin^2 x. &\quad ...(1)\end{aligned}$$

(c) Since $\sin^2 x=1-\cos^2 x$, we have from (1),

$$\begin{aligned}\cos 2x&=\cos^2 x-(1-\cos^2 x),\\&=2\cos^2 x-1. &\quad ...(2)\end{aligned}$$

(d) Again, since $\cos^2 x=1-\sin^2 x$, we may re-write (1) as

$$\begin{aligned}\cos 2x&=(1-\sin^2 x)-\sin^2 x,\\&=1-2\sin^2 x. &\quad ...(3)\end{aligned}$$

Corollary. $1-\cos x=2\sin^2(x/2),$
$1+\cos x=2\cos^2(x/2).$

Proof. The identity

$$\cos 2x=1-2\sin^2 x$$

can be rewritten as

$$1-\cos 2x=2\sin^2 x.$$

Replacing x by $(x/2)$ throughout, we haue

$$1-\cos x=2\sin^2 (x/2).$$

Again, the identity

$$\cos 2x=2\cos^2 x-1$$

can be re-written as

$$1+\cos 2x=2\cos^2 x.$$

Replacing x by $x/2$ throughout, we have

$$1+\cos x=2\cos^2 (x/2).$$

Theorem 13·11. *(Trigonometric functions of 3x). For all* $x \in \mathbf{R}$,

$$\boxed{\begin{aligned} \sin 3x &= 3\sin x-4\sin^3 x, \\ \cos 3x &= 4\cos^3 x-3\cos x. \end{aligned}}$$

Proof. By the addition theorem for sine,

$$\begin{aligned} \sin 3x &= \sin (2x+x), \\ &= \sin 2x\cos x+\cos 2x\sin x, \\ &= (2\sin x\cos x)\cos x+(1-2\sin^2 x)\sin x, \\ &= 2\sin x\cos^2 x+\sin x-2\sin^3 x, \\ &= 2\sin x(1-\sin^2 x)+\sin x-2\sin^3 x, \\ &= 3\sin x-4\sin^3 x. \end{aligned}$$

Again, by the addition theorem for cosine,

$$\begin{aligned} \cos 3x &= \cos (2x+x), \\ &= \cos 2x\cos x-\sin 2x\sin x, \\ &= (2\cos^2 x-1)\cos x-(2\sin x\cos x)\sin x, \\ &= (2\cos^2 x-1)\cos x-2\sin^2 x\cos x, \\ &= (2\cos^2 x-1)\cos x-2(1-\cos^2 x)\cos x, \\ &= 4\cos^3 x-3\cos x. \end{aligned}$$

13·5·2. Transformation of Products into Sums and Vice-versa

For all $x, y \in \mathbf{R}$, we have

$$\cos (x+y)=\cos x\cos y-\sin x\sin y, \quad \ldots(i)$$
$$\cos (x-y)=\cos x\cos y+\sin x\sin y. \quad \ldots(ii)$$

Adding (resp. subtracting) corresponding sides of (i) and (ii) and dividing throughout by 2, we have

$$\cos x \cos y = \tfrac{1}{2}[\cos(x+y)+\cos(x-y)], \quad \ldots(1)$$

$$\sin x \sin y = -\tfrac{1}{2}[\cos(x+y)-\cos(x-y)]. \quad \ldots(2)$$

Again, from

$$\sin(x+y)=\sin x \cos y+\cos x \sin y, \quad \ldots(iii)$$
$$\sin(x-y)=\sin x \cos y-\cos x \sin y, \quad \ldots(iv)$$

by adding (resp. subtracting) corresponding sides of (*iii*) and (*iv*), and dividing throughout by 2, we have

$$\sin x \cos y = \tfrac{1}{2}[\sin(x+y)+\sin(x-y)], \quad \ldots(3)$$

$$\cos x \sin y = \tfrac{1}{2}[\sin(x+y)-\sin(x-y)]. \quad \ldots(4)$$

Let us write $x+y=u$, and $x-y=v$, so that

$$x=\tfrac{1}{2}(u+v),\ y=\tfrac{1}{2}(u-v).$$

Then formula (3) and (4) can be re-written as

$$\sin u+\sin v=2\sin\tfrac{1}{2}(u+v)\cos\tfrac{1}{2}(u-v), \quad \ldots(5)$$

$$\sin u-\sin v=2\cos\tfrac{1}{2}(u+v)\sin\tfrac{1}{2}(u-v). \quad \ldots(6)$$

Also, formula (1) and (2) can be re-written as

$$\cos u+\cos v=2\cos\tfrac{1}{2}(u+v)\cos\tfrac{1}{2}(u-v), \quad \ldots(7)$$

$$\cos u-\cos v=-2\sin\tfrac{1}{2}(u+v)\sin\tfrac{1}{2}(u-v). \quad \ldots(8)$$

Example 1. *Prove the following identities :*

(*i*) $\sin(A+B)\sin(A-B)=\sin^2 A-\sin^2 B$,

(*ii*) $\cos(A+B)\cos(A-B)=\cos^2 A-\sin^2 B$.

Solution. (*i*) $\sin(A+B)\sin(A-B)$

$$=\tfrac{1}{2}[\cos\{(A+B)-(A-B)\}-\cos\{(A+B)+(A-B)\}],$$
$$=\tfrac{1}{2}[\cos 2B-\cos 2A],$$
$$=\tfrac{1}{2}[(1-2\sin^2 B)-(1-2\sin^2 A)],$$
$$=\sin^2 A-\sin^2 B.$$

(*ii*) $\cos(A+B)\cos(A-B)$

$$=\tfrac{1}{2}[\cos\{(A+B)+(A-B)\}]+\cos\{(A+B)-(A-B)\}],$$
$$=\tfrac{1}{2}[\cos 2A+\cos 2B],$$
$$=\tfrac{1}{2}[(2\cos^2 A-1)+(1-2\sin^2 B)],$$
$$=\cos^2 A-\sin^2 B.$$

Example 2. *Calculate* $\sin\frac{\pi}{12}$ *and* $\sin\frac{5\pi}{12}$.

Solution. $$\sin\frac{\pi}{12}=\sin\left(\frac{\pi}{4}-\frac{\pi}{6}\right),$$
$$=\sin\frac{\pi}{4}\cos\frac{\pi}{6}-\cos\frac{\pi}{4}\sin\frac{\pi}{6},$$
$$=\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{2}}\cdot\frac{1}{2},$$
$$=\frac{(\sqrt{3}-1)}{2\sqrt{2}}$$
$$\sin\frac{5\pi}{12}=\sin\left(\frac{\pi}{4}+\frac{\pi}{6}\right),$$
$$=\sin\frac{\pi}{4}\cos\frac{\pi}{6}+\cos\frac{\pi}{4}\sin\frac{\pi}{6},$$
$$=\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}\cdot\frac{1}{2},$$
$$=\frac{\sqrt{3}+1}{2\sqrt{2}}$$

Example 3. *Calculate* $\sin\frac{\pi}{10}$ *and* $\cos\frac{\pi}{5}$.

Solution. $$\cos\left(\frac{3\pi}{10}\right)=\sin\left(\frac{\pi}{2}-\frac{3\pi}{10}\right)=\sin\left(\frac{2\pi}{10}\right),$$

or $$4\cos^3\frac{\pi}{10}-3\cos\frac{\pi}{10}=2\sin\frac{\pi}{10}\cos\frac{\pi}{10},$$

or $$\left(\cos\frac{\pi}{10}\right)\left(4\cos^2\frac{\pi}{10}-3-2\sin\frac{\pi}{10}\right)=0,$$

or $$4\cos^2\frac{\pi}{10}-3-2\sin\frac{\pi}{10}=0,\ \text{since}\ \cos\frac{\pi}{10}\neq 0,$$

or $$4\left(1-\sin^2\frac{\pi}{10}\right)-3-2\sin\frac{\pi}{10}=0,$$

or $$4\sin^2\frac{\pi}{10}+2\sin\frac{\pi}{10}-1=0,$$

or $$\sin\frac{\pi}{10}=\frac{-2\pm\sqrt{2^2+4.4.1}}{2.4},$$
$$=\frac{-1\pm\sqrt{5}}{4}.$$

Since $0<\frac{\pi}{10}<\frac{\pi}{2}$, therefore, $\sin\frac{\pi}{10}>0$.

Consequently $\sin\frac{\pi}{10}\neq\frac{-1-\sqrt{5}}{4}$.

Hence $\sin \frac{\pi}{10}=\frac{-1+\sqrt{5}}{4}$.

Also, $\cos \frac{\pi}{5}=1-2 \sin^2 \frac{\pi}{10}$,

$$=1-2\left(\frac{\sqrt{5}-1}{4}\right)^2,$$

$$=1-\frac{6-2\sqrt{5}}{8},$$

$$=\frac{\sqrt{5}+1}{4}.$$

Thus $\cos \frac{\pi}{5}=\frac{\sqrt{5}+1}{4}$.

Example 4. *Express* $\cos x+\cos 3x+\cos 5x+\cos 7x$ *as a product.*

Solution. By grouping the first two and the last two terms together and expressing as a product, we have

$\cos x+\cos 3x+\cos 5x+\cos 7x$

$$=2 \cos \frac{x+3x}{2} \cos \frac{x-3x}{2}+2 \cos \frac{5x+7x}{2} \cos \frac{5x-7x}{2},$$

$$=2 \cos 2x \cos (-x)+2 \cos 6x \cos (-x),$$

$$=2 \cos x (\cos 2x+\cos 6x),$$

$$=2 \cos x (2 \cos 4x \cos 2x),$$

$$=4 \cos x \cos 2x \cos 2x.$$

EXERCISE 13 (*b*)

1. Prove the following identities :

(*i*) $\cos\left(\frac{\pi}{3}+x\right)=\frac{1}{2}(\cos x-\sqrt{3} \sin x)$.

(*ii*) $\sin\left(\frac{\pi}{6}+x\right)=\frac{1}{2}(\cos x+\sqrt{3} \sin x)$,

(*iii*) $\sin\left(\frac{\pi}{4}+x\right)=\frac{1}{\sqrt{2}}(\cos x+\sin x)$,

(*iv*) $\cos x-\sin x=\frac{1}{\sqrt{2}} \cos\left(\frac{\pi}{4}+x\right)$,

(*v*) $\frac{\cos (x+y)}{\cos x \cos y}=1-\tan x \tan y$,

(*vi*) $\frac{\sin (x+y)}{\cos x \cos y}=\tan x+\tan y$.

2. Express each of the following as a product :

(*i*) $\cos 5x+\cos 3x$ (*ii*) $\cos 2x-\cos 4x$

(*iii*) $\sin 3x+\sin 7x$ (*iv*) $\sin 4y-\sin 2x$

(*v*) $\sin (2x+y)-\sin (2x-y)$ (*vi*) $\cos (3x-5y)-\cos (3x+5y)$.

3. Express each of the following product as an algebraic sum of sines and cosines :

(*i*) $2 \sin 2x \cos 3x$ (*ii*) $2 \sin 3x \cos 2x$

(*iii*) $2 \cos 4x \cos 6x$ (*iv*) $2 \sin 3x \sin 5x$.

4. Prove the following identities :

(*i*) $\dfrac{\cos 3x-\cos 5x}{\sin 5x-\sin 3x}=\tan 4x$.

(*ii*) $\dfrac{\cos 2x-\cos 4x}{\cos 2x+\cos 4x}=\tan x \tan 3x$.

(*iii*) $\dfrac{\cos x-\cos 2x+\cos 3x}{\sin x-\sin 2x+\sin 3x}=\dfrac{1}{\tan 2x}$

(*iv*) $\dfrac{\sin 2x}{1+\cos 2x}=\tan x$.

5. Prove that $\cos 5x=16 \cos^5 x-20 \cos^3 x+5 \cos x$.

6. Show that $\sin^2\left(\dfrac{2\pi}{5}\right)-\sin^2\left(\dfrac{\pi}{5}\right)=\dfrac{\sqrt{5}-1}{8}$.

7. Show that $\cos^2\dfrac{\pi}{8}+\cos^2\dfrac{3\pi}{8}+\cos^2\dfrac{5\pi}{8}+\cos^2\dfrac{7\pi}{8}=2$.

13·6. THE FUNCTIONS COTANGENT, SECANT AND COSECANT

We shall now define three more trigonometric functions, namely, the cotangent, the secant, and the cosecant, abbreviated to cot, sec, and csc respectively.

Definition 13·5. $\cot x=\dfrac{\cos x}{\sin x}$,

$$\sec x=\frac{1}{\cos x},$$

$$\csc x=\frac{1}{\sin x},$$

where cot x and csc x are defined whenever $\sin x\neq 0$, *i.e., whenever x is not an integral multiple of* π *and sec x is defined whenever* $\cos x\neq 0$, *i.e., whenever x is not an odd multiple of* $\pi/2$.

All the six trigonometric functions sin, cos, tan, cot, sec, csc are expressible neatly in terms of sin and cos and therefore, theoretically speaking, we need only the two functions sin and cos. However, the introduction of the other functions lends flexibility to the treatment.

Corresponding to the fundamental identity $\cos^2 x+\sin^2 x=1$ connecting the sine and the cosine functions, we have two similar identities connecting the remaining functions.

Theorem 13·12.

$$\boxed{\begin{aligned} &\sec^2 x=1+\tan^2 x,\\ &\csc^2 x=1+\cot^2 x.\end{aligned}}$$

Proof. Dividing both sides of the identity $\cos^2 x+\sin^2 x=1$ by $\cos^2 x$, we have

$$1+\frac{\sin^2 x}{\cos^2 x}=\frac{1}{\cos^2 x},$$

i.e.,

$$1+\tan^2 x=\sec^2 x. \qquad \text{...(A)}$$

(A) holds whenever $\cos x\neq 0$, *i.e.*, whenever x is not an odd multiple of $\pi/2$.

Again, dividing both sides of the identity $\cos^2 x+\sin^2 x=1$ by $\sin^2 x$, we have

$$\frac{\cos^2 x}{\sin^2 x}+1=\frac{1}{\sin^2 x},$$

i.e.,

$$\cot^2 x+1=\csc^2 x. \qquad \text{...(B)}$$

(*b*) holds whenever $\sin x\neq 0$, *i.e.*, whenever x is not an integral multiple of π.

13·7. SOME BASIC PROPERTIES OF THE TANGENT FUNCTION

The basic properties of the tangent function are contained in the following two theorems :

Theorem 13·13. *If x is not an odd multiple of $\pi/2$, then*

$$\boxed{\begin{aligned} \tan(-x)&=-\tan x,\\ \tan\left(\frac{\pi}{2}-x\right)&=\cot x,\\ \tan\left(\frac{\pi}{2}+x\right)&=-\cot x,\\ \tan(\pi-x)&=-\tan x,\\ \tan(\pi+x)&=\tan x.\end{aligned}}$$

Proof.

(*a*) $\tan(-x)=\dfrac{\sin(-x)}{\cos(-x)}=\dfrac{-\sin x}{\cos x}=-\tan x.$

(b) $\tan\left(\frac{\pi}{2}-x\right)=\dfrac{\sin\left(\frac{\pi}{2}-x\right)}{\cos\left(\frac{\pi}{2}-x\right)}=\dfrac{\cos x}{\sin x}=\cot x.$

(c) $\tan\left(\frac{\pi}{2}+x\right)=\dfrac{\sin\left(\frac{\pi}{2}+x\right)}{\cos\left(\frac{\pi}{2}+x\right)}=\dfrac{\cos x}{-\sin x}=-\cot x.$

(d) $\tan(\pi-x)=\dfrac{\sin(\pi-x)}{\cos(\pi-x)}=\dfrac{\sin x}{-\cos x}=-\tan x.$

(e) $\tan(\pi+x)=\dfrac{\sin(\pi+x)}{\cos(\pi+x)}=\dfrac{-\sin x}{-\cos x}=\tan x.$

Corollary. If k be any integer whatsoever, $\tan(k\pi+x)=\tan x$.

Proof. If $k>0$, the result follows by the principle of mathematical induction.

If $k=0$, the result is trivially true.

If $k<0$, let $k=-l$ where $l>0$.

$$\begin{aligned}\tan(k\pi+x)=\tan(-l\pi+x)&=\tan[-(l\pi-x)],\\ &=-\tan(l\pi-x),\\ &=-\tan(-x),\\ &=\tan x.\end{aligned}$$

Theorem 13·14. (*Addition theorem for tangent*). *If none of x, y and $x+y$ is an odd multiple of $\pi/2$, then*

$$\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}.$$

Proof. As none of the numbers x, y and $x+y$ is an odd multiple of $\pi/2$, therefore, $\cos x\cos y\neq 0$, $\cos(x+y)\neq 0$. Now

$$\begin{aligned}\tan(x+y)&=\frac{\sin(x+y)}{\cos(x+y)},\\ &=\frac{\sin x\cos y+\cos x\sin y}{\cos x\cos y-\sin x\sin y},\\ &=\frac{\dfrac{\sin x\cos y}{\cos x\cos y}+\dfrac{\cos x\sin y}{\cos x\cos y}}{\dfrac{\cos x\cos y}{\cos x\cos y}-\dfrac{\sin x\sin y}{\cos x\cos y}},\end{aligned}$$

(dividing the numerator and denominator by $\cos x\cos y$),

$$=\frac{\tan x+\tan y}{1-\tan x\tan y}.$$

Corollary. $\tan(x-y)=\dfrac{\tan x-\tan y}{1+\tan x\tan y}$,

where none of the numbers x, y, $x-y$ is an odd multiple of $\pi/2$.

Proof. $\tan(x-y)=\tan[x+(-y)]$,

$$=\frac{\tan x+\tan(-y)}{1-\tan x\tan(-y)},$$

$$=\frac{\tan x-\tan y}{1+\tan x\tan y}.$$

Example 5. *Express* $\tan\left(\dfrac{\pi}{4}+x\right)$ *and* $\tan\left(\dfrac{\pi}{4}-x\right)$ *in terms of* $\tan x$, *stating the values of* x *for which the expressions are valid.*

Solution. $\tan\left(\dfrac{\pi}{4}+x\right)=\dfrac{\tan\dfrac{\pi}{4}+\tan x}{1-\tan\dfrac{\pi}{4}\tan x}$,

$$=\frac{1+\tan x}{1-\tan x}.$$

The above expression is valid for all those values of x for which $\tan\left(\dfrac{\pi}{4}+x\right)$ and $\tan x$ are defined. Now $\tan x$ is defined for all values of x except those which are odd multiples of $\dfrac{\pi}{2}$. Also, $\tan\left(\dfrac{\pi}{4}+x\right)$ is defined for all those values of x for which $\dfrac{\pi}{4}+x$ is not an odd multiple of $\pi/2$. Thus the above expression is valid whenever neither x nor $\pi/4+x$ is an odd multiple of $\pi/2$.

$$\tan\left(\frac{\pi}{4}-x\right)=\frac{\tan\dfrac{\pi}{4}-\tan x}{1+\tan\dfrac{\pi}{4}\tan x},$$

$$=\frac{1-\tan x}{1+\tan x}.$$

The above expression is valid whenever neither x nor $\pi/4-x$ is an odd multiple of $\pi/2$.

Example 6. *Express* $\sin(x+y+z)$, $\cos(x+y+z)$, *and* $\tan(x+y+z)$ *in terms of trigonometric functions of* x, y *and* z.

Solution. $\sin(x+y+z)=\sin[(x+y)+z]$,

$$=\sin(x+y)\cos z+\cos(x+y)\sin z,$$

$$=(\sin x \cos y+\cos x \sin y) \cos z$$
$$+(\cos x \cos y-\sin x \sin y) \sin z,$$
$$=\sin x \cos y \cos z+\sin y \cos x \cos z$$
$$+\cos x \cos y \sin z-\sin x \sin y \sin z. \quad ...(a)$$

$$\cos (x+y+z)=\cos [(x+y)+z],$$
$$=\cos (x+y) \cos z-\sin (x+y) \sin z,$$
$$=[\cos x \cos y-\sin x \sin y) \cos z,$$
$$-(\sin x \cos y+\cos x \sin y) \sin z,$$
$$=\cos x \cos y \cos z-\sin x \sin y \cos z$$
$$-\sin y \sin z \cos x-\sin z \sin x \cos y, \quad ...(b)$$

interchanging the last two terms.

(*a*) and (*b*) express $\sin (x+y+z)$ and $\cos (x+y+z)$ in terms of sines and cosines of x, y and z. We can re-write (*a*) and (*b*) as

$$\sin (x+y+z)=\cos x \cos y \cos z\,[\tan x+\tan y+\tan z$$
$$-\tan x \tan y \tan z], \quad ..(c)$$

and
$$\cos (x+y+z)=\cos x \cos y \cos z\,[1-\tan x \tan y$$
$$-\tan y \tan z-\tan z \tan x], \quad ...(d)$$

respectively.

Dividing corresponding sides of (*c*) by those of (*d*), we have,

$$\tan (x+y+z)=\frac{\tan x+\tan y+\tan z-\tan x \tan y \tan z}{1-\tan x \tan y-\tan y \tan z-\tan z \tan x}. \quad ...(e)$$

Observe that (*e*) is valid only when $\cos x$, $\cos y$, $\cos z$, $\cos (x+y+z)$ are all different from zero, *i.e.*, when none of x, y, z and $x+y+z$ is an odd multiple of $\pi/2$.

13·7·1. Tan 2x and tan 3x in Terms of tan x

Theorem 13·15. *Whenever x is not an odd multiple of $\pi/6$, $\pi/4$ or $\pi/2$, then*

$$\tan 2x=\frac{2 \tan x}{1-\tan^2 x},$$

$$\tan 3x=\frac{3 \tan x-\tan^3 x}{1-3 \tan^2 x}.$$

Proof. By the addition theorem for tangent, we have

$$\tan 2x=\tan (x+x)=\frac{\tan x+\tan x}{1-\tan x.\tan x},$$

$$=\frac{2 \tan x}{1-\tan^2 x}. \quad ...(a)$$

Again $\tan 3x=\tan (2x+x)$,

$$=\frac{\tan 2x+\tan x}{1-\tan 2x \tan x},$$

$$=\frac{\dfrac{2\tan x}{1-\tan^2 x}+\tan x}{1-\dfrac{2\tan x}{1-\tan^2 x}\cdot\tan x},$$

$$=\frac{2\tan x+\tan x\,(1-\tan^2 x)}{1-\tan^2 x-2\tan^2 x},$$

$$=\frac{3\tan x-\tan^3 x}{1-3\tan^2 x}. \qquad ...(b)$$

(*a*) is valid when cos $2x$ and cos x are both different from zero, *i.e.*, when neither of x and $2x$ is an odd multiple of $\pi/2$, *i.e.*, when x is not an odd multiple of $\pi/4$ or $\pi/2$.

(*b*) is valid when cos $3x$, cos $2x$ and cos x are all different from zero, *i.e.*, when none of x, $2x$ and $3x$ is an odd multiple of $\pi/2$, *i.e.*, when x is not an odd multiple of $\pi/2$, $\pi/4$ or $\pi/6$.

EXERCISE 13 (*c*)

1. Reduce each of the following expressions to a single term involving only one function :

(*i*) $\dfrac{2\tan(\pi/5)}{1-\tan^2(\pi/5)}$ (*ii*) $\dfrac{1-\tan^2(\pi/9)}{1+\tan^2(\pi/9)}$

(*iii*) $\dfrac{1-3\tan^2(\pi/7)}{3\tan(\pi/7)-\tan^3(\pi/7)}$ (*iv*) $\dfrac{2\tan(\pi/8)}{1+\tan^2(\pi/8)}$

2. Prove that $\tan x+\tan\left(\dfrac{\pi}{3}-x\right)+\tan\left(\dfrac{\pi}{3}+x\right)=3\tan 3x$.

3. Prove that $\tan 3x\tan 2x\tan x=\tan 3x-\tan 2x-\tan x$.

4. Prove that
$\tan(y-z)+\tan(z-x)+\tan(x-y)=\tan(y-z)\tan(z-x)\tan(x-y)$.

5. Prove each of the following identities :

(*i*) $\cot x-\tan x=2\cot 2x$,

(*ii*) $\cot x+\tan x=2\csc 2x$,

(*iii*) $\csc 2x-\cot 2x=\tan x$.

(*iv*) $\dfrac{\tan 2x}{1+\sec 2x}=\tan x$.

(*v*) $\tan\left(\dfrac{\pi}{4}+x\right)\tan\left(\dfrac{\pi}{4}-x\right)=1$.

13·8. GENERATED ANGLES

Recall that in plane geometry an angle is defined as the union of two rays (half-lines) radiating from a point. We now extend this definition by adding to it the requirement that an angle defined in this manner has a measure which equals the amount of rotation required to move one of these rays so as to bring it into position with the other.

Let us consider two rays p and q lying in the plane of the paper and radiating from O. If we regard p as the initial ray and q as the terminal ray, there are two possible directions of rotation of p about an axis through O perpendicular to the plane of the paper, anticlockwise and clockwise. The angle is said to be positive if the rotation is counter-clockwise and negative if the rotation is clockwise. A curved arrow is often used to indicate the direction of rotation. An angle obtained by the rotation of a ray in the manner just now described is called a **generated angle**.

To understand the concept of measure of a generated angle, let us consider a ray p which radiates from the origin O of a rectangular co-ordinate system and coincides with the positive x-axis (see Fig. 13·7). As this ray rotates, any point P on it will describe a

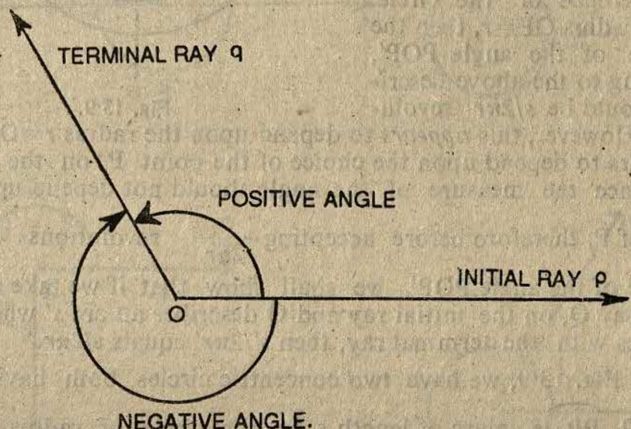


Fig. 13·7.

part or whole of the circumference of a circle of radius OP. It may, in fact, describe the circumference more than one as well. After the rotation, OP will be in some position OP′. We can use the circular arc PP′ (denoted by s, say) to measure the angle POP′. An angle such as $\angle$POP′ whose initial ray OP coincides with the positive-half of the x-axis of a rectangular co-ordinate system is said to be in standard position. We also say that it lies in the quadrant in which OP′ happens to be located.

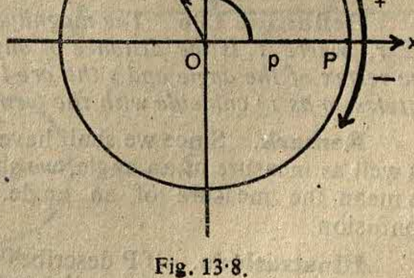


Fig. 13·8.

A natural way of measuring the magnitude of the angle POP′ is the number of revolutions described by OP as it rotates from its initial position to come to its final position. (The sign of the angle will, of course, be determined by the direction of rotation.)

Now the number of revolutions is simply the ratio of the arc described to the circumference of the circle. If the radius OP$=r$, then the measure of the angle POP′, according to the above description would be $s/2\pi r$ revolutions. However, this *appears* to depend upon the radius $r=$OP, *i.e.*, it appears to depend upon the choice of the point P on the initial ray. Since the measure of the angle should not depend upon the choice of P, therefore before accepting $\frac{s}{2\pi r}$ revolutions as the measure of the angle POP′, we shall show that if we take another point, say Q, on the initial ray and Q describes an arc s' when OQ coincides with the terminal ray, then $s'/2\pi r'$ equals $s/2\pi r$.

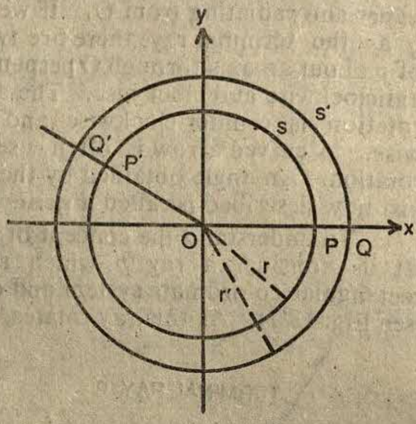


Fig. 13·9.

In Fig. 13·9, we have two concentric circles both having the centre O $\overset{\frown}{PP'}$ is an arc of length s on the circle of radius r, and $\overset{\frown}{QQ'}$ is an arc of length s' on the circle of radius r'. Since each arc length must bear the same ratio to the circumference of the circle along which it lies, therefore,

$$\frac{s}{2\pi r}=\frac{s'}{2\pi r'}$$

Thus $s/2\pi r$ is independent of the choice of the point, and we are now ready to have the following definition :

Definition 13·6. *The magnitude of an angle in revolutions is $s/2\pi r$, where r is the distance of any point P on the initial ray from the vertex of the angle and s the arc-length described by P when OP rotates so as to coincide with the terminal ray.*

Remark. Since we shall have to often talk of the magnitude as well as measure of an angle, we shall simply ush the word 'angle' to mean the measure of an angle. This should not create any confusion.

Illustrations. If P describes one-third of the circumference of a circle, its magnitude is 1/3 revolution. If the rotation is in the

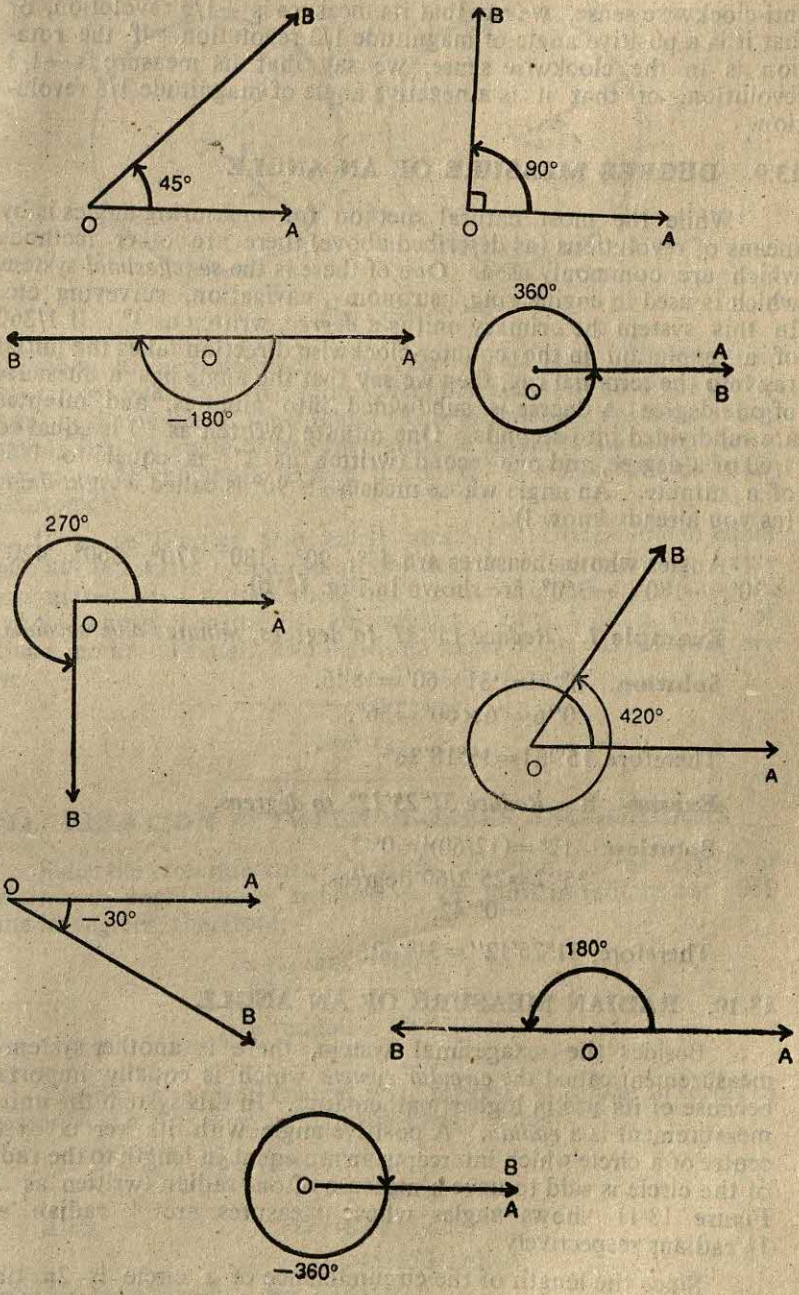

B
45°
O
A
B
90°
O
A
B
O
A
−180°
360°
A
O
B
270°
O
A
B
B
420°
O
A
O
A
−30°
B
180°
B
O
A
O
B
A
−360°

Fig. 13·10.

anti-clockwise sense, we say that its measure is $+1/3$ revolution, or that it is a positive angle of magnitude 1/3 revolution. If the rotation is in the clockwise sense, we say that its measure is $-1/3$ revolution, or that it is a negative angle of magnitude 1/3 revolution.

13·9. DEGREE MEASURE OF AN ANGLE

While the most natural method for measuring angles is by means of revolutions (as described above) there are other methods which are commonly used. One of these is the *sexagesimal* system which is used in engineering, astronomy, navigation, surveying etc. In this system the primary unit is a *degree*, written as $1°$. If 1/360 of a revolution in the counter-clockwise direction takes the initial ray into the terminal ray, then we say that the angle has a measure of one degree. A degree is subdivided into minutes, and minutes are subdivided into seconds. One minute (written as $1'$) is equal to 1/60 of a degree, and one second (written as $1''$) is equal to 1/60 of a minute. An angle whose measure is $90°$ is called a *right angle* (as you already know !).

Angles whose measures are $45°$, $90°$, $180°$, $270°$, $360°$, $420°$, $-30°$, $-180°$, $-360°$, are shown in Fig. 13·10.

Example 7. *Reduce 15°·31′ to degrees, minutes and seconds.*

Solution. $0°{\cdot}31 = {\cdot}31 \times 60' = 18'{\cdot}6$.

$0'{\cdot}6 = {\cdot}6 \times 60'' = 36''$.

Therefore $15°{\cdot}31 = 15°18'36''$.

Example 8. *Reduce 31°25′12″ to degrees.*

Solution. $12'' = (12/60)' = 0'{\cdot}2$.

$25'{\cdot}2 = 25{\cdot}2/60$ degrees,

$= 0°{\cdot}42$.

Therefore, $31°25'12'' = 31°{\cdot}42$.

13·10. RADIAN MEASURE OF AN ANGLE

Besides the sexagesimal system, there is another system of measurement called the *circular system* which is equally important because of its use in higher mathematics. In this system the unit of measurement is a *radian*. A positive angle with its vertax at the centre of a circle which intercepts an arc equal in length to the radius of the circle is said to have a measure of one radian (written as 1^c). Figure 13·11 shows angles whose measures are 1 radian and $1\frac{1}{2}$ radians respectively.

Since the length of the circumference of a circle is 2π times the radius, therefore, the length of one radian can be measured off 2π

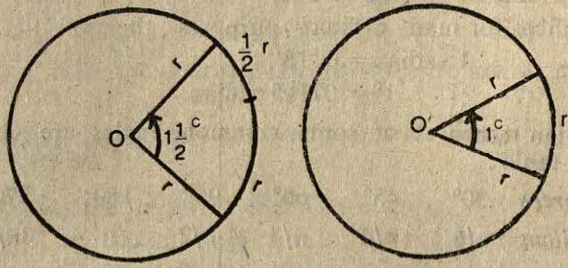


Fig. 13·11.

times along the circumference. In other words, the circumference of a circle (whatever its radius may be) subtends an angle of 2π radians at the centre of the circle. (This is precisely the same thing as saying that an angle of one revolution is equal to an angle whose measure is 2π radians.) Hence a radian is a fixed unit of angle measurement.

It is well-known that equal arcs of a circle subtend equal angles at the centre. Since an arc of length r subtends an angle whose measure is 1 radian, therefore, an arc of length s will subtend an angle whose measure is s/r radians. Thus, if in a circle of radius r, an arc of length s subtends an angle θ^c at the centre, we have

$$\boxed{\theta = s/r.}$$

13·11. RELATION BETWEEN DEGREES AND RADIANS

Since the circumference of a circle subtends at the centre of the circle an angle whose measure is 2π units in radians and 360 units in degrees, therefore,

$$2\pi \text{ radians} = 360°,$$

or $$\boxed{\pi \text{ radians} = 180°.}$$

The above relation enables us to express a radian in terms of degrees, and a degree in terms of radians. Thus

$$1 \text{ radian} = 180°/\pi,$$
$$= 57°17'44''\cdot81.$$

Also, $$1° = \pi/180 \text{ radian},$$
$$= \cdot017453293 \text{ radian},$$

where we have taken $\pi = 3\cdot1415927$.

If we take $\pi = 3{\cdot}142$,
(which suffices for many ordinary purposes), then

$$1 \text{ radian} = 57°18',$$

and

$$1° = {\cdot}07145 \text{ radian}.$$

Radian measures of some common angles are given in the following table :

Degrees	30°	45°	60°	90°	180°	270°	360°
Radians	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π	$3\pi/2$	2π

Example 9. *Express 7°30′ in radians.*

Solution. $7°30' = 7°{\cdot}5 = 7{\cdot}5 \times \frac{\pi}{180}$ radians $= \frac{\pi}{24}$ radian,

$$= {\cdot}1317 \text{ radian}.$$

Example 10. *Express 2·5 radians in sexagesimal measure correct to the nearest minute.*

Solution. $2{\cdot}5$ radians $= 2{\cdot}5\ (57°17'45'')$,

$$= 143°14', \text{ to the nearest minute.}$$

Example 11. *Find the radius of a circle in which a central angle of 50° intercepts an arc of 150 cm.*

Solution. Here $s = 150$ cm, $\theta = 50\ \pi/180$.
Hence by $r = s/\theta$, $r = 180 \times 3/\pi$ cm,

$$= 171{\cdot}8 \text{ cm (Taking } \pi = 22/7).$$

Example 12. *The minute hand of a watch is 1·5 cm long. How far does its tip move during a class period of 50 minutes ?*

Solution. In 50 minutes, the minute hand of a watch turns through 5/6 of a revolution, or $(5/3)\pi$ radians. Hence the required distance travelled is given by

$$s = 1{\cdot}5 \times 5\pi/3 \text{ cm},$$
$$= 5\pi/2 \text{ cm},$$
$$= 5 \times 3{\cdot}142/2 \text{ cm}, \qquad (\pi = 3{\cdot}142)$$
$$= 7{\cdot}855 \text{ cm}.$$

Example 13. *Find the radius of a wheel whose circumference is 110 cm ($\pi = 22/7$).*

Solution. Here $2\pi r = 110$ cm.

or $\qquad (44/7)\ r = 110$ cm,

or $\qquad r = 7 \times 110/44 \text{ cm} = 17{\cdot}5$ cm.

EXERCISE 13 (*d*)

1. Reduce to degrees, minutes and seconds :
   (*i*) $12°{\cdot}26$, (*ii*) $37°{\cdot}62$, (*iii*) $42°{\cdot}15$.

2. Write in degrees correct to two decimal places :
   (*i*) 19°17′21″. (*ii*) 194′21″, (*iii*) 172′41″.

3. Determine the quadrants in which the following angles lie :
   $-36°$, $154°$ $-225°$, $-315°$, $675°$.

4. What is the acute angle between the hands of a watch at 3·30 P.M. ?

5. Express the following angles in degrees :
   $\pi/3$ radians, $\pi/6$ radians, $\pi/4$ radians, $\pi/20$ radians,
   2^c, 3^c, 5^c, 0^c, 30^c, 10^c.

6. Express the following angles in radians :
   240°, 60°, 45°, 30°, 18°, 104°36′.

7. In a circle of diameter 20 cm, the length of a chord is 10 cm. Find the length of the arc of the chord.

8. A wheel makes 180 revolutions in one minute. Through how many radians does it turn in one second ?

9. A point on the circumference of a rotating wheel of diameter 100 cm is moving at the rate of 50 cm a second. Through how many radians does it turn in one second ?

10. Find the number of degrees in the central angle of a circle of diameter 300 cm by an arc of 22 cm ($\pi=22/7$).

11. A wheel rotates making 20 revolutions per second. If the radius of the wheel is 50 cm, what linear distance does a point of its rim traverse in three minutes ?

12. If in two circles, an arc of the same length subtends angles 60° and 75° at the centre, find the ratio of their radii.

13. A circular race track is 2 km. in circumference. If a man is running on the track at the rate of 8 metres per second, what is his angular velocity in degrees and in radians ?

14. A belt moving 15 metres per second passes over a pulley one metre in diameter. What is the angular velocity of the pulley in radians per second ?

15. The equatorial radius of the earth is 6371 km. Find the length of the circumference of the equator ($\pi=355/113$).

13.12. TRIGONOMETRIC FUNCTIONS OF ANGLES

Since the radian measure of an angle is the same real number as s (namely the arc length of the unit circle which subtends that angle at the centre) for which the trigonometric functions were originally defined, therefore, we can define any trigonometric function of s, where s is the radian measure of an angle, as the corresponding trigonometric function of s.

Definition 13·7. *If $s \in \mathbf{R}$, then any trigonometric function of the angle whose radian measure is s, is equal to the corresponding trigonometric function of s.*

Illustrations :

(*a*) Let A be an angle whose measure is 30° or $\pi/6$ radians. Then

$$\sin A = \sin 30^\circ,$$
$$= \sin (\pi/6 \text{ radians}),$$
$$= \sin (\pi/6) = 1/2.$$

(*b*) Let B be an angle whose measure is 90° or $\pi/2$ radians. Then

$$\tan B = \tan 90^\circ,$$
$$= \tan (\pi/2 \text{ radians}),$$
$$= \tan (\pi/2), \text{ which is not defined because } \cos \pi/2 = 0.$$

When we write $\sin (\pi/6)$, it could be interpreted either as the sine of the real number $\pi/6$ or as the sine of the angle whose radian measure is $\pi/6$ radians. What interpretation shall we take will always be clear from the context. (We can, if we so desire, write $\sin (\pi/6)$ for the sine of the real number $\pi/6$ and $\sin (\pi/6)^\circ$ for the sine of the angle whose radian measure is $\pi/6$ radians. But that would only complicate matters for nothing, and therefore, we shall refrain from doing so.

As a consequence of definition 13·7 and of the definitions of the six trigonometric functions of real numbers, we know what we mean by $\sin \theta$, $\cos \theta$, $\tan \theta$, $\cot \theta$, $\sec \theta$ $\csc \theta$, for any angle whose radian measure is θ.

13·13. VALUES OF TRIGONOMETRIC FUNCTIONS OF 0°, 30°, 45°, 60°, 90°.

The following table gives the values of the six trigonometric functions of angles whose degree measures are 0°, 30°, 45°, 60°, 90°. The fact of a particular trigonometric function not being defined for an angle, has been indicated by a '×' in place of the value. Since the radian measures corresponding to 0°, 30°, 45°, 60° and 90° are 0, $\pi/6$, $\pi/4$, $\pi/3$, and $\pi/2$, and we have already obtained values of the sine and cosine for all these real numbers, the entries in the following Table 13·4 can be easily verified.

Angle	*sin*	*cos*	*tan*	*cot*	*sec*	*csc*
0°	0	1	0	×	1	×
30°	1/2	$\sqrt{3}/2$	$1/\sqrt{3}$	$\sqrt{3}$	$2/\sqrt{3}$	2
45°	$1/\sqrt{2}$	$1/\sqrt{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\sqrt{3}/2$	1/2	$\sqrt{3}$	$1/\sqrt{3}$	2	$2/\sqrt{3}$
90°	1	0	×	0	×	1

Table 13·4

13·14. RESTATEMENT OF TRIGONOMETRIC IDENTITIES FOR ANGLES

An important consequence of the equality of the value of any trigonometric function of a real number x and of the same trigonometric function of the angle whose radian measure is x, is that all the trigonometric identities stated and proved in theorems 13.1—13.15 are valid identities for trigonometric functions of angles. We shall take them as having been proved for angles and shall use them as such. A useful application of some of these identities is that a trigonometric function of any angle can be expressed in terms of a trigonometric function of an angle lying between 0° and 45°. The following example will illustrate the method.

Example 14. (*a*) *Express* $\sin 415°$ *as a trigonometric function of an angle lying between* 0° *and* 45°. (*b*) *Express* $\tan(-830°)$ *as a trigonometric function of a positive acute angle not exceeding* 45°.

Solution. (*a*)
$$\begin{aligned}\sin 415° &= \sin(360°+55°),\\ &= \sin 55°, & \ldots(i)\\ &= \sin(90°-35°),\\ &= \cos 35°. & \ldots(ii)\end{aligned}$$

Note that in arriving at (*i*), we have used the identity $\sin(2\pi+x)=\sin x$ because for angles this simply means $\sin(360°+x)=\sin x$. In arriving at (*ii*) we have used $\sin(\pi/2-x)=\cos x$, because for angles it simply reads $\sin(90°-x)=\cos x$.

(*b*)
$$\begin{aligned}\tan(-830°) &= -\tan 830°, \text{ since } \tan(-x)=-\tan x,\\ &= -\tan(4\cdot 180°+110°),\\ &= -\tan 110°, \text{ since } \tan(k\pi+x)=\tan x,\\ &= -\tan(180°-70°),\\ &= -(-\tan 70°), \text{ since } \tan(\pi-x)=-\tan x,\\ &= \tan 70°,\\ &= \tan(90°-20°),\\ &= \cot 20°, \text{ since } \tan(\pi/2-x)=\cot x.\end{aligned}$$

EXERCISE 13 (*e*)

1. Evaluate each of the following expressions :
   (*i*) $\sin 45° \cos 30° + \cos 45° \sin 30°$
   (*ii*) $\cos 60° \cos 45° - \sin 60° \sin 45°$
2. Substitute the values of trigonometric functions and show that each of the following statements is true :
   (*i*) $\sin^2 30° + \cos^2 30° = 1$.
   (*ii*) $\sec^2 45° = 1 + \tan^2 45°$.
   (*iii*) $\cos 30° = \cos 60° \cos 30° + \sin 60° \sin 30°$.

(*iv*) $\tan 30^\circ = \dfrac{\tan 60^\circ - \tan 30^\circ}{1+\tan 60^\circ \tan 30^\circ}$.

3. Construct an angle of 330° and find the values of all its trigonometric functions.

4. Verify that $\sin(60^\circ - 45^\circ) \neq \sin 60^\circ - \sin 45^\circ$.

5. Without the use of tables, find the value of :
   (*i*) $\cos 48^\circ \cos 12^\circ - \sin 48^\circ \sin 12^\circ$.
   (*ii*) $\sin 72^\circ \cos 12^\circ - \cos 72^\circ \sin 12^\circ$.
   (*iii*) $\sin 32^\circ \cos 13^\circ + \cos 32^\circ \sin 13^\circ$.
   (*iv*) $\cos 68^\circ \cos 8^\circ + \sin 68^\circ \sin 8^\circ$.

6. Reduce each of the following expressions to a single term involving only one function of an angle :
   (*i*) $2 \sin 13^\circ \cos 13^\circ$. (*ii*) $\cos^2 25^\circ - \sin^2 25^\circ$.
   (*iii*) $3 \sin 15^\circ - 4 \sin^3 15^\circ$. (*iv*) $1 - 2 \sin^2 17^\circ$.

7. Find the values of :
   (*i*) $\cos 210^\circ$. (*ii*) $\sin 225^\circ$.
   (*iii*) $\tan 330^\circ$. (*iv*) $\cot(-315^\circ)$.
   (*v*) $\sec 240^\circ$. (*vi*) $\csc(-150^\circ)$.
   (*vii*) $\sin 330^\circ \tan 135^\circ + \cos 225^\circ \sin 135^\circ$.
   (*viii*) $\cos(-765^\circ) \sin(405^\circ) + \tan 300^\circ \csc 120^\circ$.

8. Express the following in terms of trigonometric functions of θ :
   (*i*) $\sin(270^\circ + \theta)$. (*ii*) $\cos(270^\circ - \theta)$.
   (*iii*) $\tan(2\pi - \theta)$. (*iv*) $\cot(\pi + \theta)$.
   (*v*) $\sec(\pi - \theta)$. (*vi*) $\csc(\pi/2 + \theta)$.
   (*vii*) $\dfrac{\sin(180^\circ+\theta)\sec(-\theta)\cot(90^\circ-\theta)}{\tan(270^\circ-\theta)\cos(360^\circ-\theta)\csc(180^\circ-\theta)}$.

9. Express the following in terms of trigonometric ratios of positive acute angles not greater than 45°.
   (*i*) $\sin(-971^\circ)$. (*ii*) $\cos(567^\circ)$.
   (*iii*) $\tan(425^\circ)$. (*iv*) $\cot(-762^\circ)$.
   (*v*) $\sec(-823^\circ)$. (*vi*) $\csc(1146^\circ)$.

10. Find the angles between 0° and 360° which have their cosines equal to $\sqrt{3}/2$.

11. Find the angle between 0° and 360° whose sine is $-\sqrt{3}/2$ and whose cotangent is equal to $1/\sqrt{3}$.

12. Express each of the following products as an algebraic sum of sines and cosines :

(*i*) $2 \sin 60° \cos 30°$. (*ii*) $2 \cos 70° \sin 30°$.
(*iii*) $2 \cos 45° \cos 15°$. (*iv*) $2 \sin 40° \sin 60°$.

13. Express each of the following as a product :
(*i*) $\sin 48°+\sin 24°$. (*ii*) $\sin 72°-\sin 12°$.

14. Prove the following identities :
(*i*) $\sin \theta+\sin (120°+\theta)+\sin (240°+\theta)=0$.
(*ii*) $\sin 20° \sin 40° \sin 60° \sin 80°=3/16$.
(*iii*) $\cos 20° \cos 40° \cos 60° \cos 80°=1/16$.
(*iv*) $\tan (\theta+30°)+\cot (\theta-30°)=1/(\sin 2\theta-\sin 60°)$.
(*v*) $\cos^3 \theta+\cos^3 (120°+\theta)+\cos^3 (240°+\theta)=\frac{3}{4} \cos 3\theta$.

15. Show that $\tan 22\frac{1}{2}°=\sqrt{\left\{\frac{2-\sqrt{2}}{2+\sqrt{2}}\right\}}$.

13·15. CONDITIONAL IDENTITIES INVOLVING TRIGONOMETRIC FUNCTIONS

In this section we shall study the technique of proving conditional identities, specially those involving the angles of a triangle. The following examples will illustrate the method.

Example 15. *If $A+B+C=180°$, then show that*
$$\sin 2A+\sin 2B+\sin 2C=4 \sin A \sin B \sin C.$$

Solution.

$$\begin{aligned}
\text{L.H.S.}&=\sin 2A+\sin 2B+\sin 2C,\\
&=2 \sin (A+B) \cos (A-B)+2 \sin C \cos C,\\
&=2 \sin (\pi-C) \cos (A-B)+2 \sin C \cos [\pi-(A+B)],\\
&\qquad\qquad\qquad\text{since } A+B+C=\pi,\\
&=2 \sin C \cos (A-B)-2 \sin C \cos (A+B),\\
&=2 \sin C [\cos (A-B)-\cos (A+B)],\\
&=2 \sin C.\ 2 \sin A \sin B,\\
&=4 \sin A \sin B \sin C,\\
&=\text{R.H.S.}
\end{aligned}$$

Example 16. *If $A+B+C=180°$, then show that*
$$\sin^2 A-\sin^2 B+\sin^2 C=2 \sin A \cos B \sin C.$$

Solution.

$$\begin{aligned}
\text{L.H.S.}&=\sin^2 A-\sin^2 B+\sin^2 C,\\
&=\sin (A+B) \sin (A-B)+\sin^2 C,\\
&=\sin (\pi-C) \sin (A-B)+\sin C.\ \sin [\pi-(A+B)],\\
&\qquad\qquad\qquad\text{since } A+B+C=\pi,\\
&=\sin C \sin (A-B)+\sin C \sin (A+B),\\
&=\sin C [\sin (A-B)+\sin (A+B)],
\end{aligned}$$

$=\sin C.\ 2\sin A\cos B,$
$=2\sin A\cos B\sin C,$
$=$R.H.S.

Example 17. *If $A+B+C=180°$, prove that*

$$\cos A+\cos B-\cos C=-1+4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}.$$

Solution.

L.H.S. $=\cos A+\cos B-\cos C,$

$=2\cos\{\tfrac{1}{2}(A+B)\}\cos\{\tfrac{1}{2}(A-B)\}-\cos C,$

$=2\cos\{\tfrac{1}{2}(\pi-C)\}\cos\{\tfrac{1}{2}(A-B)\}-\cos C,$ since $A+B+C=\pi,$

$$=2\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right)-\left(1-2\sin^2\frac{C}{2}\right),$$

$$=-1+2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right)+\sin\frac{C}{2}\right],$$

$$=-1+2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right)+\sin\left(\frac{\pi-(A+B)}{2}\right)\right],$$
since $A+B+C=\pi,$

$$=-1+2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right)+\cos\left(\frac{A+B}{2}\right)\right],$$

$$=-1+2\sin\frac{C}{2}.\,2\cos\frac{A}{2}\cos\frac{B}{2},$$

$$=-1+4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}.$$

EXERCISE 13 (*f*)

If $A+B+C=\pi$, then prove the following identities :

1. $\sin 2A-\sin 2B+\sin 2C=4\cos A\sin B\cos C.$
2. $\cos 2A+\cos 2B+\cos 2C=1-4\cos A\cos B\cos C.$
3. $\sin A+\sin B-\sin C=4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}.$
4. $\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$
5. $\sin^2 A+\sin^2 B-\sin^2 C=2\sin A\sin B\cos C.$
6. $\cos^2\frac{A}{2}+\cos^2\frac{B}{2}-\cos^2\frac{C}{2}=2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}.$
7. If $A+B+C=90°$, then prove that
$\sin^2 A+\sin^2 B+\sin^2 C+2\sin A\sin B\sin C=1.$

13·16. TABLE OF VALUES OF TRIGONOMETRIC FUNCTIONS OF ANGLES

We have seen above how values of trigonometric functions of certain angles are calculated by geometric methods. But these are not enough, for in our study of trigonometry, we shall come across trigonometric functions of other angles as well. By advanced methods the values of trigonometric functions of any angle can be computed to any desired degree of accuracy.

Table 1 at the end of the book gives the values of trigonometric functions of angles from 0° to 90° at intervals of 10′. This table is often called the *table of natural trigonometric functions* in order to distinguish it from the table of logarithmic trigonometric functions.

The values of trigonometric functions are, in most cases, non-terminating decimals, and therefore, the values in the table are only approximations (except possibly in those cases in which the values happen to be terminating decimals). The values in the table are given to four significant figures and therefore, the table is called a four-figure table.

Let us have a look at the table. The first column contains angles from 0° to 45° at intervals of 10′. The next six columns contain the values of the trigonometric functions of these angles; the particular function, the values of which are contained in any column is indicated at the top of the column. To find the value of any trigonometric function of an angle not exceeding 45°, we first read down along the first column until the angle is found and then read to the right until we reach the column which contains the name of the function that we wish to find, at the top. The last column contains angles from 45° to 90° at intervals of 10′. The six columns preceding it contain the values of the trigonometric functions of these angles; the particular function, the values of which are contained in any column, is indicated at the bottom of the column. To find the value of any trigonometric function of an angle lying between 45° and 90° we first read up along the last column until the angle is found and then read to left until we reach the column which contains the name of the function that we wish to find, at the bottom.

The table can be used for two purposes:

1. Given an angle, to find the value of any of its trigonometric functions.

2. Given the value of a trigonometric function of an angle, to find the angle.

We shall now consider some examples to illustrate the manner in which the table can be used for any of the above purposes.

Example 18. *Find*

(*i*) *sin 32°,*

(*ii*) *tan 41° 20′, and*

(*iii*) *csc 25° 40′.*

Solution. (*i*) In table 1 at the end of the book, we read down the first column until we come to 32°. Then we look across the page in the line corresponding to 32° and read from the column headed by *sin*. We thus have sin 32°=·5299.

(*ii*) In table 1, we read down the first column until we come to 41° 20′. Then we look across the page in the line corresponding to 41° 20′ and read from the column headed by *tan*. We thus have

$$\tan 41° 20' = \cdot 8796.$$

(*iii*) In table 1 we read down the first column until we come to 25° 40′. Then we look across the page in the line corresponding to 25° 40′ and read from the column headed by *csc*. We thus have

$$\csc 25° 40' = 2\cdot 309.$$

Example 19. *Find*

(*i*) *cos 68°,*

(*ii*) *cot 53°10′, and*

(*iii*) *sec 71°50′.*

Solution. (*i*) In table 1, we read up the last column until we come to 68°. Then we look across the page in the line corresponding to 68° and read from the column having *cos* at its foot. We thus have

$$\cos 68° = \cdot 3746.$$

(*ii*) In table 1, we read up the last column until we come to 53° 10′. Then we look across the page in the line corresponding to 53° 10′ and read from the column having *cot* at its foot. We thus have

$$\cot 53° 10' = \cdot 7490.$$

(*iii*) In table 1, we read up the last column until we come to 71° 50′ and read from the column having *sec* at its foot. We thus have

$$\sec 71° 50' = 3\cdot 207.$$

Example 20. *Find the positive acute angle θ, if*

(*i*) $\sin \theta = \cdot 3987$;

(*ii*) $\tan \theta = 2\cdot 177$;

(*iii*) $\sec \theta = 1\cdot 630$.

Solution. (*i*) Since the sines are given in column two, reading *down* and in column three reading *up*, therefore, we must search for the number ·3987 through these columns. We find that ·3987 appears

in column two which has *sin* at its top. This column contains the sines of angles in the first column. Reading across the page on a line with ·3987, we find in the first column the angle 23°30′.

Hence $\sin\theta = \cdot 3987 \wedge 0 < \theta < 90° \Rightarrow \theta = 23°\ 30'$.

(*ii*) Since the tangents are given in column four reading *down* and in column five reading *up*, therefore, we must search for the number 2·177 through these columns. We find that 2·177 appears in column five which has *tan* at its foot. This column contains the tangents of angles in the last column. Reading across the page on a line with 2·177, we find in the last column the angle 65° 20′.

Hence $\tan\theta = 2{\cdot}177 \wedge 0 \leqslant \theta \leqslant 90° \Rightarrow \theta = 65°\ 20'$.

(*iii*) Since the secants are given in column six reading *down* and in column seven reading *up*, therefore, we must search for the number 1·630 through these columns. We find that 1·630 appears in column seven which has *sec* at its foot. This column contains the secants of angles in the last column. Reading across the page on a line with 1·630 we find in the last column the angle 52° 10′.

Hence $\sec\theta = 1{\cdot}630 \wedge 0 < \theta < 90° \Rightarrow \theta = 52°\ 10'$.

EXERCISE 13 (*g*)

1. Use table 1 (at the end of the book) to find the value of the following :

(*i*) sin 25° 10′	(*ii*) cos 48° 20′
(*iii*) tan 31° 40′	(*iv*) cot 57° 30′
(*v*) sec 36° 20′	(*vi*) csc 59°
(*vii*) sin 74° 40′	(*viii*) cos 16° 10′
(*ix*) tan 65° 50′	(*x*) cot 33° 20′.

2. Use table 1 (at the end of the book) to find the positive acute angle θ from the given value of the function :

(*i*) $\sin\theta = \cdot 1248$	(*ii*) $\cos\theta\ \cdot 9667$
(*iii*) $\tan\theta = 1{\cdot}632$	(*iv*) $\cot\theta = \cdot 6959$
(*v*) $\sec\theta = 2{\cdot}595$	(*vi*) $\csc\theta = 1{\cdot}228$
(*vii*) $\sin\theta = \cdot 9261$	(*viii*) $\cos\theta = \cdot 3529$
(*ix*) $\tan\theta = \cdot 4986$	(*x*) $\cot\theta = \cdot 5851$.

13·17. INTERPOLATION

In the preceding section, we have seen how we can find the value of a trigonometric function of any angle which appears in the table. We shall now see how we can find an approximate value of a trigonometric function of an angle which does not appear in the table and also how we can approximately find a positive acute angle corresponding to a given functional value which does not appear in the table. The process by which this is done is called

Example 18. *Find*

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Solution. (*i*) In table 1, we read up the last column until we come to 68°. Then we look across the page in the line corresponding to 68° and read from the column having *cos* at its foot. We thus have

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$$\cot 53° 10' = \cdot 7490.$$

(*iii*) In table 1, we read up the last column until we come to 71° 50′ and read from the column having *sec* at its foot. We thus have

$$\sec 71° 50' = 3\cdot 207.$$

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(*i*) $\sin \theta = \cdot 3987$;

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Solution. (*i*) Since the sines are given in column two, reading *down* and in column three reading *up*, therefore, we must search for the number ·3987 through these columns. We find that ·3987 appears

in column two which has *sin* at its top. This column contains the sines of angles in the first column. Reading across the page on a line with ·3987, we find in the first column the angle 23°30′.

Hence $\sin\theta = \cdot 3987 \wedge 0 < \theta < 90° \Rightarrow \theta = 23°\ 30'$.

(*ii*) Since the tangents are given in column four reading *down* and in column five reading *up*, therefore, we must search for the number 2·177 through these columns. We find that 2·177 appears in column five which has *tan* at its foot. This column contains the tangents of angles in the last column. Reading across the page on a line with 2·177, we find in the last column the angle 65° 20′.

Hence $\tan\theta = 2{\cdot}177 \wedge 0 \leqslant \theta \leqslant 90° \Rightarrow \theta = 65°\ 20'$.

(*iii*) Since the secants are given in column six reading *down* and in column seven reading *up*, therefore, we must search for the number 1·630 through these columns. We find that 1·630 appears in column seven which has *sec* at its foot. This column contains the secants of angles in the last column. Reading across the page on a line with 1·630 we find in the last column the angle 52° 10′.

Hence $\sec\theta = 1{\cdot}630 \wedge 0 < \theta < 90° \Rightarrow \theta = 52°\ 10'$.

EXERCISE 13 (*g*)

1. Use table 1 (at the end of the book) to find the value of the following :

(*i*) sin 25° 10′	(*ii*) cos 48° 20′
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13·17. INTERPOLATION

In the preceding section, we have seen how we can find the value of a trigonometric function of any angle which appears in the table. We shall now see how we can find an approximate value of a trigonometric function of an angle which does not appear in the table and also how we can approximately find a positive acute angle corresponding to a given functional value which does not appear in the table. The process by which this is done is called

interpolation. Interpolation is an extremely important technique whenever we have to use tables.

In the present case it depends on the observation that small changes in the value of trigonometric functions are proportional to small changes in the angle. This can be readily seen by going through a few consecutive entries (at random) in any column of table 1. (Let us warn the reader that the above assumption is not true in general but is approximately true only for small changes.)

We shall illustrate the technique by considering a few examples.

Example 21. *Find the value of sin 34°16′.*

Solution. The given angle is not listed in Table 1. It lies between the angles 34° 10′ and 34° 20′ which happen to be consecutive entries in the table. From the table we can see that $\sin\theta$ increases as θ increases from 0° to 90°. Therefore,

$$\sin 34°10' < \sin 34°16' < \sin 34°20'.$$

From the table, we find

$$\sin 34°10' = \cdot 5616,$$

$$\sin 34°16' = ?$$

and $$\sin 34°20' = \cdot 5640.$$

As the angle increases by 10′, its sine increases by ·0024, *i.e.*, 24 thousandths. Now the angle 34°16′ is 6/10 of the way from 34°10′ to 34°20′. Therefore, sin 34°16′ is also expected to be 6/10 of the way from ·5616 to ·5640.

Now $\left(\frac{6}{10} \text{ of}\right)(24) = 14\frac{2}{5} \simeq 14$. (We have rounded off $14\frac{2}{5}$ to 14 because $14\frac{2}{5}$ is closer to 14 than 15). Since the sine is *increasing*, therefore, we must *add* (to ·5616, 14 thousandths). Thus

$$\sin 34°\ 16' = \cdot 5630.$$

Remark. The difference ·0024 between two consecutive entries ·5616 and ·5640 is called the **tabular difference.** The number 14 thousandths is called the **correction.** The correction is to be added in the cases of sin, tan and sec, because they are increasing functions as θ increases from 0° to 90°. Since the functions cos, cot and csc are decreasing functions as θ increases from 0° to 90°, therefore, for these functions, the correction has to subtracted from the functional value of the smaller of the two consecutive entries between which the angle in question lies.

Example 22. *Find cos 69°13′.*

Solution. The given angle 69° 13′ is not listed in table 1. It lies between the consecutive entries 69°10′ and 69°20′ in the table.

Now $$\cos 69°\ 10' = \cdot 3557,$$

$$\cos 69°\ 13' = ?$$

and $$\cos 69°\ 20' = \cdot 3529.$$

The tabular difference is $\cdot 0028$. Since the given angle is 3/10 of the way from $69°10'$ to $69°20'$, therefore, the correction is $\cdot 0028\times(3/10)=\cdot 0008$. Since $\cos\theta$ decreases from $0°$ to $90°$, therefore, the correction has to be *subtracted* from $\cdot 3557$, so as to give

$$\cos 69°13'=\cdot 3557-\cdot 0008=\cdot 3549.$$

Example 23. *Find the positive acute angle θ for which $\tan\theta=1{\cdot}689$.*

Solution. Since the tangents are given in column four reading *down* and in column five reading *up*, therefore, we must search for the number $1{\cdot}689$ through these columns but there are two consecutive entries $1{\cdot}686$ and $1{\cdot}698$ in column five between which the given number lies. Since the entries in column five give tangents of angles in the last column, therefore, reading the entries in the last column against the above entries, we have

$$\tan 59°\,20'=1{\cdot}686.$$

$$\tan\ ?\quad =1{\cdot}689,$$

and $$\tan 59°\,30°=1{\cdot}698.$$

Since $1{\cdot}689$ is 3/12 of the way from $1{\cdot}686$ to $1{\cdot}698$, and since the tangent of an angle increases as θ increases from $0°$ to $90°$, therefore, we must add $(3/12)\times 10'=3'$ (rounded off to the nearest minute) to $59°\,20'$. We thus find that $\tan 59°\,23'=1{\cdot}689$.

Hence the required angle is $59°\,23'$.

Remark. After some practice, the reader should inculcate the habit of performing the interpolation mentally.

EXERCISE 13 (*h*)

1. Find the value of the following (use table 1 and interpolate) :

(*i*) $\sin 28°\,23'$	(*ii*) $\cos 51°\,47,$	(*iii*) $\tan 39°\,56'$
(*iv*) $\cot 48°\,38'$	(*v*) $\sec 66°\,25'$	(*vi*) $\csc 78°\,42'$
(*vii*) $\sin 46°\,53'$	(*viii*) $\cos 33°\,46'$	(*ix*) $\tan 56°\,37'$
(*x*) $\cot 41°\,29'$	(*xi*) $\sec 15°\,24'$	(*xii*) $\csc 35°\,44'$.

2. Find, to the nearest minute, the positive acute angle θ, for which :

(*i*) $\sin\theta=\cdot 0627$	(*ii*) $\cos\theta=\cdot 5380$	(*iii*) $\tan\theta=\cdot 4329$
(*iv*) $\cot\theta=1{\cdot}636$	(*v*) $\sec\theta=2{\cdot}537$	(*vi*) $\csc\theta=1{\cdot}395$
(*vii*) $\sin\theta=\cdot 9479$	(*viii*) $\cos\theta=\cdot 4052$	(*ix*) $\tan\theta=\cdot 5682$
(*x*) $\cot\theta=\cdot 4162$	(*xi*) $\sec\theta=1{\cdot}823$	(*xii*) $\csc\theta=2{\cdot}184$.

TEST YOUR UNDERSTANDING XIII

In each of the following problems, four alternatives are given. Put a tick-mark (✓) against the correct alternative.

1. $\tan(\pi - x)$ equals

   (a) $\tan x$ (b) $\cot x$

   (c) $-\tan x$ (d) $-\cot x$.

2. $\sin\frac{\pi}{4}\sin\frac{3\pi}{4}\sin\frac{5\pi}{4}\sin\frac{7\pi}{4}$ equals

   (a) $1/16$ (b) $1/4$

   (c) $-1/4$ (d) $1/8$.

3. $\sin(A+B)\sin(A-B)$ equals

   (a) $\sin^2 A - \cos^2 B$ (b) $\cos^2 A - \sin^2 B$

   (c) $\sin^2 A - \sin^2 B$ (d) $\cos^2 A - \cos^2 B$.

4. The value of $\sin 18° \cos 36°$ is

   (a) $1/4$ (b) $(\sqrt{5}-1)/4$

   (c) $(\sqrt{5}+1)/4$ (d) $1/2$.

5. The value of $\sin 75°$ is

   (a) $(\sqrt{3}+1)/2\sqrt{2}$ (b) $(\sqrt{3}-1)/2\sqrt{2}$

   (c) $(\sqrt{3}+1)/2$ (d) $(\sqrt{3}-1)/2$.

6. $\sin\left(\frac{\pi}{2}+x\right)$ equals

   (a) $\cos x$ (b) $-\cos x$

   (c) $\sin x$ (d) $-\sin x$.

7. $\frac{\cos 9° - \sin 9°}{\cos 9° + \sin 9°}$ equals

   (a) $\tan 54°$ (b) $\tan 36°$

   (c) $\tan 81°$ (d) $\tan 54°$.

8. The value of $\frac{1-\tan^2 15°}{1+\tan^2 15°}$ is

   (a) $\sqrt{3}$ (b) $1/2$

   (c) $1/\sqrt{3}$ (d) $\sqrt{3}/2$.

9. If $\sin^2 x = \frac{1}{4}$, and x lies in the second quadrant, then x equals

   (a) $5\pi/6$ (b) $2\pi/3$

   (c) $3\pi/4$ (d) $7\pi/12$.

10. An arc of a circle of length 22 cm subtends an angle of 60° at the centre of the circle. The radius of the circle is

(*a*) 22 cm (*b*) 42 cm

(*c*) 21 cm (*d*) 11 cm.

REVIEW EXERCISE XIII

1. If $m \tan(\theta-30)^\circ=n \tan(\theta+120^\circ)$, show that

$$\cos 2\theta=\frac{m+n}{2(m-n)}.$$

2. If $A+B=225°$, prove that

$$\frac{\cot A}{1+\cot A}\cdot\frac{\cot B}{1+\cot B}=\frac{1}{2}.$$

3. Show that $\tan 70^\circ=\tan 20^\circ+2\tan 50^\circ$.

4. Show that $\tan 75^\circ-\tan 30^\circ-\tan 75^\circ \tan 30^\circ=1$.

5. If $\alpha+\beta=\gamma$, prove that

$$\cos^2\alpha+\cos^2\beta+2\cos\alpha\cos\beta\cos\gamma=\sin^2\gamma.$$

6. If $\cos\theta=\frac{1}{2}\left(a+\frac{1}{a}\right)$, show that

$$\cos 3\theta=\frac{1}{2}\left(a^3+\frac{1}{a^3}\right).$$

7. Prove the following identities :

(*i*) $\tan^4\theta-\sec^4\theta=1-2\sec^2\theta$.

(*ii*) $\frac{\sec\theta}{1+\cos\theta}=\frac{\sec\theta-1}{\sin^2\theta}$.

8. Is the following equation an identity :

$$\cos^3\theta+\sin^3\theta=1\ ?$$

(Give reasons for your answer.)

9. If $2\tan\beta+\cot\beta=\tan\alpha$, prove that $\cot\beta=2\tan(\alpha-\beta)$.

10. If $\tan\theta=\frac{b}{a}$, prove that $a\cos 2\theta+b\sin 2\theta=a$.

11. Prove that $\tan 10^\circ+\tan 70^\circ-\tan 50^\circ=\sqrt{3}$.

12. Prove that $\sin 12^\circ \sin 48^\circ \sin 54^\circ=\frac{1}{8}$.

SUMMARY

1. $\tan x=\dfrac{\sin x}{\cos x}$, $\cot x=\dfrac{\cos x}{\sin x}$, $\sec x=\dfrac{1}{\cos x}$, $\csc x=\dfrac{1}{\sin x}$.

2. $\cos^2 x+\sin^2 x=1$.
   $1+\tan^2 x=\sec^2 x$.
   $1+\cot^2 x=\csc^2 x$.

3. $\sin(2k\pi+x)=\sin x$, $\cos(2k\pi+x)=\cos x$ for all $k\in Z$.

4. $\sin(-x)=-\sin x$, $\cos(-x)=\cos x$.
   $\tan(-x)=-\tan x$, $\cot(-x)=-\cot x$.
   $\sec(-x)=\sec x$, $\csc(-x)=-\csc x$.

5. $\sin\left(\dfrac{\pi}{2}-x\right)=\cos x$, $\sin\left(\dfrac{\pi}{2}+x\right)=\cos x$,

   $\cos\left(\dfrac{\pi}{2}-x\right)=\sin x$, $\cos\left(\dfrac{\pi}{2}+x\right)=-\sin x$.

6. $\sin(\pi-x)=\sin x$, $\sin(\pi+x)=-\sin x$.
   $\cos(\pi-x)=-\cos x$, $\cos(\pi+x)=-\cos x$.
   $\tan(\pi-x)=-\tan x$, $\tan(\pi+x)=\tan x$.

7. $\sin(x+y)=\sin x\cos y+\cos x\sin y$.
   $\sin(x-y)=\sin x\cos y-\cos x\sin y$.
   $\cos(x+y)=\cos x\cos y-\sin x\sin y$.
   $\cos(x-y)=\cos x\cos y+\sin x\sin y$.

8. $\tan(x+y)=\dfrac{\tan x+\tan y}{1-\tan x\tan y}$, $\tan(x-y)=\dfrac{\tan x-\tan y}{1+\tan x\tan y}$.

   $\tan(x+y+z)=\dfrac{\tan x+\tan y+\tan z-\tan x\tan y\tan z}{1-\tan y\tan z-\tan z\tan x-\tan x\tan y}$.

9. $\sin 2x=2\sin x\cos x$, $\sin 3x=3\sin x-4\sin^3 x$.
   $\cos 2x=\cos^2 x-\sin^2 x=2\cos^2 x-1=1-2\sin^2 x$.

   $\sin 2x=\dfrac{2\tan x}{1+\tan^2 x}$, $\cos 2x=\dfrac{1-\tan^2 x}{1+\tan^2 x}$.

   $\tan 2x=\dfrac{2\tan x}{1-\tan^2 x}$, $\tan 3x=\dfrac{3\tan x-\tan^3 x}{1-3\tan^2 x}$.

10. $\sin x\cos y=\frac{1}{2}[\sin(x+y)+\sin(x-y)]$,
    $\cos x\sin y=\frac{1}{2}[\sin(x+y)-\sin(x-y)]$,
    $\cos x\cos y=\frac{1}{2}[\cos(x+y)+\cos(x-y)]$,
    $\sin x\sin y=-\frac{1}{2}[\cos(x+y)-\cos(x-y)]$.

11. $\sin x+\sin y=2\sin\dfrac{x+y}{2}\cos\dfrac{x+y}{2}$,

    $\sin x-\sin y=2\cos\dfrac{x+y}{2}\sin\dfrac{x-y}{2}$,

$$\cos x+\cos y=2\cos\frac{x+y}{2}\cos\frac{x-y}{2},$$

$$\cos x-\cos y=-2\sin\frac{x+y}{2}\sin\frac{x-y}{2}.$$

12. $1^\circ=60'$, $1'=60''$.

13 π radians $=180^\circ$; 1 radian $=57^\circ 17' 44{\cdot}81$ (approx.),
$1^\circ={\cdot}017453293$ radian (approx.).

14. *Angle in degrees*	*Angle in radians*	*sin*	*cos*	*tan*	*cot*	*sec*	*csc*
0°	0	0	1	0	$\times$	1	$\times$
30°	$\pi/6$	$1/2$	$\sqrt{3}/2$	$1/\sqrt{3}$	$\sqrt{3}$	$2/\sqrt{3}$	2
45°	$\pi/4$	$1/\sqrt{2}$	$1/\sqrt{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$	$1/\sqrt{3}$	2	$2/\sqrt{3}$
90°	$\pi/2$	1	0	$\times$	0	$\times$	1

15. $\sin 18^\circ=\frac{\sqrt{5}-1}{4}$, $\cos 36^\circ=\frac{\sqrt{5}+1}{4}$.

HISTORICAL NOTE

Trigonometry literally means the ***science of triangle measurement.*** Its origin dates back to 1500 B.C. According to available evidence, the Egyptians, the Babylonians. the Chinese and the Greeks used some primitive notions of trigonometry for measuring heights by measuring shadows. Real progress in the development of trigonometry was due to the fact that it was an important tool for the astronomers. In this sense, the origin of trigonometry dates back to Hipparchus (c. 140 B C.). The Hindus made notable contributions to Trigonometry.

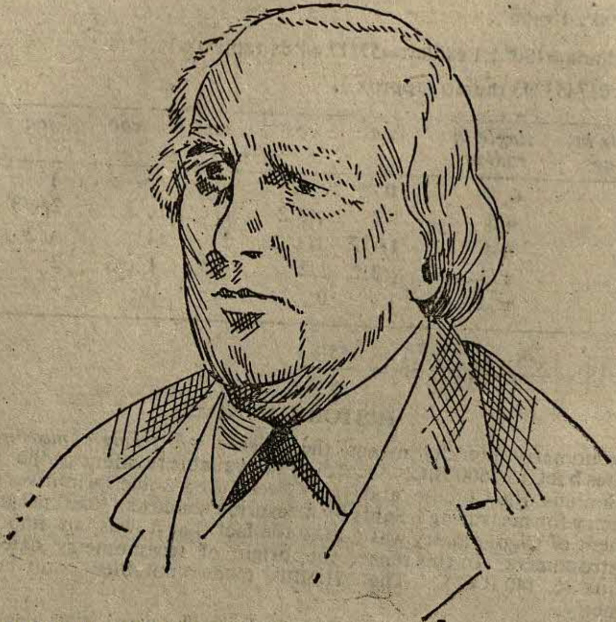
The sine function had its origin in India. The word sine itself has also descended from the Hindu word *jiva*. The Arab used the meaningless word *jiva* which was phonetically similar to *jiva*. In course of time *jiba* became *jaib* which was translated to *sinus* in Latin by *Gherado* of Cremona (c. 1150) because both the words mean 'a fold', and from *sinus* descends the English *sine*. Trigonometry was a useful and accurate tool for the Hindu astronomers.

The earliest tables for the sine function that have survived the vagaries of time are those in *Surya-Siddhanta* (c. 400) and the *Aryabhatiyam* of Aryabhanta. Here the sines of angles upto 90° are given for twenty-four equal intervals of $3\frac{3}{4}^\circ$ each. The values given in these tables are remarkably close to the modern values, showing how advanced we were in those days.

The addition theorem for the sine function was known to the Greeks. Bhaskara (c. 1150) also gives this theorem.

The formula for $\sin 2x$ was first given as a rule by Abul-Wefa. Vieta (1591 A.D.) first gave the formula for $\sin 3x$ and $\cos 3x$ in terms of $\sin x$ and $\cos x$. The expressions for $\sin 2x$ and $\cos 2x$ in terms of $\tan x$ were given by Lambert (1765 A.D.).

□ □



KARL WILHELM THEODOR WEIERSTRASS (1815-1897)

Karl Wilhelm Theodor Weierstrass was born on October 31, 1815 at Ostenfelde in the district of Munster, Germany. He had a uniformly brilliant record at school, dotted all along with prizes. He started his career at the age of twenty-six in 1841 as a secondary school teacher and continued as such for nearly fifteen years.

Weierstrass shot in into prominence only after the publication, in 1854, of his research work in Crelle's journal. He got immediate recognition. The University of Konigsberg conferred on him the degree of doctor of science, *honoris causa*. In 1856 he was appointed professor of mathematics at Berlin. His fame soon spread all over Europe and America. During the next forty years he was acknowledged as one of the leading mathematicians of the world.

Weierstrass was an inspiring teacher. His lectures were models of clarity and perfection. He remained a bachelor all his life. He died in his eighty-second year on February 19, 1897 at his home in Berlin after a long illness followed by influenza.

CHAPTER 14

Graphs of Trigonometric Functions

14·1. INTRODUCTION

Graphing of functions is of great importance in mathematics. By drawing the graph of a function one can exhibit many properties of the function. In the present chapter we shall devote ourselves to graphing of some trigonometric functions. We shall need the concepts of periodicity, and intervals of monotonicity for graphing trigonometric functions. We shall briefly discuss these concepts before we actually draw the graphs.

14·2. PERIODICITY OF TRIGONOMETRIC FUNCTIONS

Let f be a function whose domain D is a subset of **R**. A real number p $(\neq 0)$ is said to be a **period** of f if

(*i*) $x \in D \Leftrightarrow x+p \in D$, and

(*ii*) $f(x+p)=f(x)$, for all $x \in D$.

For a constant function, every non-zero number is a period. A function f is said to be **periodic** if it has at least one period.

It is obvious that if p is a period, then every positive intergal multiple of p is also a period,

i.e., $\quad f(x+kp)=f(x)$, for all $k \in \mathbf{Z}$.

From condition (*i*) in the definition of a period, it follows that if x is in the domain of a periodic function with p as a period, then so also is $x-p$, for

$$x=(x-p)+p, \text{ so that}$$
$$x \in D \Leftrightarrow (x-p) \in D.$$

Also, by condition (*ii*), we then have

$$f(x)=f(x-p+p)=f(x-p).$$

Therefore, $-p$ is also a period. Since every positive integral multiple of a period is also a period, therefore, $-kp$ is also a period for each positive integer k.

Thus, if p is a period, then kp (where k is any non-zero integer) is also a period. *The least positive period of a periodic function is said to be the period of the function.*

Periodic functions are of great importance in physics, mechanics and engineering for the study of various periodic phenomena such as vibration, motion of machines, alternating electric currents, etc. As we shall presently see, the trigonometric functions are periodic and one of the most important reasons for the study of trigonometric functions is their periodicity.

Theorem 14.1. *All trigonometric functions are periodic. Also,*

(*a*) *the period of the functions cos, sin sec, csc is* 2π, *and*

(*b*) *the period of the functions tan and cot is* π.

Proof. (*i*) Since $\cos x = \cos(x+2\pi)$ for all $x \in \mathbf{R}$, 2π is a period of the cosine function. We shall next show that it is the period, *i.e.*, it is the least positive period. If not, let a positive number $p<2\pi$ be a period. Then $\cos(x+p)=\cos x$, for all $x \in \mathbf{R}$. Putting $x=0$ in this relation we have $\cos p=1$. Since $0<p<2\pi$, the terminal side of the angle having a measure p radians cannot coincide with the positive x-axis and consequently $\cos p \neq 1$. This gives a contradiction. Hence 2π is the period of the cosine function.

(*ii*) Since $\sin x=\sin(x+2\pi)$, for all $x \in \mathbf{R}$, 2π is a period of the sine function. We shall now show that 2π is the least positive period. If not, let a positive number $p<2\pi$ be a period. Then $\sin(x+p)=\sin x$ for all x. Putting $x=\pi/2$ in this relation, we have $\sin \pi/2+p)=\cos p=1$; but this is impossible since $0<p<2\pi$. Hence 2π is the period of the sine function.

(*iii*) Since $\tan(x+\pi)=\tan x$ for all $x \in \mathbf{R}$, π is a period of the tangent function. We shall now show that π is the period. If not, let a positive number $p<\pi$ be a period. Then $\tan(x+p)=\tan x$ for all $x \in \mathbf{R}$. Putting $x=0$ in this relation, we have $\tan p=0$ which is impossible since $0<p<\pi$. Hence π is the period of the tangent function.

The rest of the theorem can be proved in the same manner as above.

The property of periodicity simplifies the study of a trigonometric function because by studying the properties of such a function over an interval of length equal to the period, we can know its properties everywhere in the domain of the function.

14·3. INTERVALS OF MONOTONY OF TRIGONOMETRIC FUNCTIONS

A function f is said to be **strictly increasing** in an interval if for any two numbers x_1, x_2 in the interval,

$$x_2>x_1 \Rightarrow f(x_2)>f(x_1),$$

and **strictly decreasing** if $x_2>x_1 \Rightarrow f(x_2)<f(x_1)$.

A function f is said to be **strictly monotonic** if it is either strictly increasing or strictly decreasing.

Note. If a function f has the property that for any numbers x_2, x_1 in an interval, $x_2 > x_1 \Rightarrow f(x_2) \geqslant f(x_1)$, then we say that f is **increasing** in the interval. Similarly, we speak of **decreasing** and **monotonic** functions.

Examples. (*i*) Let f be the function defined by setting $f(x) = x^2$ for all $x \in \mathbf{R}$. Then f is strictly increasing in $[0, 1]$ and strictly decreasing in $]-1, 0[$.

(*ii*) Let f be the function defined by setting

$$f(x) = \begin{cases} 0 \text{ if } x \leqslant 0 \\ x^2 \text{ if } x > 0. \end{cases}$$

Then f is increasing, but not strictly increasing, in $[-1, 1]$. However, f is strictly increasing in $[0, 1]$.

(*iii*) Let f be the function defined by setting

$$f(x) = x^2 - 1, \text{ for all } x \in \mathbf{R}.$$

Then f is neither increasing nor decreasing in $[-1, 1]$.

We now propose to determine the intervals in which a given trigonometric function is monotonic.

14·3·1. Intervals of Monotony of the Function cos

Let x_1, x_2 be two real numbers such that $x_2 > x_1$.

We have $\cos x_2 - \cos x_1 = -2 \sin [\frac{1}{2}(x_2 - x_1)] \sin [\frac{1}{2}(x_2 + x_1)]$.
...(1)

If x_1, x_2 be both in the interval $]0, \pi[$, then $\frac{1}{2}(x_2 + x_1)$ is also in the same interval and so also is $\frac{1}{2}(x_2 - x_1)$, and consequently $\sin [\frac{1}{2}(x_2 - x_1)]$ and $\sin [\frac{1}{2}(x_2 + x_1)]$ are both positive. Therefore, from (1) we find that $\cos x_2 - \cos x_1 < 0$, *i.e.*, $\cos x_2 < \cos x_1$. Since for al x_1, x_2 in $]0, \pi[$, $x_2 > x_1 \Rightarrow \cos x_2 < \cos x_1$, therefore, in the interval $]0, \pi[$, cos is a strictly decreasing function.

Also at $x = 0$, the cosine function has the value 1 which is thtl greatest value that $\cos x$ can have for any value of x, and at $x = \pi$e the cosine function has the value -1 which is the least value tha, $\cos x$ can have for any value of x. Thus on the closed interval $[0, \pi]$, cos strictly decreases from 1 to -1.

As $\cos(\pi + x) = -\cos x$, it follows that cos is a strictly increasing function in $[\pi, 2\pi]$.

Since the cosine function is periodic with period 2π, therefore, it will have the same behaviour in all intervals of the form $[2k\pi, (2k+1)\pi]$, k being any integer, as in $[0, \pi]$. Similarly it will have the same behaviour in all intervals of the form $[(2k+1)\pi, (2k+2)\pi]$, k being any integer, as in $[\pi, 2\pi]$.

Thus *for each integer k, $\cos x$ strictly decreases from $+1$ to -1 on the interval $[2k\pi, (2k+1)\pi]$, and strictly increases from -1 to $+1$ on the interval $[(2k+1)\pi, (2k+2)\pi]$.*

Thus *for the cosine function the intervals of strict decrease are* $[2k\pi, (2k+1)\pi]$ *and the intervals of strict increase are* $[(2k+1)\pi, (2k+2)\pi]$.

14·3·2. Intervals of Monotony of the Function sin

Since $\sin x = \cos(x - \pi/2)$, therefore, from the results obtained above in respect of the function cos, it follows that $\sin x$ strictly decreases from 1 to -1 as $x-\pi/2$ increases from 0 to π, and strictly increases from -1 to $+1$ as $x-\pi/2$ increases from π to 2π, *i.e.*, $\sin x$ strictly decreases from 1 to -1 as x increases from $\pi/2$ to $3\pi/2$, and strictly increases from -1 to 1 as x increases from $3\pi/2$ to $5\pi/2$, or from $-\pi/2$ to $\pi/2$, using the periodicity.

By periodicity, the behaviour of the sine function in the interval $[\pi/2, 3\pi/2]$ is the same as that in the interval $[2k\pi+\pi/2, 2k\pi+3\pi/2]$, k being any integer. Thus *for each integer* k, $\sin x$ *strictly decreases from 1 to -1 on the interval* $[2k\pi+\pi/2, 2k\pi+3\pi/2]$. Similarly we can show that *for each integer* k, $\sin x$ *strictly increases from -1 to 1 on the interval* $[2k\pi-\pi/2, 2k\pi+\pi/2]$.

Thus *for the sine function the intervals of strict decrease are* $[2k\pi+\pi/2, 2k\pi+3\pi/2]$, *and the intervals of strict increase are* $[2k\pi-\pi/2, 2k\pi+\pi/2]$, *k being any integer.*

14·3·3. Interval of Monotony of the Function tan

Let us consider two numbers x_1 and x_2 in $\mathbf{R}^*$ such that $x_2 > x_1$.

We have

$$\tan x_2 - \tan x_1 = \frac{\sin(x_2 - x_1)}{\cos x_2 \cos x_1}.$$

If $-\pi/2 < x_1 < x_2 < \pi/2$, then $0 < x_2 - x_1 < \pi$, so that $\sin(x_2 - x_1) > 0$, $\cos x_1 > 0$, $\cos x_2 > 0$. Therefore, $\tan x_2 > \tan x_1$, *i.e.*, tan is strictly increasing in $]-\pi/2, \pi/2[$.

As tan is periodic with period π, therefore, *for each integer* k, *tan is strictly increasing in the interval* $]k\pi-\pi/2, k\pi+\pi/2[$.

14·3·3. Intervals of Monotony of the Functions cot, sec and csc

It can be easily proved that

(*i*) *for each integer k, cot is strictly decreasing on the interval* $]k\pi, (k+1)\pi[$;

(*ii*) *for each integer k, sec is strictly decreasing on each of the intervals* $]2k\pi-\pi/2, 2k\pi[$ *and* $]2k\pi+\pi, 2k\pi+3\pi/2[$ *and is strictly increasing on each of the intervals* $]2k\pi, 2k\pi+\pi/2[$ *and* $]2k\pi+\pi/2, 2k\pi+\pi[$;

(*iii*) *for each integer k, csc is strictly decreasing on each of the intervals* $]2k\pi, 2k\pi+\pi/2[$ *and* $]2k\pi+3\pi/2, (2k+2)\pi[$, *and is strictly*

increasing on each of the intervals $]2k\pi+\pi/2, 2k\pi+\pi[$ *and* $]2k\pi+\pi, 2k\pi+3\pi/2[$.

14·4. RANGES OF TRIGONOMETRIC FUNCTIONS

14·4·1. Ranges of cos and sin

We have seen that for all $x \in \mathbf{R}$, $-1 \leqslant \cos x \leqslant 1$, so that the range of cos is contained in the interval $[-1, +1]$. We shall now show that the range is actually this interval. To this end we sum show that every number in this interval is a value of the function.

Let p be any real number such that $-1 \leqslant p \leqslant 1$. Let us choose $q \geqslant 0$ such that $p^2+q^2=1$. The point P (p, q) lies on the unit circle.

Let θ be the measure of angle XOP (having OX as the initial side and OP as the terminal side) in radians. Then $\cos \theta = p$. [Fig. 14·1 (*a*) and 14·1 (*b*)].

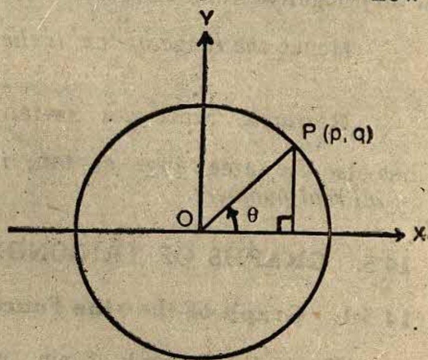


Fig. 14 1 (*a*)

Thus range of cos is $[-1, 1]$.

Corollary. *The range of the sin function is* $[-1, 1]$.

Proof. We already know that $-1 \leqslant \sin x \leqslant 1$ for all x. Also, $\sin(\pi/2-x)=\cos x$. Since every real number in $[-1, 1]$ is $\cos y$ for some y, it follows that every real number in $[-1, 1]$ is $\sin x$ for some x.

14·4·2. Range of tan and cot

Let a number $k \geqslant 0$ be given.

Then $0 \leqslant \dfrac{1}{\sqrt{(k^2+1)}} \leqslant 1$.

Since the range of cos is the interval $[-1, 1]$, therefore, we choose a real number x such that $0 \leqslant x < \pi/2$ and

$$\cos x = \frac{1}{\sqrt{(1+k^2)}}.$$

Since $\sin^2 x+\cos^2 x=1$, and for all x in $]0, \pi/2[$, $\sin x$ is non-negative, therefore, it follows that

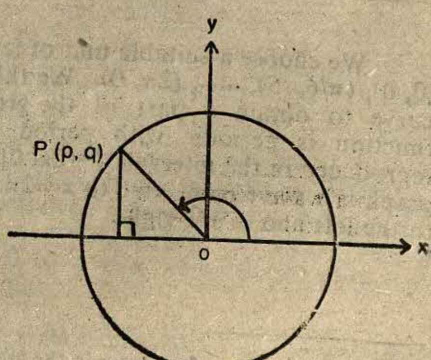


Fig. 14·1 (*b*)

$$\sin x = \frac{k}{\sqrt{(1+k^2)}}, \tan x = k.$$

Thus for every $k \geqslant 0$ we can find an $x \in]0, \pi/2[$ such that $\tan x = k$. Thus every non-negative real number is in the range of tan. Again, since $\tan(-x) = -\tan x$, therefore, it follows that every negative number is also in the range of tan.

Hence *the range of tan is the set* **R** *of all real numbers.*

Remark. Since $\cot x = \tan\left(\frac{\pi}{2} - x\right)$, it follows that cot has also the same range as tan, *i.e., the range of cot is the set* **R** *of all real numbers.*

14·5. GRAPHS OF TRIGONOMETRIC FUNCTIONS

14·5·1. Graph of the sine Function

To draw the graph of sin, we use the equation $y = \sin x$, x being the independent variable and y being the dependent variable. We give to x several values in the interval $[0, 2\pi]$ and find the corresponding values of y. The following table shows, correct to two places of decimals, the values of x from 0 to 2π by steps of $\pi/6$.

x	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π	$7\pi/6$	$4\pi/3$	$3\pi/2$	$5\pi/3$	$11\pi/6$	2π
$\sin x$	0	·5	·87	1	·87	·5	0	−·5	−·87	−1	−·87	−·5	0

We choose a suitable unit of length and then plot the points $(0, 0)$, $(\pi/6, \cdot 5)$, ..., $(2\pi, 0)$. We then join these points by a smooth curve to obtain a part of the graph (Fig. 14·2)*. Since the sine function is periodic with period 2π, therefore, this part can be reproduced in the intervals $[-2\pi, 0]$, $[2\pi, 4\pi]$, etc. The complete graph is a wave from $x=0$ to $x=2\pi$ repeated infinitely many times to the left and to the right.

*For the sake of neatness in the graph, the points are not shown in the graph.

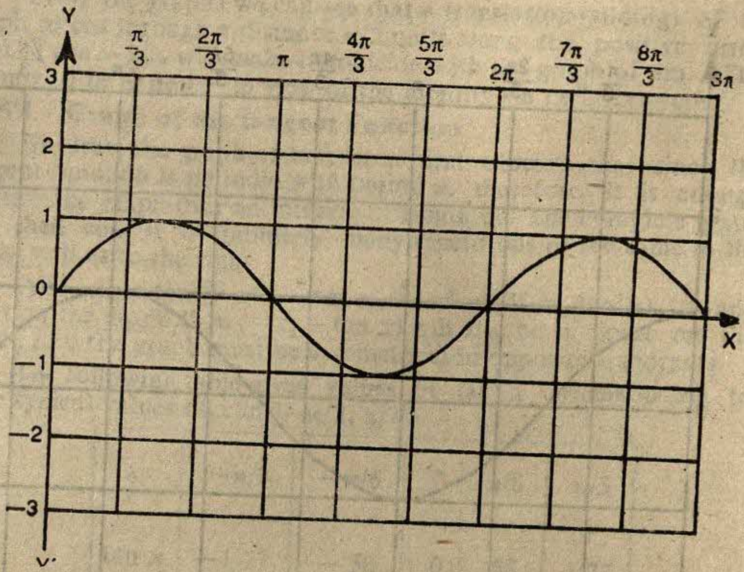


Fig. 14·2. Graph of sine (incomplete graph)

Remark. The graph of the sine function of angles can be drawn in a similar manner.

14·5·2. Graph of the cosine Function

To draw the graph of cos, we begin as in the case of sin, by preparing a table of values giving values of cos x as x varies from 0 to 2π in steps of $\pi/6$.

x	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π	$7\pi/6$	$4\pi/3$	$3\pi/2$	$5\pi/3$	$11\pi/6$	2π
cos x	1	·87	·5	0	−·5	−·87	−1	−·87	−·5	0	·5	·87	1

By plotting the points (0, 1), ..., (2π, 1) and joining them by a smooth curve, we obtain a part of the graph (Fig. 14·3). Since the cosine function is periodic with period 2π, therefore, this part can be reproduced in the interval [-2π, 0], [2π, 4π], etc. The complete graph is a wave from $x=0$ to $x=2\pi$ repeated infinitely many times to the left and to the right.

Fig. 14·4 shows the graphs of sin and cos drawn together for comparison.

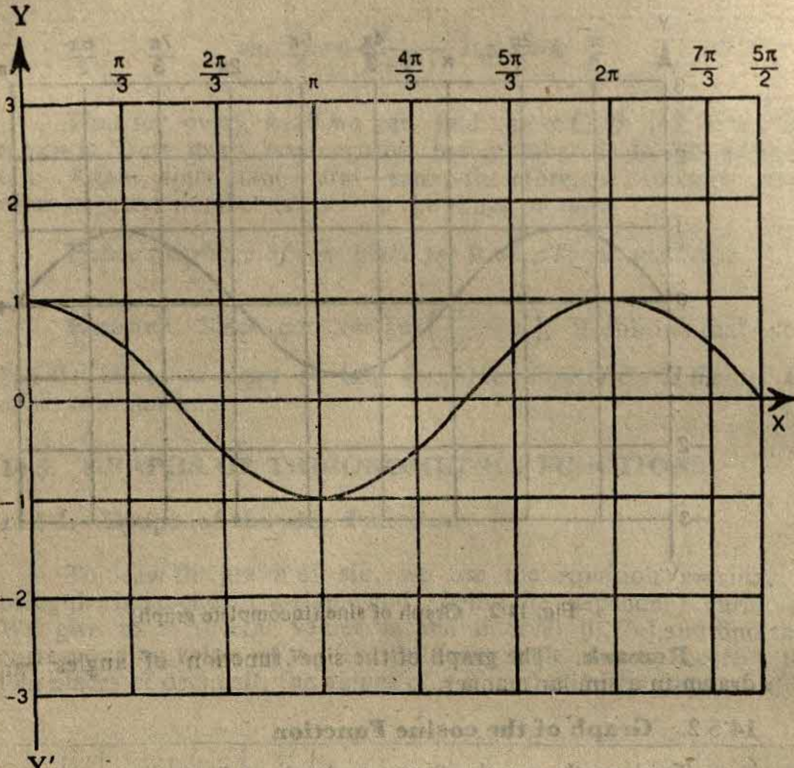


Fig. 14·3. Graph of cos (Incomplete graph)

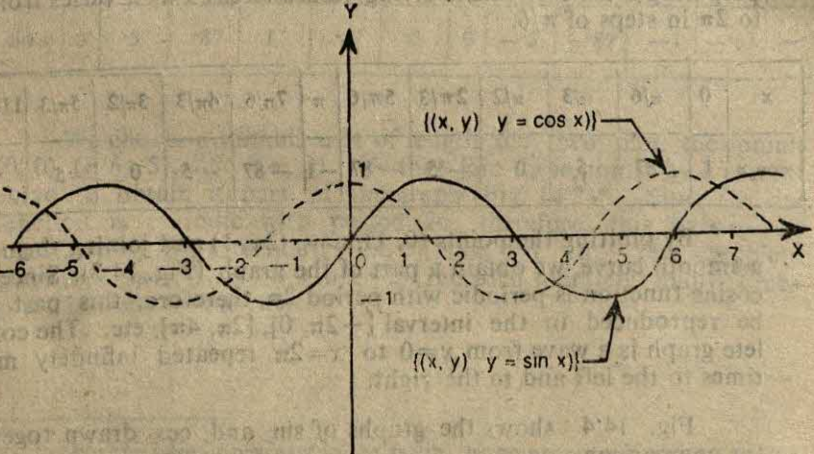


Fig. 14·4. Graphs of sin and cos (Incomplete graphs)

From the graphs we can see that a translation (sliding) of the graph of cos through a distance $\pi/2$ units along the positive direction of the x-axis will make it coincide with the graph of sin. This is only to be expected in view of the identity $\sin(\pi/2+x)=\cos x$.

14·5·3. Graph of the tangent Function

To draw the graph of tan, let us first observe that since the tangent function is periodic with period π, therefore, it is enough to draw the graph over an interval of length π. The complete graph will then consist of infinitely many repetitions of the same to the left as well as to the right.

Also, since $\tan(-x)=-\tan x$, therefore, if $(x, \tan x)$ be any point on the graph then $(-x, -\tan x)$ will also be a point on the graph, *i.e.*, the graph must be symmetrical in opposite quadrants.

The following table gives values of $\tan x$ corresponding to some typical values of x in $]-\pi/2, \pi/2[$.

x	$-\pi/3$	$-\pi/6$	0	$\pi/6$	$\pi/3$
$\tan x$	$-1·73$	$-·58$	0	·58	1·73

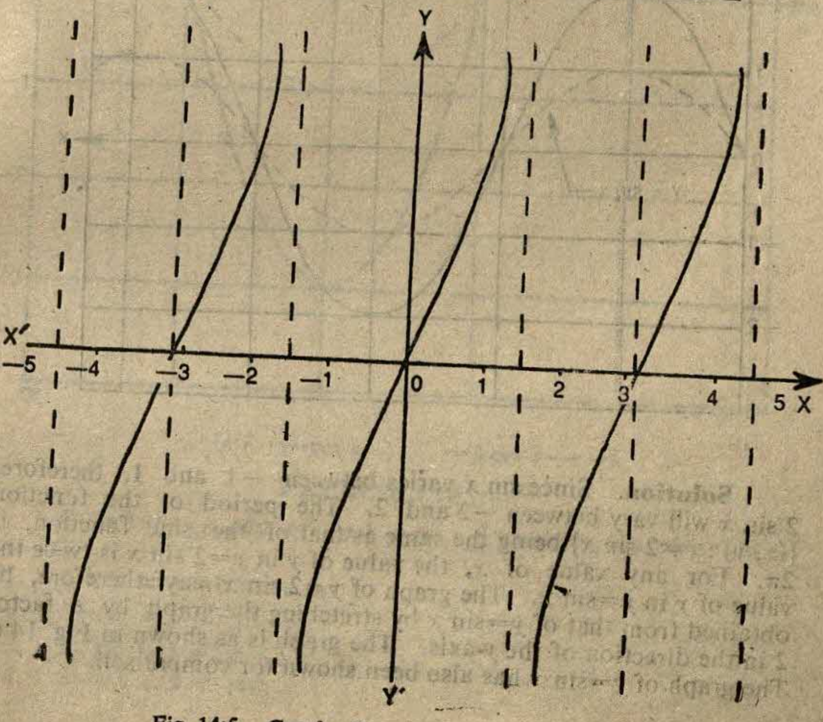


Fig. 14·5. Graph of tan (incomplete graph)

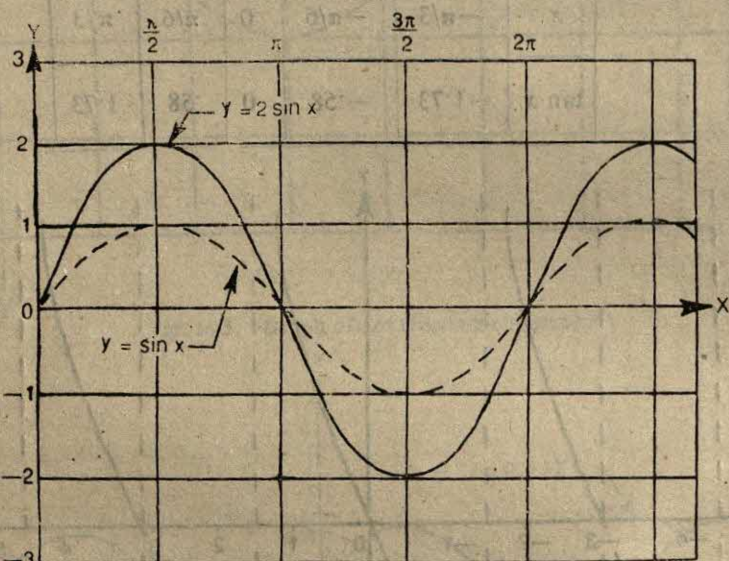
Since $\tan x$ is not defined when $x=\pm\ \pi/2$, therefore, there will be no points on the graph corresponding to these values of x. (By periodicity, there will be no points on the graph corresponding to $x=k\pi\pm\pi/2$, k being any integer). A part of the graph is as shown in Fig. 14·5.

Thoughout the entire domain, the graph rises from the left to the right. This is just the graphical demonstration of the fact that $\tan x$ is strictly increasing on the interval $]k\pi-\pi/2, k\pi+\pi/2[$, k being any integer.

14·5·4. Graphs of y=a sin x and y=a cos x

We have already drawn the graphs of the sine and the cosine functions. We now propose to consider graphs of more general functions.

Example 1. *Sketch the curve $y=2 \sin x$.*



Solution. Since $\sin x$ varies between -1 and 1, therefore, $2 \sin x$ will vary between -2 and 2. The period of the function $\{(x, y) : y=2 \sin x\}$ being the same as that of the sine function, is 2π. For any value of x, the value of y in $y=2 \sin x$ is twice the value of y in $y=\sin x$. The graph of $y=2 \sin x$ may, therefore, be obtained from that of $y=\sin x$ by stretching the graph by a factor 2 in the direction of the y-axis. The graph is as shown in Fig. 14·6. The graph of $y=\sin x$ has also been shown for comparison.

Example 2. *Sketch the curve $y=3 \cos x$.*

Solution. The function $\{(x, y) : y=3 \cos x\}$ is periodic with 2π as the period. The range of the function is $[-3, 3]$. The graph is obtained by stretching each ordinate of $y=\cos x$ by a multiple 3. A sketch is as shown in Fig. 14·7.

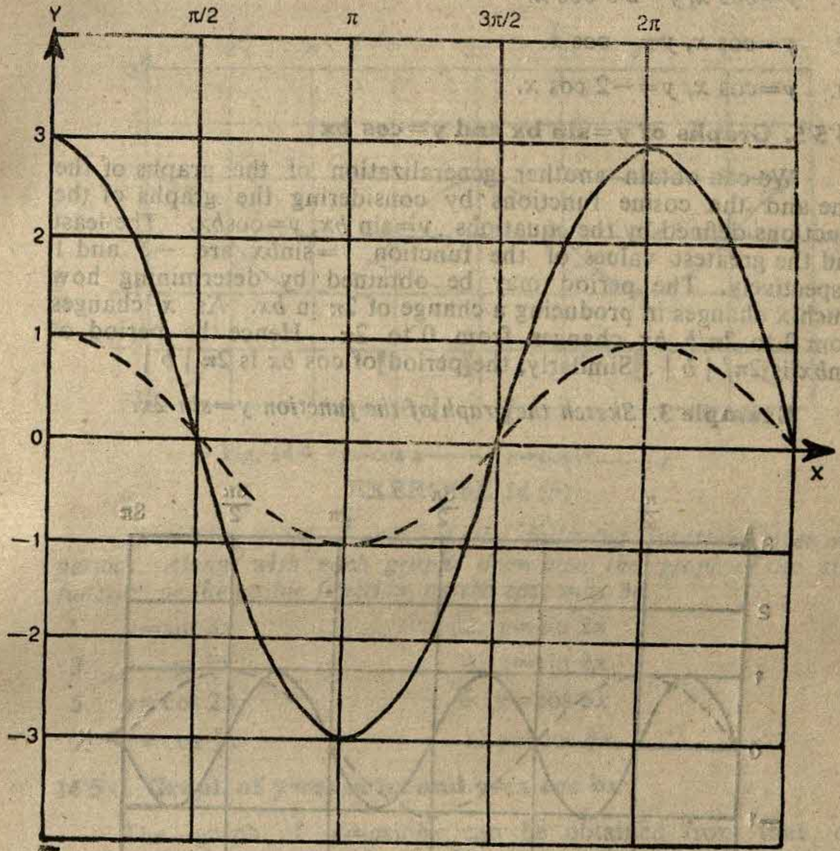


Fig. 14·7. ($y=\cos x$........., $y=3 \cos x$———).

EXERCISE 14 (*a*)

Sketch the graphs of each pair of equations in the interval $[0, 2\pi]$:

1. $y=\sin x$, $y=1·5 \sin x$.
2. $y=\sin x$, $y=3 \sin x$.
3. $y=\sin x$, $y=2·5 \sin x$.
4. $y=\sin x$, $y=-\sin x$.

5. $y=\sin x$, $y=-2\sin x$.

6. $y=\cos x$, $y=1{\cdot}5\cos x$.

7. $y=\cos x$, $y=2\cos x$.

8. $y=\cos x$, $y=2{\cdot}5\cos x$.

9. $y=\cos x$, $y=-\cos x$.

10. $y=\cos x$, $y=-2\cos x$.

14·5·5. Graphs of y=sin bx and y=cos bx

We can obtain another generalization of the graphs of the sine and the cosine functions by considering the graphs of the functions defined by the equations $y=\sin bx$, $y=\cos bx$. The least and the greatest values of the function $y=\sin bx$ are -1 and 1 respectively. The period may be obtained by determining how much x changes in producing a change of 2π in bx. As x changes isom 0 to $2\pi/b$, bx changes from 0 to 2π. Hence the period of frn bx is $2\pi/|b|$. Similarly, the period of cos bx is $2\pi/|b|$.

Example 3. *Sketch the graph of the function $y=\sin 2x$.*

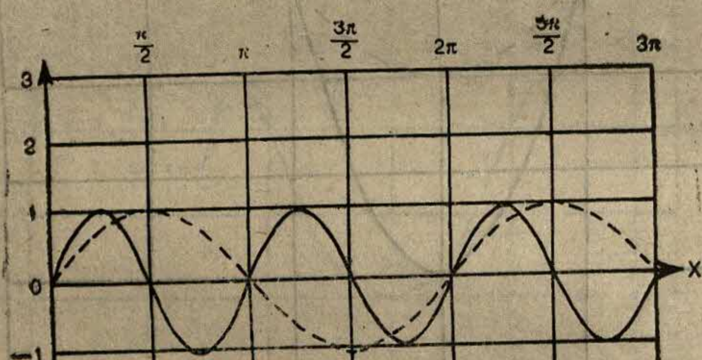


Fig. 14·8. ($y=\sin 2x$——, $y=\sin x$.........)

Solution. The least and the greatest values are -1 and 1 respectively. The period is $2\pi/2$, *i.e.*, π. The graph is similar to that of $\sin x$ and is obtained from it by compressing it by a factor 1/2 along the x-axis. The graph is as shown in Fig. 14·8.

Example 4. *Sketch the graph of the function $y=\cos\frac{1}{2}x$.*

Solution. The extreme values are -1 and 1. The period is $2\pi/\frac{1}{2}$, *i.e.*, 4π. The graph is similar to that of the cosine function and is obtained from it by stretching it along the x-axis by a factor 2. The graph is as shown in Fig. 14.9.

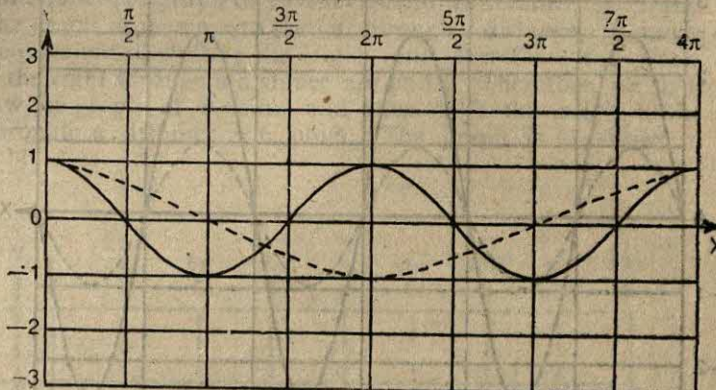


Fig. 14·9. ($y=\cos x$———, $y=\cos\frac{1}{2}x$.........)

EXERCISE 14 (*b*)

Sketch the graph of each of the following functions over one period. Along with each graph, draw also the graph of the sine function or the cosine function, as the case may be.

1. $y=\sin 3x$
2. $y=\sin 2x$
3. $y=\sin \frac{1}{2}x$
4. $y=\sin \frac{3}{2}x$
5. $y=\cos 2x$
6. $y=\cos 3x$
7. $y=\cos \frac{3}{2}x$
8. $y=\cos \frac{5}{2}x$.

14·5·6. Graph of $y=a\sin bx$ and $y=a\cos bx$

The graph of $y=a\sin bx$ can be obtained from that of $y=\sin x$ by first drawing the graph of $y=\sin bx$ (as in Example 3) and then stretching the graph of $y=\sin bx$ by a factor a along the y-axis (as in Example 1). The extreme values of $y=a\sin bx$ are $\pm a$, and its period is $2\pi/|b|$. The same is also true of the curve $y=a\cos bx$.

The following example will illustrate the method.

Example 5. *Sketch the curve $y=2{\cdot}5 \sin 2x$.*

Solution. We shall first draw the graph of $y=\sin 2x$, and then stretch the graph along the y-axis by a factor $2{\cdot}5$.

The graph is as shown in the Fig. 14·10.

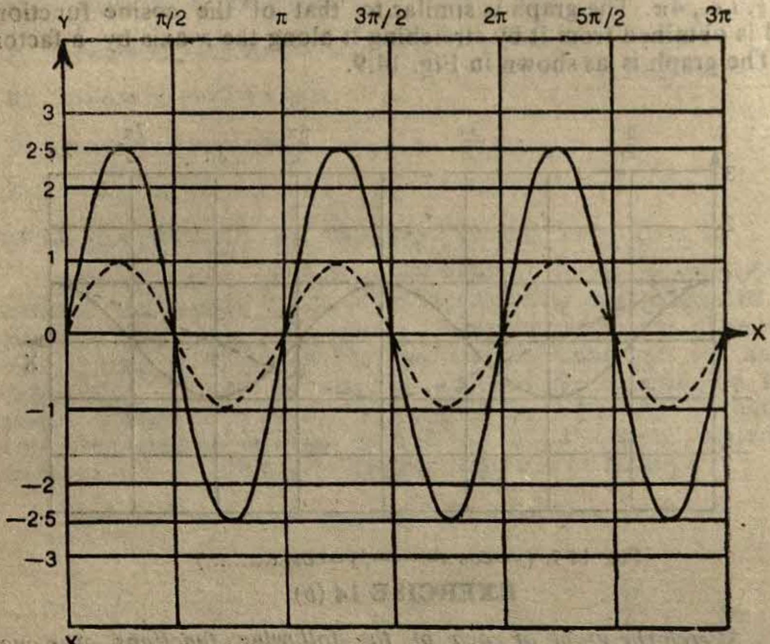


Fig. 14·10. ($y=2.5 \sin 2x$———, $y=\sin 2x$.........)

EXERCISE 14 (c)

Sketch the graph of each of the following curves over one period :

1. $y=2 \sin 2x$
2. $y=1·5 \sin 2x$
3. $y=2 \sin 3x$
4. $y=2·5 \sin 3x$
5. $y=1·5 \sin \frac{1}{2}x$
6. $y=3 \sin \frac{1}{2}x$
7. $y=2 \cos 3x$
8. $y=2 \cos \frac{1}{2}x$
9. $y=1·5 \cos 2x$
10. $y=2·5 \cos 2x$
11. $y=2·5 \cos 3x$
12. $y=3 \cos 2x$.

14·5·7. Graphs of y=sin (x+c) and y=cos (x+c)

The graph of $y=\sin (x+c)$ is similar to that of $y=\sin x$ and is obtained by translating it through a distance c units to the left

The extreme values are ± 1, and the period is 2π. The graph of $y=\cos(x+c)$ is similar to that of $y=\cos x$ and can be obtained from it by translating it to the left through a distance c units. The extreme values are again ± 1 and the period is 2π.

Example 6. *Sketch the graph of* $y=\sin(x+\pi/6)$.

Solution. The graph of $y=\sin(x+\pi/6)$ is obtained by translating the graph of $y=\sin x$ to the left through a distance $\pi/6$ units. This is equivalent to shifting the origin (and therefore, the y-axis also) to the right through a distance $\pi/6$ units. Therefore, we shall first draw the graph of $y=\sin x$ and then shift the origin to the right through a distance $\pi/6$ units. The graph is as shown in Fig. 14·11.

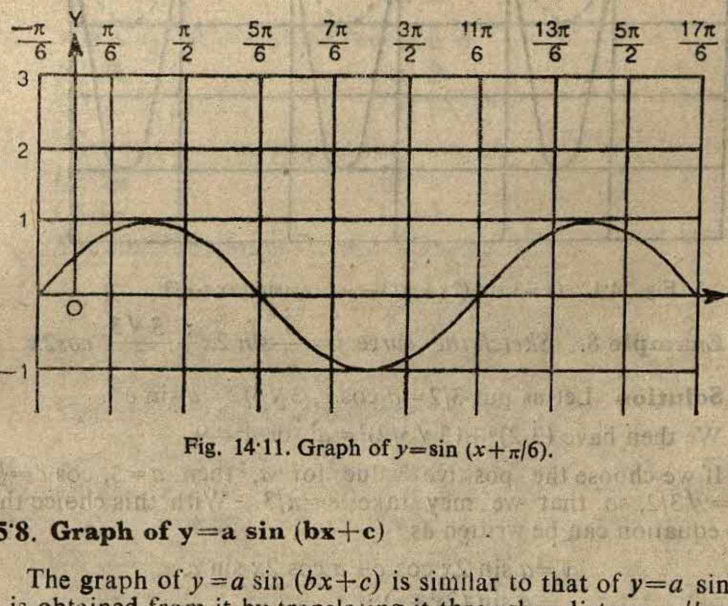


Fig. 14·11. Graph of $y=\sin(x+\pi/6)$.

14·5·8. Graph of y = a sin (bx + c)

The graph of $y=a\sin(bx+c)$ is similar to that of $y=a\sin bx$ and is obtained from it by translating it through a distance c/b units to the left (or equivalently, shifting the origin and the y-axis through a distance c/b units to the right). The following example will illustrate the method.

Example 7. *Sketch the curve* $y=3\sin(2x+\pi/3)$.

Solution. Since $3\sin(2x+\pi/3)=3\sin\{2(x+\pi/6)\}$, therefore, the graph is the same as that of $y=3\sin 2x$ with the origin shifted to the right through a distance $\pi/6$ units. The graph is shown in the Fig. 14·12.

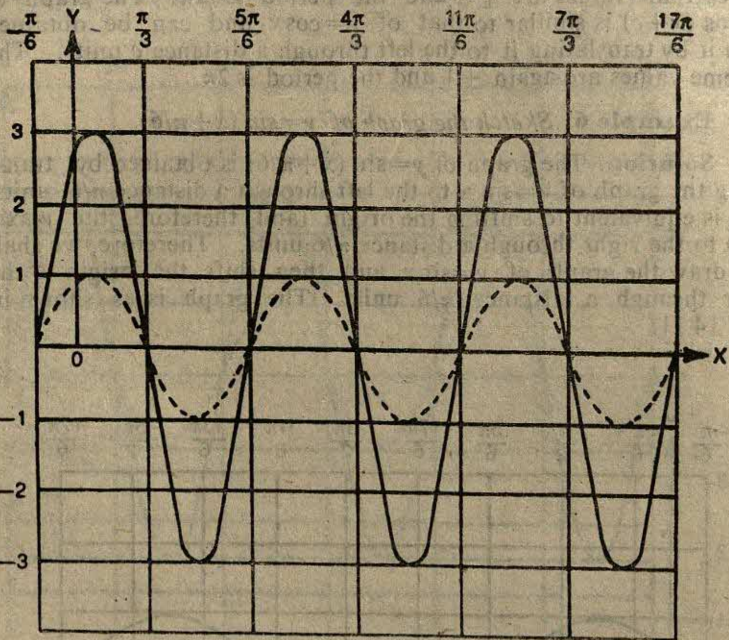


Fig. 14·12. ($y=3\sin(2x+\pi/3$— —), $y=\sin(2x+\pi/3$......).

Example 8. *Sketch the curve* $y=\frac{3}{2}\sin 2x+\frac{3\sqrt{3}}{2}\cos 2x.$

Solution. Let us put $3/2=a\cos c$, $3\sqrt{3}/2=a\sin c$.

We then have $(3/2)^2+(3\sqrt{3}/2)^2=a^2$, or $a^2=9$.

If we choose the positive value for a, then $a=3$, $\cos c=\frac{1}{2}$, $\sin c=\sqrt{3}/2$, so that we may take $c=\pi/3$. With this choice the given equation can be written as

$$\begin{aligned} y&=a\sin 2x\cos c+a\cos 2x\sin c,\\ &=a\sin(2x+c),\\ &=3\sin(2x+\pi/3). \end{aligned}$$

The equation being the same as in Example 7, the desired graph is as shown in Fig 14·12.

EXERCISE 14 (*d*)

Sketch the graph of each of the following curves :

1. $y=\sin(x+\pi/4)$.
2. $y=\sin(x-\pi/6)$.
3. $y=\cos(x+\pi/3)$.
4. $y=\cos(x-\pi/4)$.

5. $y=\sin(2x+\pi/4)$.
6. $y=\sin(2x-\pi/3)$.
7. $y=\cos(2x+\pi/3)$.
8. $y=\cos(3x-\pi/2)$.
9. $y=1\cdot5\cos(2x+\pi/3)$.
10. $y=2.5\sin(2x+\pi/3)$.
11. $y=3\cos(2x-\pi/3)$.
12. $y=2\sin(3x-3\pi/4)$.
13. $y=\cos x+\sqrt{3}\sin 2x$.
14. $y=4\cos 2x-3\sin 2x$.
15. $y=\sqrt{2}(\cos x+\sin x)$

TEST YOUR UNDERSTANDING XIV

In each of the following problems four alternatives are given. Put a tick-mark (✓) against the correct alternative.

1. The domain of the sine function is
   (*a*) **R** (*b*) **R**$-\{0\}$
   (*c*) $[-1, 1]$ (*d*) **Z**.

2. The range of the cosine function is
   (*a*) **R** (*b*) **Z**
   (*c*) $[-1, 1]$ (*d*) $[0, 1]$.

3. The period of the tangent function is
   (*a*) 2π (*b*) π
   (*c*) $\frac{1}{2}\pi$ (*d*) 1.

4. The sine function is strictly increasing in
   (*a*) $[0, \pi]$ (*b*) $\left[0, \frac{\pi}{2}\right]$
   (*c*) $\left[\frac{\pi}{2}, \pi\right]$ (*d*) $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$.

5. The cosine function is strictly decreasing in
   (*a*) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (*b*) $\left[\pi, \frac{3\pi}{2}\right]$
   (*c*) $\left[\frac{-\pi}{2}, 0\right]$ (*d*) $\left[0, \frac{\pi}{2}\right]$

6. The tangent function is increasing
   (*a*) in **R** (*b*) only in $]0, \infty[$
   (*c*) only in $]-\infty, 0[$ (*d*) nowhere.

7. The range of the cotangent function is
   (*a*) $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ (*b*) $[-1, 1]$
   (*c*) $[0, 1]$ (*d*) **R**.

8. The period of the function f defined by $f(x)=\sin x+\cos x$ for all $x \in$ **R** is

(a) 2π (b) π
(c) π^2 (d) 1.

9. The domain of the tangent function is
(a) $[-1, 1]$ (b) $\mathbf{R}$
(c) $[-\pi, \pi]$ (d) $]0, \infty[$.

10. The period of the function f defined by $f(x)=\sin 2x$ for all $x \in \mathbf{R}$ is
(a) 2π (b) $\pi/2$
(c) $\pi+2$ (d) π.

REVIEW EXERCISES XIV

Sketch the graphs of each pair of equations in the interval $[0, 2\pi]$:

1. $y=\sin x$, $y=2 \sin x$.
2. $y=\cos x$, $y=3 \cos x$.
3. $y=\cos x$, $y=-2\cdot5 \cos x$.

Sketch the graph of each of the following curves over one period. Alongwith each graph, draw also the graph of the sine function or the cosine function, as the case may be :

4. $y=\cos \frac{1}{2} x$. 5. $y=\sin\frac{3}{2}x$. 6 $y=\cos 4x$.

Sketch the graph of each of the following curves over one period :

7. $y=\frac{1}{2} \sin 2x$. 8. $y=3 \sin 2x$ 9. $y=-2 \cos 2x$.

10. Sketch the graph of $y=\sin\left(x+\dfrac{\pi}{3}\right)$ from $x=0$ to $x=2\pi$.

SUMMARY

1. The functions sin, cos, tan, cot, sec, cse are all periodic. The period of the functions sin, cos, sec and csc is 2π. The period of the functions tan and cot is π.
2. For each integer k, $\sin x$ strictly increases from -1 to $+1$ on the interval $[2k\pi-\pi/2, 2k\pi+\pi/2]$ and strictly decreases from $+1$ to -1 in the interval $[2k+\pi/2, 2k\pi+3\pi/2]$.
3. For each integer k, $\cos x$ strictly increases from -1 to $+1$ on the interval $[(2k-1)\pi, 2k\pi]$ and strictly decreases from $+1$ to -1 on the interval $[2k\pi, (2k+1)\pi]$.
4. For each integer k, $\tan x$ is strictly increasing on the interval $]k\pi-\pi/2, k\pi+\pi/2[$.
5. For each integer k, $\cot x$ is strictly decreasing on the interval $]k\pi, (k+1)\pi[$.
6. For each integer k, $\sec x$ is strictly decreasing on each of the intervals $]2k\pi-\pi/2, 2k\pi[$ and $]2k\pi+\pi, 2k\pi+3\pi/2[$ and is strictly increasing on each of the intervals $]2k\pi, 2k\pi+\pi/2[$ and $]2k\pi+\pi/2, 2k\pi+\pi[$.

7. For each integer k, csc x is strictly decreasing on each of the intervals $]2k\pi, 2k\pi+\pi/2[$ and $]2k\pi+3\pi/2, (2k+2)\pi[$, and is strictly increasing on each of the intervals $]2k\pi+\pi/2, 2k\pi+\pi[$ and $]2k\pi+\pi, 2k\pi+3\pi/2[$.

8. The range of sin and cos is $[-1, 1]$.

9. The range of sec and csc is $]-\infty, -1[\cup]1, \infty[$.

□ □

AUGUSTIN-LOUIS CAUCHY (1789-1857)

Augustin-Louis Cauchy was born in Paris on August 21, 1789. In 1811, Cauchy submitted his first Memoir on the theory of polyhedra. As a child he was taught French and Latin by his father. He showed extra-ordinary promise even at an early age As a boy of 22 he proved that the only regular polyhedra are those having 4, 6, 8, 12 or 20 faces.

In 1821, Cauchy published his course of lectures which he gave at the ecole Polytechnique. This is the work which set standards in rigorous mathematics.

Cauchy's mathematical productivity was incredible. He started two journals of his own for the publication of his expository and original work in pure and applied mathematics. These works were eagerly bought and studied and did much to reform mathematical taste before 1860. During the last 19 years of his life, Cauchy produced over 500 papers on all branches of mathematics including mechanics, physics and astronomy Many of these works were long treatises. In fact, Cauchy invited some criticism also for overproduction and hasty composition. His total output is 789 papers filling twenty-four large volumes However, his reputation as a mathematician has risen steadily. The methods he introduced, his whole programme inaugurating the first period of modern rigour and his unique originality have made an impact on mathematics that is destined to last for a long time to come.

He died rather unexpectedly in his sixty-eighth year on May 23, 1857.

CHAPTER 15

Properties of Triangles and Their Applications

15·1. INTRODUCTION

We shall apply the theory of trigonometric functions to discuss the properties of triangles, *i.e.*, the relations between sides, angles, circum-radius, etc. of a triangle. We shall use the properties obtained by us to solution of triangles and to simple problems on heights and distances.

Let ABC be a triangle. By the angle A we shall mean that angle between the sides AB and AC which lies between 0^c and 180°. The angles B and C are similarly defined. Thus we shall always have $A>0$, $B>0$, $C>0$ and $A+B+C=180°$. The sides BC, CA, AB opposite to the vertices A, B, C will be denoted by a, b, c respectively.

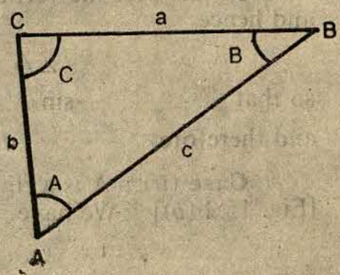


Fig. 15·1.

Thus we shall have

$$0 < a < b+c,$$
$$0 < b < c+a,$$
$$0 < c < a+b.$$

The length of the radius of the circumscribed circle (or the circum-circle) will be denoted by R. The area of the triangle will be denoted by $\triangle$ and the semi-perimeter by s, so that $2s=a+b+c$.

15·2. RELATIONS BETWEEN SIDES AND ANGLES OF A TRIANGLE

Theorem 15·1. (The Law of Sines). *In any triangle, sides are proportional to the sines of opposite angles* :

$$\boxed{\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}} \qquad ...(1)$$

Proof. Let us draw the circum-circle of the triangle ABC [Figs. 15·2 (*a*), 15·2 (*b*), 15·2 (*c*)]. Let us consider any vertex, say, A. We shall consider three cases according as the angle A is

(*i*) an acute angle, [Fig. 15·2 (*a*)]

(*ii*) a right angle, [Fig. 15·2 (*b*)]

or (*iii*) an obtuse angle. [Fig. 15·2 (*c*)]

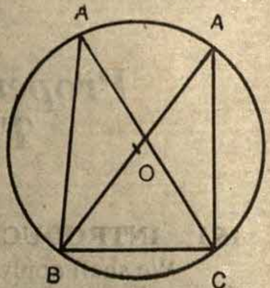


Fig. 15·2 (*a*)

Case (*i*) : A is acute. Through any other vertex, say B, draw the diameter BA′ of the circumscribed circle [Fig. 15·2 (*a*)]. Then the triangle A′BC is right-angled with the right angle at C. From this triangle we obtain $a=2R \sin A'$.

In this case the vertices A and A′ are on the same side of BC and hence

$$\angle A'=\angle A,$$

so that $$\sin A'=\sin A,$$

and therefore, $$a=2R \sin A.$$

Case (*ii*) : A is a right angle. In this case BC is a diameter [Fig. 15·2 (*b*)]. We have

$$a=2R.$$
$$=2R \sin A, \text{ as } \sin A=\sin 90^\circ=1.$$

Case (*iii*) : A is obtuse. Through any other vertex, say B, draw the diameter BA′ of the circumscribed circle [Fig. 15·2 (*c*)]. Then the triangle A′BC is right-angled with the right angle at C. From this triangle we obtain

$$a=2R \sin A'.$$

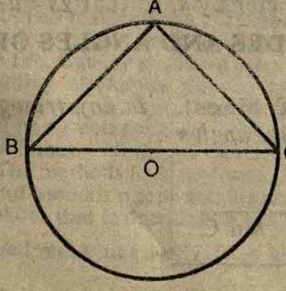


Fig. 15·2 (*b*)

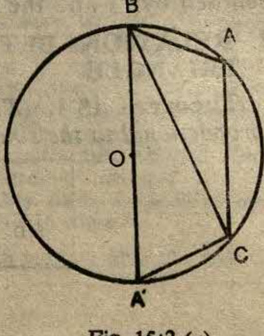


Fig. 15·2 (*c*)

In this case the vertices A and A′ are on the opposite sides of BC and hence

$$\angle A+\angle A'=180°,$$

so that $\sin A'=\sin(180°-A)=\sin A,$

and therefore $a=2R\sin A.$

Thus in all the three cases

$$a=2R\sin A. \qquad ...(2)$$

Similar relations can be obtained for other sides. Thus

$$a=2R\sin A,\ b=2R\sin B,\ c=2R\sin C,$$

which give

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R. \qquad ...(3)$$

Corollary. The circum-radius

$$R=\frac{a}{2\sin A}=\frac{b}{2\sin B}=\frac{c}{2\sin C} \qquad ...(3a)$$

Example 1. *Show that*

$$\begin{aligned} a&=b\cos C+c\cos B,\\ b&=c\cos A+a\cos C,\\ c&=a\cos B+b\cos A. \end{aligned} \qquad ...(4)$$

Proof. We have

$$\begin{aligned} a&=2R\sin A,\\ &=2R\sin(B+C), \text{ as } A+B+C=180°,\\ &=2R\sin B\cos C+2R\sin C\cos B,\\ &=b\cos C+c\cos B. \end{aligned}$$

The other two relations ore obtained in a similar manner.

Theorem 15·2 (*The Law of Cosines*). *In any triangle*

$$\boxed{\begin{aligned} a^2&=b^2+c^2-2bc\cos A,\\ b^2&=c^2+a^2-2ca\cos B,\\ c^2&=a^2+b^2-2ab\cos C. \end{aligned}} \qquad ...(5)$$

We shall prove only the first relation. Others follow in a similar manner.

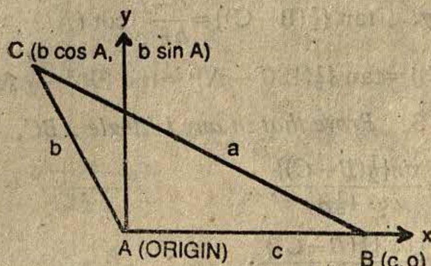


Fig. 15·3.

Proof. Let us suppose that the origin of a rectangular co-ordinate system is at A and AB is along the positive x-axis (Fig. 15·3). Then AC is the terminal side of the angle BAC$=\angle$A. AC is of length b. B has the coordinates $(c, 0)$ and C has the coordinates $(b \cos A, b \sin A)$. Therefore,

$$a^2 = BC^2 = (b \cos A - c)^2 + (b \sin A - 0)^2,$$
$$= b^2 (\cos^2 A + \sin^2 A) + c^2 - 2bc \cos A,$$
$$= b^2 + c^2 - 2bc \cos A.$$

Corollary. We have immediately from (5)

$$\boxed{\begin{aligned} \cos A &= \frac{b^2+c^2-a^2}{2bc}, \\ \cos B &= \frac{c^2+a^2-b^2}{2ca}, \\ \cos C &= \frac{a^2+b^2-c^2}{2ab}. \end{aligned}} \quad \ldots(6)$$

Example 2. *Show that in any triangle ABC*

$$\frac{\tan \{\frac{1}{2}(B-C)\}}{\tan \{\frac{1}{2}(B+C)\}} = \frac{b-c}{b+c}.$$

Solution. We have

$$\frac{b-c}{b+c} = \frac{2R (\sin B - \sin C)}{2R (\sin B + \sin C)}, \quad \ldots(7)$$
$$= \frac{2 \cos\{\frac{1}{2}(B+C)\} \sin \{\frac{1}{2}(B-C)\}}{2 \sin\{\frac{1}{2}(B+C)\} \cos \{\frac{1}{2}(B-C)\}}$$
$$= \frac{\tan\{\frac{1}{2}(B-C)\}}{\tan\{\frac{1}{2}(B+C)\}}.$$

Corollary. $\tan \{\frac{1}{2}(B-C)\} = \frac{b-c}{b+c} \cot (A/2),$

for $\tan \{\frac{1}{2}(B+C)\} = \tan \{\frac{1}{2}(180° - A)\} = \tan (90° - \frac{1}{2} A) = \cot \frac{1}{2} A.$

Example 3. *Prove that in any triangle ABC,*

(*i*) $\frac{b-c}{a} = \frac{\sin\{\frac{1}{2}(B-C)\}}{\cos (\frac{1}{2}A)},$

(*ii*) $\frac{b+c}{a} = \frac{\cos \{\frac{1}{2}(B-C)\}}{\sin (\frac{1}{2}A)}.$

Solution. We have

$$\frac{b-c}{a}=\frac{2R\,(\sin B-\sin C)}{2R\,\sin A},$$

$$=\frac{\sin B-\sin C}{\sin A},$$

$$=\frac{2\cos\{\tfrac{1}{2}(B+C)\}\sin\{\tfrac{1}{2}(B-C)\}}{2\sin(\tfrac{1}{2}A)\cos(\tfrac{1}{2}A)},$$

$$=\frac{\sin\tfrac{1}{2}(B-C)}{\cos(\tfrac{1}{2}A)},$$

since $\cos\tfrac{1}{2}(B+C)=\sin(\tfrac{1}{2}A).$

(*ii*) We have

$$\frac{b+c}{a}=\frac{2R\,(\sin B+\sin C)}{2R\,\sin A},$$

$$=\frac{\sin B+\sin C}{\sin A},$$

$$=\frac{2\sin\{\tfrac{1}{2}(B+C)\}\cos\{\tfrac{1}{2}(B-C)\}}{2\sin(\tfrac{1}{2}A)\cos(\tfrac{1}{2}A)},$$

$$=\frac{\cos\{\tfrac{1}{2}(B-C)\}}{\sin(\tfrac{1}{2}A)},$$

since $\sin\tfrac{1}{2}(B+C)=\cos\tfrac{1}{2}A.$

Note. The above formulae are very useful in checking solutions of triangles, since all six parts of a triangle appear in each formula.

Half-angle Formulae. We have

$$\cos A=\frac{b^2+c^2-a^2}{2bc},$$

and $$\cos A=2\cos^2\left(\frac{A}{2}\right)-1.$$

Hence

$$2\cos^2\tfrac{1}{2}A=1+\frac{b^2+c^2-a^2}{2bc},$$

$$=\frac{(b+c)^2-a^2}{2bc},$$

$$=\frac{(b+c+a)(b+c-a)}{2bc},$$

$$=\frac{2s \cdot 2(s-a)}{2bc}.$$

Therefore,

$$\cos(A/2)=\pm\sqrt{\left\{\frac{s(s-a)}{bc}\right\}}.$$

Since $0 < A < 180°$, therefore $0 < A/2 < 90°$, and consequently $\cos(A/2)$ is positive. We must, therefore, take the positive sign before the radical.

Hence $\cos(A/2)=\sqrt{\left\{\frac{s(s-a)}{bc}\right\}}$. ...(8a)

Similarly, it can be shown that

$$\cos(B/2)=\sqrt{\left\{\frac{s(s-b)}{ca}\right\}}, \quad \text{...(8b)}$$

and $$\cos(C/2)=\sqrt{\left\{\frac{s(s-c)}{ab}\right\}}. \quad \text{...(8c)}$$

Again, from

$$\cos A=1-2\sin^2(A/2),$$

we get

$$2\sin^2(A/2)=1-\frac{b^2+c^2-a^2}{2bc},$$

$$=\frac{a^2-(b-c)^2}{2bc},$$

$$=\frac{(a-b+c)(a+b-c)}{2bc},$$

$$=\frac{2(s-b).2(s-c)}{2bc}.$$

Therefore,

$$\sin(A/2)=\pm\sqrt{\left\{\frac{(s-b)(s-c)}{bc}\right\}}.$$

Since $0 < A < 180°$, therefore, $0 < A/2 < 90°$ and consequently $\sin(A/2)$ is positive. We must, therefore, take the positive sign before the radical.

Hence $\sin (A/2)=\sqrt{\left\{\dfrac{(s-b)(s-c)}{bc}\right\}}.$ ...(9*a*)

Similarly

$$\sin (B/2)=\sqrt{\left\{\frac{(s-c)(s-a)}{ca}\right\}}, \qquad \text{...(9}b\text{)}$$

and

$$\sin (C/2)=\sqrt{\left\{\frac{(s-a)(s-b)}{ab}\right\}}. \qquad \text{...(9}c\text{)}$$

From formulae (8*a*, 8*b*, 8*c*) and (9*a*, 9*b*, 9*c*) we obtain, by division

$$\boxed{\begin{aligned}\tan (A/2)&=\sqrt{\left\{\frac{(s-b)(s-c)}{s(s-a)}\right\}},\\ \tan (B/2)&=\sqrt{\left\{\frac{(s-c)(s-a)}{s(s-b)}\right\}},\\ \tan (C/2)&=\sqrt{\left\{\frac{(s-a)(s-b)}{s(s-c)}\right\}}.\end{aligned}} \qquad \text{...(10)}$$

Corollary. $\sin A=\dfrac{2}{bc}\sqrt{\{s(s-a)(s-b)(s-c)\}},$ ...(11)

and two similar relations.

We have

$$\sin A=2 \sin (A/2) \cos (A/2),$$

$$=2.\sqrt{\left\{\frac{(s-b)(s-c)}{bc}\right\}}\cdot\sqrt{\left\{\frac{s(s-a)}{bc}\right\}}, \text{ by}$$

(8*a*) and (9*a*)

$$=\frac{2}{bc}\sqrt{\left\{s(s-a)(s-b)(s-c)\right\}}.$$

Expressions for sin B and sin C are obtained in a similar manner.

Example 4. *If the sides of a triangle are in A.P., prove that the cotangents of half the angles are also in A.P.*

Solution. We are given that

$$a+c=2b,$$

i.e., $$2s-b=2b, \qquad \ldots(i)$$

and we are required to prove that

$$\cot\left(\frac{A}{2}\right)+\cot\left(\frac{C}{2}\right)=2\cot\left(\frac{B}{2}\right). \qquad \ldots(ii)$$

Now, $$\cot\frac{A}{2}+\cot\frac{C}{2}=\frac{\cos\left(\frac{A}{2}\right)\sin\left(\frac{C}{2}\right)+\cos\left(\frac{C}{2}\right)\sin\left(\frac{A}{2}\right)}{\sin\left(\frac{A}{2}\right)\sin\left(\frac{C}{2}\right)},$$

$$=\frac{\sin\left(\frac{A+C}{2}\right)}{\sin\left(\frac{A}{2}\right)\sin\left(\frac{C}{2}\right)},$$

$$=\frac{\cos\left(\frac{B}{2}\right)}{\sin\left(\frac{A}{2}\right)\sin\left(\frac{C}{2}\right)},$$

$$=2\cot\left(\frac{B}{2}\right)\left(\frac{1}{2}\cdot\frac{\sin\frac{B}{2}}{\sin\left(\frac{A}{2}\right)\sin\left(\frac{C}{2}\right)}\right) \qquad \ldots(iii)$$

Again, $$\frac{1}{2}\cdot\frac{\sin\left(\frac{B}{2}\right)}{\sin\left(\frac{A}{2}\right)\sin\left(\frac{C}{2}\right)}$$

$$=\frac{1}{2}\sqrt{\left\{\frac{(s-c)(s-a)}{ca}\right\}}\sqrt{\left\{\frac{bc}{(s-b)(s-c)}\right\}}\cdot\sqrt{\left\{\frac{ab}{(s-a)(s-b)}\right\}},$$

$$=\frac{1}{2}\sqrt{\left\{\frac{b^2}{(s-b)^2}\right\}}=1, \text{ by } (i).$$

(*iii*) now becomes

$$\cot\frac{A}{2}+\cot\frac{C}{2}=2\cot\frac{B}{2}.$$

15·3. AREA OF A TRIANGLE

The area Δ of a triangle ABC=(1/2) base × altitude.

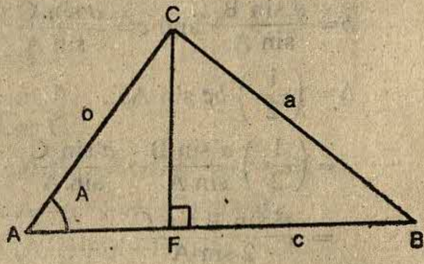


Fig. 15·4.

In the triangle ABC, there must be at least two acute angles. Let us suppose that A and B are the acute angles. Let us draw the perpendicular CF from C to AB (Fig. 15·4).

Then the altitude (to the base AB)

$$=\text{CF},$$

$$=b \sin \text{A}.$$

Therefore, $\Delta=(\frac{1}{2})\, c\times b \sin \text{A}$,

$$=(\tfrac{1}{2})\, bc \sin \text{A}.$$

Now we have from the Law of Sines

$$\frac{a}{\sin \text{A}}=\frac{b}{\sin \text{B}}=\frac{c}{\sin \text{C}},$$

whence

$$bc \sin \text{A}=ca \sin \text{B}=ab \sin \text{C}.$$

Thus $\Delta=(1/2)\, bc \sin \text{A}=(1/2)\, ca \sin \text{B}=(1/2)\, ab \sin \text{C}$, ...(*i*) so that the area of a triangle is equal to half the rectangle contained by two adjacent sides multiplied by the sine of the included angle.

Hence, from (11)

$$\boxed{\Delta=\sqrt{\{s\,(s-a)\,(s-b)\,(s-c)\}}.} \qquad \cdots(i)$$

Example 5. *Show that in any triangle ABC,*

$$\Delta=\frac{a^2 \sin B \sin C}{2 \sin (B+C)}.$$

Use the above result to show that if $B=45^\circ$, $C=60^\circ$, and $a=2(\sqrt{3}+1)$ cm, then the area of the triangle ABC is $6+2\sqrt{3}$ sq. cm.

Solution. Since $\dfrac{a}{\sin \text{A}}=\dfrac{b}{\sin \text{B}}=\dfrac{c}{\sin \text{C}}$,

therefore $$b=\frac{a \sin B}{\sin A} \text{ and } c=\frac{a \sin C}{\sin A}.$$

Hence $$\Delta=\left(\frac{1}{2}\right) bc \sin A,$$

$$=\left(\frac{1}{2}\right)\frac{a \sin B}{\sin A}\cdot\frac{a \sin C}{\sin A}\cdot \sin A,$$

$$=\frac{a^2 \sin B \sin C}{2 \sin A},$$

$$=\frac{a^2 \sin B \sin C}{2 \sin (B+C)}. \qquad ...(1)$$

Substituting $B=45°$, $C=60°$, and $a=2(\sqrt{3}+1)$ cm, in (1), we have

$$\Delta=\frac{\{2(\sqrt{3}+1)\}^2 \sin 45° \sin 60°}{2 \sin (45°+60°)} \text{ sq. cm},$$

$$=\frac{4\left(\sqrt{3}+1\right)^2\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}}{2\left(\frac{1}{\sqrt{2}}\cdot\frac{1}{2}+\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}\right)} \text{ sq. cm},$$

$$=2\sqrt{3}(\sqrt{3}+1) \text{ sq. cm},$$

$$=6+2\sqrt{3} \text{ sq. cm}.$$

Example 6. *In any triangle ABC, prove that*

$$\Delta=2R^2 \sin A \sin B \sin C.$$

Solution. Since $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R,$

therefore, $$a=2R \sin A,\ b=2R \sin B,\ c=2R \sin C.$$

Now $$\Delta=\left(\frac{1}{2}\right) bc \sin A,$$

$$=\left(\frac{1}{2}\right) 2R \sin B.\ 2R \sin C.\ \sin A,$$

$$=2R^2 \sin A \sin B \sin C.$$

EXERCISE 15 (*a*)

In any triangle ABC, prove that

1. $a \cos A+b \cos B+c \cos C=4R \sin A \sin B \sin C.$
2. $a \sin A-b \sin B=c \sin (A-B).$
3. $a^2 \sin (B-C)=(b^2-c^2) \sin A.$
4. $a \cos A+b \cos B+c \cos C=2a \sin B \sin C.$

5. $a \sin (B-C)+b \sin (C-A)+c \sin (A-B)=0$.

6. $c=a \cos B+b \cos A$.

7. $\dfrac{c-b \cos A}{b-c \cos A}=\dfrac{\cos B}{\cos C}$.

8. $2(bc \cos A+ca \cos B+ab \cos C)=a^2+b^2+c^2$.

9. $(b^2+c^2-a^2) \tan A=(c^2+a^2-b^2) \tan B$.

10. $a^2=(b-c)^2 \cos^2 (A/2)+(b+c)^2 \sin^2 (A/2)$.

11. $c^2=(a+b)^2-4ab \cos^2 (C/2)$.

12. $(b^2-c^2) \cot A+(c^2-a^2) \cot B+(a^2-b^2) \cot C=0$.

13. If $C=60°$, then

$$\frac{1}{a+c}+\frac{1}{b+c}=\frac{3}{a+b+c}.$$

14. $\tan \{(1/2)(A-B)\}=\dfrac{a-b}{a+b} \cot (C/2)$.

15. $\dfrac{c+a}{c-a}=\tan \dfrac{C+A}{2} \cdot \cot \dfrac{C-A}{2}$.

16. $\sin (B/2)=\sqrt{\left\{\dfrac{(s-c)(s-a)}{ca}\right\}}$.

17. $\sin (C/2)=\sqrt{\left\{\dfrac{(s-a)(s-b)}{ab}\right\}}$.

18. $\cot (A/2)+\cot (B/2)+\cot (C/2)=\dfrac{a+b+c}{a+b-c} \cot (C/2)$.

19. $(b-c) \cot (A/2)+(c-a) \cot (B/2)+(a-b) \cot (C/2)=0$.

20. $\dfrac{a+b+c}{a-b+c}=\cot (A/2) \cot (C/2)$.

21. $(b+c-a)\{\cot (B/2)+\cot (C/2)\}=2a \cot (A/2)$.

22. If $\cot (C/2)=\dfrac{a+b}{c}$, the triangle is right-angled.

23. If $a \cos A=b \cos B$, then the triangle is either isosceles or right-angled.

24. If $b+c=3a$, then $\cot (B/2) \cot (C/2)=2$.

25. If $3 \tan (A/2) \tan (C/2)=1$, then a, b, c are in A.P.

26. Deduce the formulae (5) of Article 15.2 from the formulae (4) of Art. 15.2.

27. Find the area of the triangle ABC when

(i) $a=5$ cm, $b=12$ cm, and $c=13$ cm.

(ii) $a=14$ cm, $b=48$ cm, and $c=50$ cm.

(*iii*) $a=45$ cm, $b=108$ cm, and $c=117$ cm.

(*iv*) $a=\sqrt{3}\ m$, $b=\sqrt{2}\ m$, and $c=\dfrac{\sqrt{6}+\sqrt{2}}{2}m$.

(*v*) $a=10$ cm, $b=12$ cm, and $C=30°$.

28. If one angle of a triangle be 60°, the area be $10\sqrt{3}$ sq. cm and the length of one of the sides containing the angle be 8 cm, find the length of the other side.

29. In any triangle ABC, prove that

(*i*) $s(s-c)\tan(C/2)=\Delta$.

(*ii*) $4\Delta\cot A=b^2+c^2-a^2$.

(*iii*) $\Delta=\dfrac{c^2\sin A\sin B}{2\sin(A+B)}$.

(*iv*) $abc.\,s.\sin(A/2)\sin(B/2)\sin(C/2)=\Delta^2$.

15·4. SOLUTION OF TRIANGLES

The three sides and the three angles of a triangle are called the parts of the triangle. Given the measures of three parts, at least one of which is a side, we can compute the possible measures of the remaining parts. The process of computing the measures of the unknown parts from those of the given parts is known as solving a triangle. Since we are going to be interested only in the measures of the parts of a triangle, therefore, we shall not distinguish between congruent triangles. Also, we shall use the phrase 'a part of a triangle' to mean the measure, of that part. We shall say that three given parts determine a triangle if there exists a triangle with given parts and if all the other parts can be computed uniquely by means of the given parts. Thus, for example, the three sides determine a triangle provided no side is as great as the sum of the other two ; three angles do not determine a triangle because if the sum of the angles be different from 180°, then no triangle exists with the given angles as parts, and if the sum of the angles be 180°, then infinitely many triangles exist with the given angles as parts.

Solution of triangles has lots of applications in navigation, astronomy, surveying, architecture, engineering, and in many other sciences. In astronomy, the radius of the earth, the radius of the moon and the distance of the moon from the earth are computed by the process of solving triangles. In surveying (and elsewhere too) heights and distances which cannot be measured directly are computed by solving triangles. In fact, till the beginning of the present century, the study of trigonometry was motivated solely by its utility in solving triangles. (Unfortunately, most text-books still cling to this practice !). In modern times, however, the trigonometric functions have found important applications in numerous other fields such as simple harmonic motion, the theory of vibrating strings,

Fourier series, etc. and this has led the mathematicians to emphasize the analytical aspect of trigonometry more than the computational aspect. Even then, solution of triangles continues to be sufficiently important.

In the following sections we propose to discuss various methods of solving triangles and their elementary applications to heights and distances, navigation, etc. We shall divide our study into three parts: solution of right-angled triangles, solution of oblique triangles, and applications. However, we shall first set out some general directions which will prove useful while solving triangles.

15 5. SOME GENERAL DIRECTIONS

The following points should be remembered while solving a triangle :

1. By using the given parts, construct a triangle roughly to scale. Mark the known parts of the triangle and indicate the unknown parts. A proper figure helps in having a clearer understanding of the problem and a better planning of the solution. It guides us very much in certain cases (for example in case IV, p. 521). The figure also gives a rough estimate of the parts required to be computed and at the same time it serves as a check against gross errors.

2. While solving triangles, we must remember that the given parts are generally the result of certain measurements and, as such, are only approximate quantities. Also, the computed parts cannot be any more accurate than the given parts. We have therefore, to see as to what degree of accuracy in the computed parts can possibly be there when the given parts have a certain degree of accuracy. A computed side can be accurate to at the most as many significant digits as the given side or sides contain. Therefore, we should always round off a computed side to as many significant digits as the given side or sides contain. Similarly, a computed angle can be accurate to the nearest degree, nearest multiple of $10'$, or nearest minute at the most, according as the given angle or angles are accurate to the nearest degree, the nearest multiple of $10'$ or nearest minute.

   Since in any problem angles and sides are involved together, therefore, we should know as to what accuracy in angles is comparable to a given accuracy in the sides. The following table gives the accuracy in an angle (or angles) corresponding to a given accuracy in a side (or sides).

Sides	Angles
Two significant digits	Nearest degree
Three significant digits	Nearest multiple of $10'$
Four significant digits	Nearest minute
Five significant digits	Nearest tenth of a minute

According to the preceding table, which we shall adhere to, a computed angle should be rounded off to the nearest degree, nearest multiple of 10′, nearest minute or nearest tenth of a minute according as the sides have two, three, four or five significant digits. We shall throughout assume that the given parts of a triangle to be solved are of corresponding accuracies.

3. Solution of triangles involves a lot of multiplications and divisions which may be done either without any computational aids, or we may choose to use a slide rule, logarithmic tables or calculators/computers to perform the desired computations. If the results are required only to two significant digits, then we need not use any computational aids. If the results are required to not more than three significant digits, we may use the slide rule. If a higher degree of accuracy is desired, then we may use logarithm tables. Calculators/computers give most accurate results, but we may not have access to them and therefore, we shall not discuss their use for solution of triangles.

   The slide rule can also be used as a check on the working when other methods are used for computation, unless of course the answers are to be more accurate than the slide rule can give.

4. For each part to be computed, choose a formula which determines that part. The choice of the formula will generally depend on the mode of computation. Different formulae may be needed according as we wish to use logarithms or we wish to do without them.

5. If is always better, if possible, to find a part from the given parts rather than from the computed parts. This gives better results and also avoids carrying forward any error in computation.

6. It is extremely important to check the results in problems on solution of triangles as elsewhere too. While a figure to scale serves as a rough check and the use of the slide rule gives a check when a high degree of accuracy is not desired, the solution can always be checked either by computing a part by two different methods or by substituting the parts in a formula which holds for triangles and which has not been used in the process of solution. While using the latter method for checking the solutions, the following formulae are particularly useful :

   (*i*) $\dfrac{a+b}{c}=\dfrac{\cos\frac{1}{2}(A-B)}{\sin\frac{1}{2}C}$, (*ii*) $\dfrac{a-b}{c}=\dfrac{\sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}C}$.

   The above fromulae, known as Molleweide's formulae, have the redeeming feature that each of them involves all the six parts of a triangle. (The derivation of these formulae is left as an exercise for the reader).

7. The working should be arranged in a neat and systematic manner. A completed outline of the working should be prepared before doing any computations.

15·6. SOLUTION OF RIGHT-ANGLED TRIANGLES

If one of the angles of a triangle is given to be right angle, then the triangle can be solved easily without having recourse to the formulae derived earlier.

Suppose it is known that in a triangle ABC, $\angle C=90°$. Let us also suppose that two other parts are given, at least one of them being a side. Then the following different cases arise :

Case I : Given a and b. The solution is unique.

We use the formulae

$\tan A=a/b$, to find A ;

$B=90°-A$, to find B ;

$c=b/\cos A$, to find c

or $c=\sqrt{(a^2+b^2)}$, to find c. (Use whichever is simpler.)

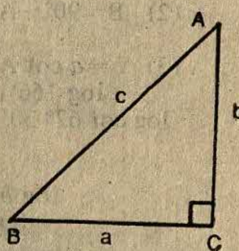


Fig. 15·5.

Case II : Given a and c. There is no solution unless $c > a$.

If $c > a$, there is a unique solution.

We use the formulae

$\sin A=a/c$, to find A ;

$B=90°-A$, to find B ;

$b=a \cot A$ to find b,

or $b=\sqrt{(c^2-a^2)}$, to find b. (Use whichever is simpler.)

Case III : Given a and A. There is no solution unless A is acute. If A is acute, there is a unique solution.

We use the formulae

$B=90°-A$, to find B ;

$b=a \cot A$, to find b ;

$c=a/\sin A$, to find c,

or $c=\sqrt{(a^2+b^2)}$, to find c. (Use whichever is simpler).

Case IV : Given c and A. There is no solution unless A is acute. If A acute, there is a unique solution.

We use the formulae

$B=90°-A$, to find B ;

$a=c \sin A$, to find a ;

$b=c \cos A$, to find b,

or $b=\sqrt{(c^2-a^2)}$, to find b. (Use whichever is simpler.)

Example 7. *Given $a=166{\cdot}1$, $c=187{\cdot}3$, $C=90°$, solve the triangle.*

Solution.

(1) $\sin A = \dfrac{a}{c} = \dfrac{166{\cdot}1}{187{\cdot}3}$.

Preparing an outline of the logarithmic solution and filling in the numbers we have

$\log 166{\cdot}1 = 2{\cdot}2204$

$\log 187{\cdot}3 = 2{\cdot}2725$ S

$\log \sin A = 9{\cdot}9479 - 10$

$A = 62°30'$.

(2) $B = 90° - A = 90° - 62°\,30'$,

$= 27°\,30'$.

(3) $b = a \cot A = 166{\cdot}1 \cot 62°\,30'$

$\log 166{\cdot}1 = 2{\cdot}2204$

$\log \cot 62°\,30' = 9{\cdot}7165 - 10$ A

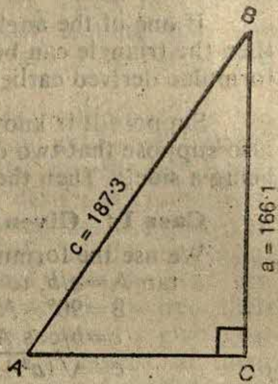


Fig. 15·6.

$\log b = 1{\cdot}9369$

$b = 86{\cdot}48$.

The computed parts are

$A = 62°\,30'$, $B = 27°\,30'$, $b = 86{\cdot}48$.

Check. To check the solution we may compute b by the formula $b = c \cos A = 187{\cdot}3 \cos 62°\,30'$.

$\log 187{\cdot}3 = 2{\cdot}2725$

$\log \cos 62°\,30' = 9{\cdot}6644 - 10$ A

$\log b = 1{\cdot}9369$

$b = 86{\cdot}48$

which agrees with the value of b found by another method.

15·7. SOLUTION OF OBLIQUE TRIANGLES

A triangle, none of whose angles is a right angle, is called an oblique triangle. Since the ratio of two sides of an oblique triangle does not represent any circular function of an angle of triangle, therefore, certain formulae are needed for solving oblique triangles. The formulae needed are the cosine formulae, the sine formulae, Napier's formulae and the half-angle formulae, all of which have been established earlier.

If three parts, at least one of which is a side, be given, the triangle can be solved. The various possibilities can be divided into the following four cases :

(*i*) Given three sides (SSS).

(*ii*) Given two sides and the included angle (SAS).

(*iii*) Given two sides and the angle opposite to one of them (SSA).

(*iv*) Given one side and two angles (SAA).

We shall now discuss each of these cases separately.

15·8. CASE I. THREE SIDES GIVEN (SSS)

To solve a triangle ABC when a, b and c are given. There will be no solution unless the sum of each pair of sides exceeds the remaining side. There will be a unique solution if the sum of each pair of sides is greater than the third. The angles can be found by using the cosine formulae :

$$\cos A=\frac{b^2+c^2-a^2}{2bc},$$

$$\cos B=\frac{c^2+a^2-b^2}{2ca},$$

$$\cos C=\frac{a^2+b^2-c^2}{2ab}$$

The above formulae are, however, not convenient and therefore, used only if the solution is to be obtained without the aid of logarithms.

For logarithmic computation, the following half-angle formulae are useful :

$$\tan\frac{A}{2}=\sqrt{\left\{\frac{(s-b)(s-c)}{s(s-a)}\right\}},$$

$$\tan\frac{B}{2}=\sqrt{\left\{\frac{(s-c)(s-a)}{s(s-b)}\right\}},$$

$$\tan\frac{C}{2}=\sqrt{\left\{\frac{(s-a)(s-b)}{s(s-c)}\right\}}.$$

We can determine A/2, B/2, C/2 from the above formulae and A, B, C can be determined therefrom. In practice, however, it is most convenient to use the following versions of the above formule

$$r=\sqrt{\left\{\frac{(s-a)(s-b)(s-c)}{s}\right\}},$$

$$\tan (A/2)=r(s-a),\ \tan (B/2)=r(s-b),\ \tan (C/2)=r\,(s-c).$$

Example 9 illustrates the procedure.

The solution can also be obtained without using any formulae as in Example 10. Whichever method we may use, the solution is checked by finding A+B+C and examining its deviation from 180°.

Example 8. *Solve the triangle ABC with $a=4$, $b=7$, $c=9$.*

Solution. $a^2=16$, $2ab=56$,

$b^2=49$, $2bc=126$,

$c^2=81$, $2ca=72$,

so that

$$\left.\begin{aligned} \cos A&=\frac{b^2+c^2-a^2}{2bc}=\cdot 9048, \text{ or } A=25^\circ \\ \cos B&=\frac{c^2+a^2-b^2}{2ca}=\cdot 6667, \text{ or } B=48^\circ \\ \cos C&=\frac{a^2+b^2-c^2}{2ab}=-\cdot 2859, \text{ or } C=106^\circ \end{aligned}\right\} \text{ to the nearest degree.}$$

Check. $A+B+C=25^\circ+48^\circ+106^\circ=179^\circ$.

The check is satisfactory. The deficit of 1° in the sum is not at all surprising since the value of each angle has been computed to the nearest degree.

Example 9. *In a triangle ABC, $a=210$, $b=218$, $c=360$. Solve the triangle.*

Solution. We have to first compute s, $s-a$, $s-b$ and $s-c$. We can then obtain $\log r$ from these values by using the formula

$$\log r=\tfrac{1}{2}\{\log(s-a)+\log(s-b)+\log(s-c)-\log s\}.$$

Now, $s=\tfrac{1}{2}(a+b+c)=\tfrac{1}{2}(210+218+360)=394.$

$s-a=184$	$\log(s-a)=2\cdot 2648$
$s-b=176$	$\log(s-b)=2\cdot 2455$
$s-c=34$	$\log(s-c)=1\cdot 5315$
	add: $6\cdot 0418$
	$\log s=2\cdot 5955$
	subtract: $2\log r=3\cdot 4463$
	$\log r=1\cdot 7232$ (approx.)
$\tan(A/2)=\dfrac{r}{s-a}$	$\tan(B/2)=\dfrac{r}{s-b}$
$\log r=11\cdot 7232-10$	$\log r=11\cdot 7232-10$
$\log(s-a)=2\cdot 2648$	$\log(s-b)=2\cdot 2455$
S	S
$\log\tan(A/2)=9\cdot 4584-10$	$\log\tan(B/2)=9\cdot 4777-10$
$A/2=16^\circ$	$B/2=16^\circ\ 40'$
$A=32^\circ$	$B=33^\circ\ 20'$

$$\tan\frac{C}{2}=\frac{r}{s-c}$$

$$\log r = 11{\cdot}7232 - 10$$

$$\log (s-c) = \underline{\;1{\cdot}5315\;}\,\text{S}$$

$$\log \tan (C/2) = 10{\cdot}1917 - 10$$

$$C/2 = 57^\circ\ 20'$$

$$C = 114^\circ\ 40'.$$

Check : $A+B+C=32^\circ+33^\circ\ 20'+114^\circ\ 40'=180^\circ$.

Example 10. *Solve the triangle ABC when $a=20$, $b=17$, $c=27$.*

Solution. Through C draw $CD \perp AB$.

Let $AD=p$, $DB=q$.
Then $p+q=AD+DB$,
$=AB=c=27$. ...(*i*)

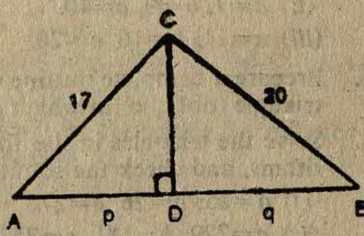


Fig. 15·7.

Also, from the right-angled triangles ADC and CDB, we have

$$CD^2=(17)^2-p^2,$$
$$=(20)^2-q^2,$$

or $q^2-p^2=(20)^2-(17)^2=111$. ...(*ii*)

From (*i*) and (*ii*), we have

$$q-p=\frac{37}{9}. \qquad ...(iii)$$

From (*i*) and (*iii*), we get

$$p=\frac{103}{9},\ q=\frac{140}{9}. \qquad ...(iv)$$

From the right-angled triangle ADC, we have

$$\cos A=\frac{p}{17}=\frac{103}{153}={\cdot}6732,$$

so that $\angle A=47^\circ\ 40'$, to the nearest $10'$.

Again, from the right-angled triangle CDB, we have

$$\cos B=\frac{q}{20}=\frac{7}{9}={\cdot}7778$$

so that $\angle B=39^\circ$, to the nearest $10'$.

$$\angle C=180^\circ-(\angle A+\angle B)=93^\circ\ 20'.$$

EXERCISE 15 (*b*)

1. Solve by using the cosine formula, the triangles in problems (*i*) to (*iv*) :
   (*i*) $a=\sqrt{6}, b=2, c=\sqrt{3}+1$.
   (*ii*) $a=50, b=52, c=54$.
   (*iii*) $a=8, b=9, c=10$.
   (*iv*) $a=16, b=20, c=35$.
2. For what values of x can $x^2-1, 2x+1, x^2+x+1$ be the lengths of the sides of a triangle ? If x has any one of these values, prove that the greatest angle of the triangle is 120°.
3. Solve, without using the cosine formulae and the half-angle formulae, the triangles in the following problems :
   (*i*) $a=9, b=5, c=8$.
   (*ii*) $a=7, b=9, c=10$.
   (*iii*) $a=11, b=16, c=20$.
4. Prepare a complete outline of the logarithmic solution of the triangle (oblique) in which a, b, c are given.
5. Solve the triangles in the following problems by using logarithms, and check the solution in each case :
   (*i*) $a=25, b=26, c=27$.
   (*ii*) $a=229, b=181, c=257$.
   (*iii*) $a=4584, b=3624, c=5140$.
   (*iv*) $a=18{\cdot}77, b=25{\cdot}32, c=29{\cdot}65$.
6. Check the solutions of problems 1(*iii*), 3(*i*), 3(*ii*) by constructing the triangles accurately.

15·9. CASE II. TWO SIDES AND THE INCLUDED ANGLE GIVEN (SAS)

To solve a triangle ABC when b, c, and A are given. In this case the solution always exists and is unique. If logarithms are not to be used, then we first determine a by the cosine formula

$$a^2=b^2+c^2-2\,bc\cos A$$

and then determine B and C by the sine formulae

$$\frac{\sin B}{b}=\frac{\sin A}{a},\ i.e.,\ \sin B=\frac{b\sin A}{a}$$

and

$$\frac{\sin C}{c}=\frac{\sin A}{a},\ i.e.,\ \sin C=\frac{c\sin A}{a}.$$

If logarithmic computation is desired (see Example 13), then we use the formulae

$$\tan\left\{\frac{1}{2}(B-C)\right\}=\frac{b-c}{b+c}\tan\left\{\frac{1}{2}(B+C)\right\}$$

and $$a=\frac{b \sin A}{\sin B}.$$

The first of these formulae is given in suitable form when $b > c$. If $b < c$, the equivalent form obtained by interchanging b and B with c and C respectively is used. If $b=c$, then $B=C=90° - A/2$, and the first formula is not needed. Also, then

$$a=2b \sin (A/2).$$

To check the solution we may use one or more of the following:

(*i*) find A+B+C;

(*ii*) find B from the formula $\sin B=\frac{b \sin C}{c}$;

(*iii*) find a from the formula $a=\frac{c \sin A}{\sin C}$.

Example 11. *Solve the triangle ABC, given $b=\sqrt{3}+1$, $c=2$, $A=60°$.*

Solution. We shall first find a. By the cosine formula for a,

$$a^2=b^2+c^2-2bc \cos A,$$
$$=(\sqrt{3}+1)^2+2^2-2\,.\,(\sqrt{3}+1)\,.\,2\,.\,\tfrac{1}{2},$$
$$=6,$$

so that $$a=\sqrt{6}.$$

We shall next find C. By the sine formula,

$$\frac{a}{\sin A}=\frac{c}{\sin C},$$

i.e., $$\sin C=\frac{c \sin A}{a}=\frac{2.(\sqrt{3}/2)}{\sqrt{6}}=\frac{1}{\sqrt{2}},$$

so that $$C=45° \text{ or } 135°.$$

Since the sum of the angles A and C cannot exceed 180°, therefore, the value C=135° is not admissible.

Thus $C=45°$.

Now we shall find the remaining angle B.

$$B=180°-(A+C)=180°-(60°+45°)=75°.$$

Hence $a=\sqrt{6}$, $B=75°$, $C=45°$.

Check. By the sine formula,

$$\sin B=\frac{b \sin C}{c}=\frac{(\sqrt{3}+1)}{2\sqrt{2}},$$
$$=\frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}}+\frac{1}{2}\cdot\frac{1}{\sqrt{2}},$$
$$=\sin 60° \cos 45°+\cos 60° \sin 45°.$$

$=\sin(60°+45°)$,
$=\sin 105°$,
$=\sin 75°$,

so that $B=75°$ which agrees with the value of B found above by another method.

Example 12. *Solve the triangle ABC, given $b=5$, $c=3$, $A=120°$.*

Solution. We shall first find a. By the cosine formula for a,

$$a^2=b^2+c^2-2bc\cos A,$$
$$=5^2+3^2-2\cdot5\cdot3\cdot(-1/2)=49,$$

so that $a=7$.

We shall next find C. By the sine formula,

$$\sin C=\frac{c\sin A}{a},$$
$$=\frac{3\sqrt{3}}{14},$$
$$=\cdot3712,$$

so that $C=22°$, to the nearest degree.

We shall now find B. By the sine formula,

$$\sin B=\frac{b\sin A}{a},$$
$$=\frac{5\sqrt{3}}{14}=\cdot6186$$

so that $B=38°$, to the nearest degree.

Check. $A+B+C=120°+38°+22°=180°$.

Remarks. 1. Since the sides are given to one significant digit and the angle is given to the nearest degree, therefore, we have determined the angles to the nearest degree.

2. $\sin C=\cdot3712$ actually gives $C=22°$ or $158°$.

Since $A=120°$, the value $C=158°$ is obviously inadmissible. We therefore, wrote $C=22°$, without caring at all for the value $158°$.

Example 13. *Two sides and the included angle of a triangle have measures 98·17, 67·13 and 107° 36′. Solve the triangle.*

Solution. Let us denote the given parts by b, c and A as shown in Fig. 15·8. Then

$$b=98\cdot17$$
$$c=67\cdot13$$

$$b-c=31\cdot04$$
$$b+c=165\cdot30$$
$$B+C=180°-107°\ 36'$$
$$=72°\ 24'$$

$\frac{1}{2}(B+C)=36° 12'$

$$\Rightarrow \tan\frac{1}{2}(B-C)=\frac{b-c}{b+c}\tan\frac{1}{2}(B+C),$$

$$=\frac{31{\cdot}04}{165{\cdot}30}\tan 36° 12'.$$

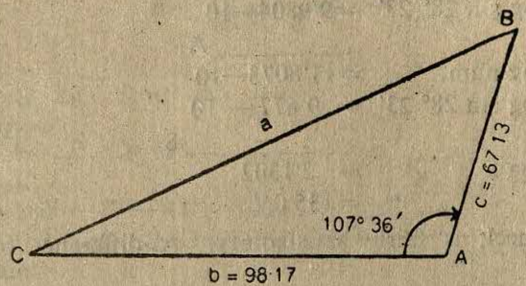


Fig. 15·8.

$$\begin{aligned} \log 31{\cdot}04 &= 1{\cdot}4920 \\ \log\tan 36° 12' &= 9{\cdot}8644-10 \\ & \qquad\qquad A \\ \log \text{num.} &= 11{\cdot}3564-10 \\ \log 165{\cdot}30 &= 2{\cdot}2183 \\ & \qquad\qquad S \\ \log\{\tan\tfrac{1}{2}(B-C)\} &= 9{\cdot}1381-10 \\ \Rightarrow \quad (B-C) &= 7° 49' \end{aligned}$$

so that $B=44° 1'$,

$C=28° 23'$.

To determine a, we have

$$a=\frac{b\sin A}{\sin B}=\frac{98{\cdot}17\sin 107° 36'}{\sin 44° 1'}.$$

Now

$$\begin{aligned} \log 98{\cdot}17 &= 1{\cdot}9920 \\ \log\sin 107° 36' &= 9{\cdot}9804-10 \\ & \qquad\qquad A \\ \log \text{num.} &= 11{\cdot}9724-10 \\ \log\sin 44° 1' &= 9{\cdot}8419-10 \\ & \qquad\qquad S \\ \log a &= 2{\cdot}1305 \\ a &= 135{\cdot}1 \end{aligned}$$

Check. To check the solution, we may determine a by using the formula

$$a=\frac{c \sin A}{\sin C}.$$

$$a=\frac{c \sin A}{\sin C}=\frac{67{\cdot}13 \sin 107^\circ\, 36'}{\sin 28^\circ\, 23'}$$

$$\begin{array}{ll}
\log 67{\cdot}13 & =1{\cdot}8269 \\
\log \sin 28^\circ\, 23' & =9{\cdot}9804-10 \\
 & \overline{\phantom{=11{\cdot}8073-10}}\ \text{A} \\
\log \text{num.} & =11{\cdot}8073-10 \\
\log \sin 28^\circ\, 23' & =\ 9{\cdot}6771-10 \\
 & \overline{\phantom{=11{\cdot}8073-10}}\ \text{S} \\
\log a & =\ 2{\cdot}1302 \\
a & =135{\cdot}0.
\end{array}$$

The check is fairly satisfactory. A difference of ·1 is not surprising.

EXERCISE 15 (c)

1. Solve, by using the cosine formula and the sine formulae, the triangles in the following problems. Check your solution in each case.
   (*i*) $b=1$, $c=\sqrt{3}$, $A=30^\circ$.
   (*ii*) $b=\sqrt{3}+1$, $c=\sqrt{3}-1$, $A=60^\circ$.
   (*iii*) $a=11$, $b=20$, $C=37^\circ$.
   (*iv*) $c=23$, $a=12$, $B=35^\circ$.
2. Prepare a complete outline of the logarithmic solution of the triangle in which the following parts are given :
   (*i*) a, b, C with $a>b$. (*ii*) a, b, C with $b>a$.
   (*iii*) b, c, A with $b>c$. (*iv*) b, c, A with $c>b$.
   (*v*) c, a, B with $c>a$. (*vi*) c, a, B with $a>c$.
3. Solve the triangles in the following problems by using logarithms Check the solution in each case.
   (*i*) $b=540$, $c=420$, $A=52^\circ\, 10'$.
   (*ii*) $a=35{\cdot}21$, $b=21{\cdot}35$, $C=50^\circ\, 48'$.
   (*iii*) $c=164{\cdot}3$, $a=242{\cdot}5$, $B=54^\circ\, 36'$.
   (*iv*) $a=158$, $b=237$, $C=66^\circ\, 40'$.
4. The ratio of the two sides a and b of a triangle ABC is 7 : 3 and the angle included by these sides is 60°. Find, to the nearest 10′, the angles A and B.
5. Check the solutions of problems 3 (*iii*) and 3 (*iv*) by constructing the triangles to scale.

15·10. CASE III. ONE SIDE AND TWO ANGLES GIVEN (SAA)

To solve the triangle ABC when a, B, C are given. The triangle exists if $B+C<180°$ and is then unique.

The third angle is given by the formula

$$A=180°-(B+C).$$

b and c are then determined by the sine formulae

$$b=\frac{a \sin B}{\sin A} \text{ and } c=\frac{a \sin C}{\sin A}$$

respectively.

Example 14. *Solve the triangle ABC, given*
$a=15$, $B=37°$, $C=43°$.

Solution.

(1) $A=180°-(37°+43°)=100°$.

(2) To find b, we have by using the sine formula

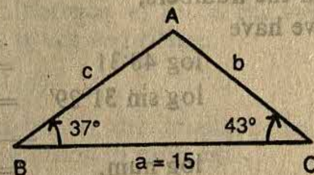


Fig. 15·9.

$$b=\frac{a \sin B}{\sin A}=\frac{15 \sin 37°}{\sin 100°}=\frac{15(\cdot 6018)}{\cdot 9848}=9 \cdot 2.$$

(3) To find c, we have by using the sine formula

$$c=\frac{a \sin C}{\sin A}=\frac{15 \sin 43°}{\sin 100}=\frac{15(\cdot 6820)}{\cdot 9848}=10.$$

The computed parts are $A=110°$, $b=9 \cdot 2$, $c=11$. (The values of b and c have been rounded off to two figure accuracy.)

Check. To check the results, let us find c by using the formula

$$c=\frac{b \sin C}{\sin B}.$$

$$c=\frac{9 \cdot 2 \sin 43°}{\sin 37°}=\frac{9 \cdot 2(\cdot 6820)}{\cdot 6018}=10,$$

when rounded off to two-figure accuracy.

Remark. We could also have used the cosine formula to check the result.

Example 15. *Solve the triangle ABC, given $c=48 \cdot 31$, $A=31°29'$, $C=74°43'$.*

Solution. (1) $B=180°-(31° 29'+74° 43')$

$$=73° 48'.$$

(2) $$a=\frac{c \sin A}{\sin C}=\frac{48 \cdot 31 \sin 31° 29''}{\sin 74° 43'}.$$

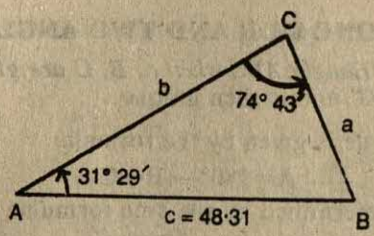


Fig. 15·10.

On preparing an outline of the logarithmic solution and filling in the numbers,

we have

$$\begin{array}{ll} \log 48{\cdot}31 & = 1{\cdot}6840 \\ \log \sin 31°29' & = \bar{1}{\cdot}7178 - 10 \\ \hline & \qquad\qquad\qquad \text{A} \\ \log \text{num.} & = 3{\cdot}4018 - 10 \\ \log \sin 74°\,43' & = \bar{1}{\cdot}9843 - 10 \\ \hline & \qquad\qquad\qquad \text{S} \\ \log a & = 1{\cdot}4175 \\ a & = 26{\cdot}15. \end{array}$$

(3) $b=\dfrac{c \sin B}{\sin C}=\dfrac{48{\cdot}31 \sin 73°\,48'}{\sin 74°\,43'}.$

On preparing an outline of the logarithmic solution and filling in the numbers,

we have

$$\begin{array}{ll} \log 48{\cdot}31 & = 1{\cdot}6840 \\ \log \sin 73°\,48' & = \bar{1}{\cdot}9824 - 10 \\ \hline & \qquad\qquad\qquad \text{A} \\ \log 48{\cdot}31 & = 3{\cdot}6664 - 10 \\ \log \sin 74°43' & = \bar{1}{\cdot}9843 - 10 \\ \hline & \qquad\qquad\qquad \text{S} \\ \log b & = 1{\cdot}6821 \\ b & = 48{\cdot}09. \end{array}$$

The computed parts are B=73° 48′, a=26·15, b=48·09.

Check. To check the results, let us find b by using the formula

$$b=\frac{a \sin B}{\sin A}.$$

$$b=\frac{a \sin B}{\sin A}=\frac{(26{\cdot}15) \sin 73° 48'}{\sin 31° 29'},$$

log 26·15	=1·4174
log sin 73° 48′	=1·9824−10
	A
log num.	=3·3998−10
log sin 31° 22′	=1·7178−10
	S
log b	=1·6820
b	=48·08.

The check is satisfactory. The difference of ·01 in the values of b obtained by two different methods is not surprising.

EXERCISE 15 (*d*)

1. Solve, without the use of logarithms, the triangles in the following problems. Check your solution in each case.
   (*i*) B=31°, C=43°, a=28.
   (*ii*) A=80°, B=53°, b=152.
   (*iii*) C=36° 20′, A=45° 50′, c=140.
   (*iv*) A=61° 10′, B=36° 20′, a=152.

2. Prepare an outline of the logarithmic solution of the triangle in which the following parts are given :
   (*i*) B, C, a. (*ii*) B, C, b.
   (*iii*) B, C, c. (*iv*) C, A, a.
   (*v*) C, A, b. (*vi*) C, A, c.

3. Solve the triangles in the following problems by using logarithms. Check your solution in each case.
   (*i*) b=39, A=81°, B=27°.
   (*ii*) c=136, B=34° 20′, C=67° 30′.
   (*iii*) a=15·72, A=41° 30′, B=72° 45′.
   (*iv*) b=28·94, C=61° 48′, B=84° 23′.

4. Check the solutions of problems 3 (*ii*) and 3 (*iii*) by using the formula

$$\frac{a+b}{c}=\frac{\cos \frac{1}{2}(A-B)}{\sin \frac{1}{2} C}.$$

15·11. CASE IV. TWO SIDES AND ONE ANGLE GIVEN (SSA)

To solve the triangle ABC when a, b, A are given. There may exist no, one or two triangles depending on the relation between the given parts as we shall see below. Because of the possibility of having two triangles, this case is called the ***Ambiguous Case.***

To discuss the existence and uniqueness of the solution we shall proceed geometrically.

We construct angle A and cut off $AC=b$. This fixes the vertex C. With C as centre we draw an arc in order to locate (if possible) B.

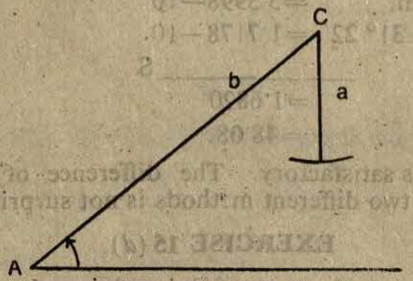


Fig. 15·11 (*a*).

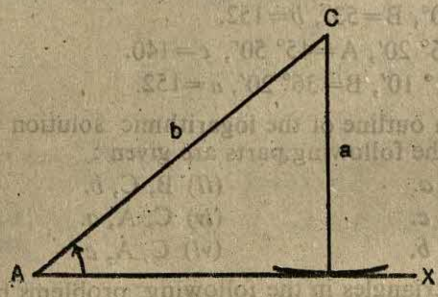


Fig. 15·11 (*b*).

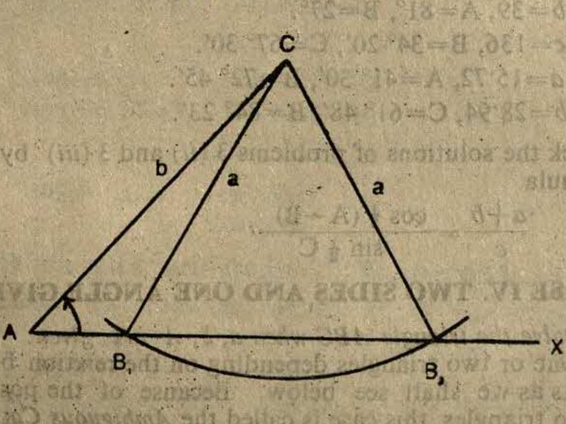


Fig. 15·11 (*c*).

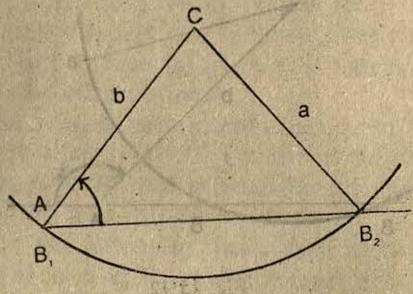


Fig. 15·11 (d).

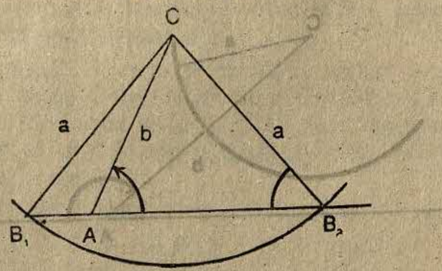


Fig. 15·11 (e).

For the sake of convenience let us consider the cases $A < 90°$, $A > 90°$ separately.

(α) $A < 90°$: Several possibilities arise :

(i) If $a < p$ (where $p = b \sin A$) is the perpendicular from C on AX, then the arc does not cut AX and no triangle is possible [Fig. 15·11 (a)].

(ii) If $a = p$, then the arc touches AX. Therefore, one triangle is possible and it is right-angled [Fig. 15·11 (b)].

(iii) If $a > p$, then the arc cuts AX at two points, both these points lie to the right of A if $a < b$ [Fig. 15·11 (c)], one of them lies to the right of A and the other coincides with A if $a = b$ [Fig. 15·11 (d)], and one of them lies to the right of A and the other to the left of A if $a > b$ [Fig. 15·11 (e)]. Thus two triangles are possible if $a < b$ and only one triangle is possible if $a \geqslant b$. Because of the possibility of two triangles, the case $a < b$, $a > b \sin A$, A acute, is called the *Ambiguous Case*.

(β) $A > 90°$: The following possibilities arise :

(i) If $a \leqslant b$, the arc does not cut AX at any point to the right of A and no triangle is possible (Fig. 15·12 and Fig. 15·13).

(ii) If $a > b$, then the arc cuts AX at two points, only one of which lies to the right of A and therefore, only one triangle is possible (Fig. 15·14). This completes the discussion.

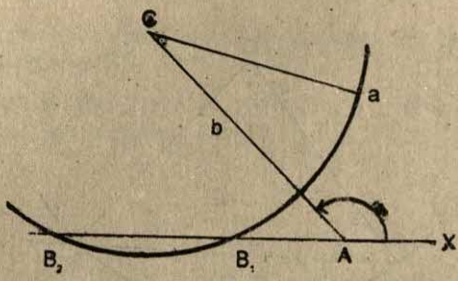


Fig. 15·12.

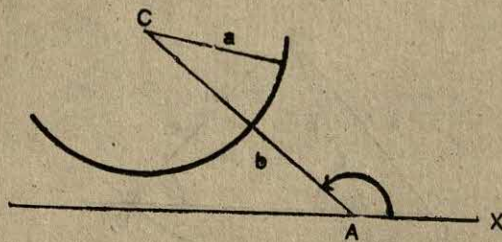


Fig. 15·13.

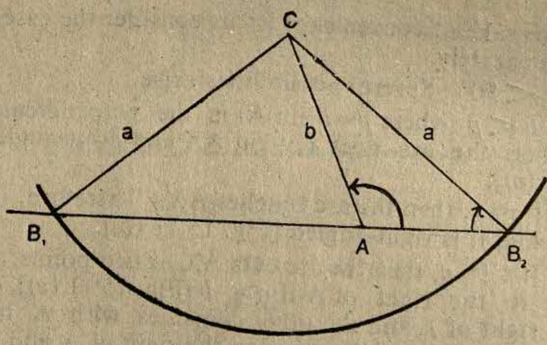


Fig. 15·14.

To solve the triangle when a, b, A are given we have to use the sine formula. For the sake of convenience we shall discuss the case $A < 90°$, and $A > 90°$ separately.

(*a*) $A < 90°$: The following possibilities arise :

(*i*) If $a < b \sin A$, then from the formula

$$\frac{a}{\sin A}=\frac{b}{\sin B} \qquad ...(1)$$

$\sin B > 1$ and consequently no solution is possible.

(*ii*) If $a=b \sin A$, then from (1), $\sin B=1$. so that $B=90°$. Therefore, there is one solution and the triangle is right-angled.

(*iii*) If $a > b \sin A$, then (1) gives two values of B, one of which is acute and the other obtuse.

If $a \leqslant b$, then $A \leqslant B$, so that only the acute value of B is permissible, and consequently there is only one solution.

If $a < b$, then $A < B$, so that both the values of B are possible and consequently there may be two solutions.

(β) $A > 90°$: The following possibilities arise :

(*i*) If $a \leqslant b$, then $A \leqslant B$, so that B must also be an obtuse angle, which is impossible. Hence no solution is possible.

(*ii*) If $a > b$, then only the acute value of B is permissible and therefore, only one triangle is possible.

Having determined B (whenever there exists a permissible value of B), we determine C by the formula $C=180°-(A+B)$. The remaining side c is then found as in the SAS case. In the ambiguous case the values of C and c corresponding to the two values of B have to be found separately.

If we use logarithms, then it may be helpful to observe that

$\log \sin B > 0 \Rightarrow \sin B > 1 \Rightarrow$ no solution.

$\log \sin B = 0 \Rightarrow \sin B = 1 \Rightarrow B=90°$.

$\log \sin B < 0 \Rightarrow \sin B < 1$, *i.e.*, there are two values of B say B_1 and B_2. We then consider these values as discussed above.

Remark. Case IV can be disposed of by using the cosine formula also.

If a, b, A are given, then the cosine formula for a gives

$$a^2=b^2+c^2-2bc \cos A,$$

or

$$c^2-2bc \cos A+b^2-a^2=0. \qquad \text{...(1)}$$

Solving (1) as a quadratic in c, we have

$$c=\frac{2b \cos A \pm \sqrt{\{4b^2 \cos^2 A-4(b^2-a^2)\}}}{2},$$

$$=b \cos A \pm \sqrt{(a^2-b^2 \sin^2 A)}. \qquad \text{...(2)}$$

Since c is the length of a side of a triangle, therefore, it must be positive. We have therefore to determine as to how many of the values of c given by (2) are positive for any given set of a, b and A.

Two different possibilities arise :

(*i*) $A < 90°$: If $A < 90°$, cos A is positive so that $b \cos A$ is positive.

Three sub-cases arise :

(*a*) If $a < b \sin A$, then $a^2 < b^2 \sin^2 A$, so that $a^2-b^2 \sin^2 A < 0$. The two values of c are imaginary and no triangle is possible.

(β) If $a=b \sin A$, then $a^2=b^2 \sin^2 A$, so that $a^2-b^2 \sin^2 A=0$. There is only one value of c $(=b \cos A)$ from (2) which is positive. Therefore, only one triangle is possible.

(γ) If $a > b \sin A$, then $a^2 > b^2 \sin^2 A$ so that

$a^2-b^2 \sin^2 A > 0$. In this case (2) gives two real and distinct values of c. One of these values, namely

$$b \cos A + \sqrt{\{a^2-b^2 \sin^2 A\}}$$

is surely positive ; the other value

$$b \cos A - \sqrt{\{a^2-b^2 \sin^2 A\}}$$

is positive if

$$b \cos A > \sqrt{\{a^2-b^2 \sin^2 A\}},$$

i.e., if $\quad b^2 \cos^2 A > a^2-b^2 \sin^2 A,$

i.e., if $\quad b^2 > a^2,$

i.e., if $\quad b > a.$

Therefore two triangles are possible if $b > a$ and only one triangle is possible if $b \leqslant a$.

(*ii*) $A > 90°$: If $A > 90°$, $\cos A$ is negative so that $b \cos A$ is negative.

The value $b \cos A - \sqrt{\{a^2-b^2 \sin^2 A\}}$ is surely negative.

The value $b \cos A + \sqrt{\{a^2-b^2 \sin^2 A\}}$ is positive if

$$\sqrt{\{a^2-b^2 \sin^2 A\}} > -b \cos A,$$

i.e., if $\quad a^2-b^2 \sin^2 A > b^2 \cos^2 A,$

i.e., if $\quad a^2 > b^2$

i.e., if $\quad a > b.$

Thus we find that when $A > 90°$, no triangle is possible when $a \leqslant b$ and only one triangle is possible when $a > b$.

SUMMARY OF THE VARIOUS POSSIBILITIES

(*i*) $A < 90°$

(α) $a < b \sin A$	No triangle
(β) $a = b \sin A$	One triangle
(γ) $a > b \sin A$	Two triangles if $a < b$; one triangle if $a \geqslant b$.

(*ii*) $A > 90°$

(α) $a \leqslant b$	No triangle
(β) $a > b$	One triangle.

Example 16. *Given $a=19$, $b=31$, $A=56°$, either show that there is no solution or find all solutions.*

Solution. We shall use the sine formula to find B.

$$\sin B = \frac{b \sin A}{a} = \frac{31 \sin 56^\circ}{19}$$

$$= \frac{31(\cdot 8290)}{19} = 1\cdot 3525.$$

Since $\sin B > 1$, there does not exist any solution.

Remark. The above example illustrates the case $A < 90^\circ$, $a < b \sin A$.

Example 17. *Given $a=31\cdot 1$, $b=62\cdot 2$, $A=30^\circ$, either show that there is no solution or find all solutions.*

Solution. (1) We shall use the sine formula to find B

$$\sin B = \frac{b \sin A}{a} = \frac{(62\cdot 2)\ \frac{1}{2}}{31\cdot 1} = 1.$$

Since $\sin B = 1$, therefore, $B = 90^\circ$. There exists only one solution.

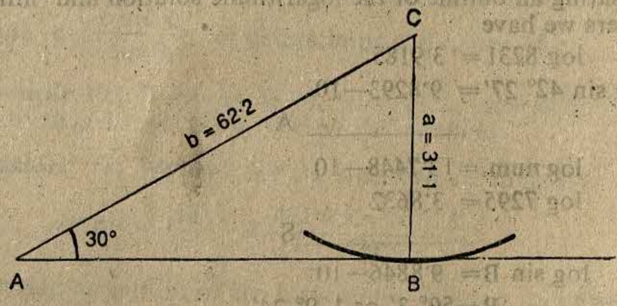


Fig. 15·15.

(2) $C = 180^\circ - (30^\circ + 90^\circ) = 60^\circ$.

(3) To find c, we shall use the sine formula

$$c = \frac{b \sin C}{\sin B} = 62\cdot 2 \sin 60^\circ,$$

$$= (62\cdot 2) . (0\cdot 8660) = 53\cdot 9,$$

when rounded off to three-figure accuracy.

Remark. The above example illustrates the case $A < 90^\circ$, $a = b \sin A$.

Example 18. *Given $a=7295$, $b=8231$, $A=42^\circ\ 27'$, either show that there is no solution or find all solutions.*

Solution. (1) We shall use the sine formula to find B.

$$\sin B = \frac{b \sin A}{a}.$$

$$=\frac{8231\ (\sin 42^\circ\ 27')}{7295}.$$

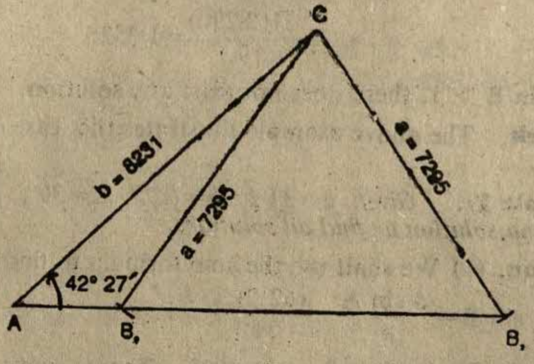


Fig. 15·16.

Preparing an outline of the logarithmic solution and filling in the numbers we have

$$\begin{aligned}
\log 8231 &= 3{\cdot}9185\\
\log \sin 42^\circ\ 27' &= 9{\cdot}8293-10\\
&\overline{}\ \text{A}\\
\log \text{num.} &= 13{\cdot}7448-10\\
\log 7295 &= 3{\cdot}8632\\
&\overline{}\ \text{S}\\
\log \sin B &= 9{\cdot}8846-10\\
B &= 50^\circ\ 3' \text{ or } 129^\circ\ 24'
\end{aligned}$$

Let $B_1=50^\circ\ 3'$,

$B_2=29^\circ\ 3'$.

Since $A+B_2 < 180^\circ$, therefore both, the values are admissible and there are two solutions.

(2) $C_1=180^\circ-(A+B_1)$,

$=180^\circ-(42^\circ\ 27'+50^\circ\ 3')$,

$=87^\circ\ 30'$.

$C_2=180^\circ-(A+B_2)$

$=180^\circ-(42^\circ\ 27'+129^\circ\ 57')$,

$=7^\circ\ 36'$.

(3) $c_1=\dfrac{a \sin C_1}{\sin A}$,

$=\dfrac{7295 \sin 87^\circ\ 30'}{\sin 42^\circ\ 27'}$.

$c_2=\dfrac{a \sin C_2}{\sin A}$,

$=\dfrac{7295 \sin 7^\circ\ 36'}{\sin 42^\circ\ 27'}$.

$$\begin{aligned} \log 7295 &= 3{\cdot}8632 \\ \log \sin 87^\circ 30' &= 9{\cdot}9996-10 \\ & \qquad\qquad\text{A} \\ \log \text{num.} &= 13{\cdot}8628-10 \\ \log \sin 42^\circ 27' &= 9{\cdot}8293-10 \\ & \qquad\qquad\text{S} \\ \log c_1 &= 4{\cdot}0335 \\ c_1 &= 10800. \end{aligned}$$

$$\begin{aligned} \log 7295 &= 3{\cdot}8632 \\ \log \sin 7^\circ 36' &= 9{\cdot}1214-10 \\ & \qquad\qquad\text{A} \\ \log \text{num.} &= 12{\cdot}9846-10 \\ \log \sin 42^\circ 27' &= 9{\cdot}8293-10 \\ & \qquad\qquad\text{S} \\ \log c_2 &= 3{\cdot}1553 \\ c_2 &= 1430. \end{aligned}$$

Thus there exist two solutions. The computed parts are

$$B_1=50^\circ 3', \quad C_1=87^\circ 30', \quad c_1=10800\,;$$

$$B_2=129^\circ 57', \quad C_2=7^\circ 36', \quad c_2=1430.$$

Remark. The above example illustrates the ambiguous case $A<90^\circ$, $a>b \sin A$ and $a<b$. The solutions can be checked by using the formula $c=\dfrac{b \sin C}{\sin B}$ for determining the values of c.

Example 19. *Given $a=531{\cdot}4$, $b=421{\cdot}9$, $A=70^\circ 15'$, either show that there is no solution or find all the solutions.*

Solution. (1) We shall use the sine formula to find B.

$$\sin B=\frac{b \sin A}{a}=\frac{421{\cdot}9 \sin 70^\circ 15'}{531{\cdot}4}.$$

Preparing an outline of the logarithmic solution and filling in the numbers we have

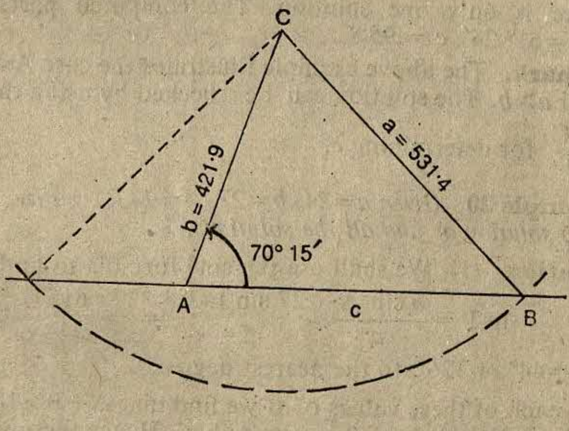


Fig. 15·17.

$$\begin{aligned}
\log 421{\cdot}9 &= 2{\cdot}6252\\
\log \sin 70^\circ 15' &= 9{\cdot}9736-10 \quad \text{A}\\
\log \text{num.} &= 12{\cdot}5988-10\\
\log 531{\cdot}4 &= 2{\cdot}7254 \quad \text{S}\\
\log \sin B &= 9{\cdot}8734-10
\end{aligned}$$

$$B=48^\circ 21' \quad \text{or} \quad 131^\circ 39'.$$

Since $131^\circ 39'+70^\circ 15'$ exceeds 180°, therefore, the value $131^\circ 39'$ is inadmissible, so that $B=48^\circ 21'$.

Therefore, there is only one solution.

(2) $C=180^\circ-(A+B)=180^\circ-(70^\circ 15'+48^\circ 21')=61^\circ 24'$.

(3) We shall use the sine formula to determine c:

$$c=\frac{a \sin C}{\sin A}=\frac{531{\cdot}4 \sin 61^\circ 24'}{\sin 70^\circ 15'}.$$

Preparing an outline of the logarithmic solution and filling in the numbers we have

$$\begin{aligned}
\log 531{\cdot}4 &= 2{\cdot}7254\\
\log \sin 61^\circ 24' &= 9{\cdot}9435-10 \quad \text{A}\\
\log \text{num.} &= 12{\cdot}6689-10\\
\log \sin 70^\circ 15' &= 9{\cdot}9736-10 \quad \text{S}\\
\log c &= 2{\cdot}6953\\
c &= 495{\cdot}8.
\end{aligned}$$

There is only one solution. The computed parts are $B=48^\circ 21'$, $C=61^\circ 24'$, $c=495{\cdot}8$.

Remark. The above example illustrates the case $A<90^\circ$, $a>b \sin A$, and $a>b$. The solution can be checked by using the formula $c=\dfrac{b \sin C}{\sin B}$ for determining c.

Example 20. *Given $a=24$, $b=27$, $A=142^\circ$, either show that there is no solution or find all the solutions.*

Solution. (1) We shall use the sine formula to find B.

$$\sin B=\frac{b \sin A}{a}=\frac{27 \sin 142^\circ}{24}=\frac{27.({\cdot}6157)}{24}={\cdot}6927,$$

so that $B=44^\circ$ or 136°, to the nearest degree.

For each of these values of B we find that $A+B>180^\circ$. Therefore, none of the values of B is admissible. Hence there is no solution.

Remark. The above example illustrates the case $A>90°$, $a<b$.

Example 21. *Given $a=27{\cdot}3$, $b=19{\cdot}1$, $A=108°\ 20'$, show that there is only one solution and find B.*

Solution. By the sine formula we have

$$\sin B=\frac{b \sin A}{a}=\frac{19{\cdot}1 \sin 108°\ 20'}{27{\cdot}3}.$$

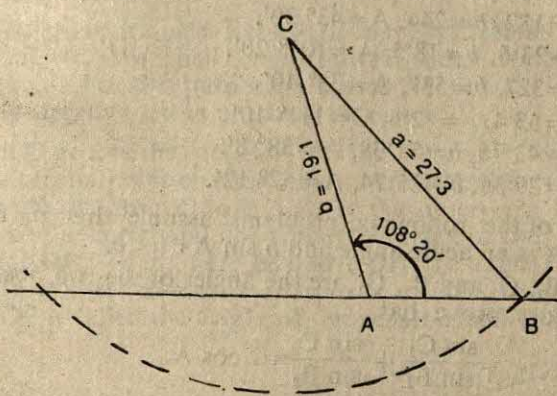


Fig. 15·18.

Preparing an outline of the logarithmic solution and filling in the numbers we have

$$\begin{aligned} \log 19{\cdot}1 &= 1{\cdot}2810 \\ \log \sin 108°\ 20' &= 9{\cdot}9774-10 \\ \hline & \qquad\qquad \text{A} \\ \log \text{num.} &= 11{\cdot}2584-10 \\ \log 27{\cdot}3 &= 1{\cdot}4362 \\ \hline & \qquad\qquad \text{S} \\ \log \sin B &= 9{\cdot}8222-10 \end{aligned}$$

$B=41°\ 37'$ or $138°23'$.

Since $A>90°$, the s cond one of the two values of **B** is obviously inadmissible. Also, since $41°37'+108°20' < 180°$, therefore, $41°37'$ is in admissible value of B.

Hence there is only one solution and $B=41°\ 37'$.

Remark. The above example illustrates the case $A>90°$, $a>b$.

EXERCISE 15 (*e*)

1. In each of the following problems, either show that there is no solution or find all the solutions. Do not use logarithms in problems (*i*) to (*vi*).

(*i*) $a=28, b=17, A=108°$. (*ii*) $a=32, b=25, A=29°$.
(*iii*) $a=69, b=49, B=37°$. (*iv*) $a=24, b=31, A=128°$.
(*v*) $a=82, b=130, A=44°$. (*vi*) $a=44, b=71, A=32°$.
(*vii*) $a=75{\cdot}2, b=84{\cdot}8, A=65°\ 30'$.
(*viii*) $a=32{\cdot}2, b=25{\cdot}1, A=51°\ 10'$.
(*ix*) $a=182, b=244, A=43°\ 30'$.
(*x*) $a=23{\cdot}6, b=18{\cdot}5, A=101°\ 20'$.
(*xi*) $a=327, b=537, A=33°\ 40'$.
(*xii*) $b=53{\cdot}4, c=42{\cdot}6, C=111°\ 10'$.
(*xiii*) $a=43{\cdot}75, b=71{\cdot}38, A=38°\ 8'$.
(*xiv*) $a=29{\cdot}36, b=25{\cdot}74, A=52°\ 12'$.

2. In each of the following problems assume that a, b, A are given, A is an acute angle and $b \sin A < a < b$.

(*i*) If B_1, C_1 and B_2, C_2 are the angles of the two possible triangles, prove that
$$\frac{\sin C_1}{\sin B_1}+\frac{\sin C_2}{\sin B_2}=2\cos A.$$

(*ii*) Show that if c_1, c_2 be the two values of the third side, then
$$|c_1-c_2|=2\sqrt{\{a^2-b^2\sin^2 A\}}.$$

(*iii*) Show that if c_1, c_2 be the two values of the third side, then
$$c_1^2-2c_1c_2\cos 2A+c_2^2=4a^2\cos^2 A.$$

(*iv*) Show that if c_1, c_2 be the two values of the third side, then
$$(c_1-c_2)^2+(c_1+c_2)^2\tan^2 A=4a^2.$$

SUMMARY

The following table gives a summary of the formulae to be used for solving triangles in the various cases discussed above.

Given parts	**Formulae to be used**	
	without use of logarithms	**with use of logarithms**
SSS	Cosine formulae	Half-angle formulae
SAS	Cosine formulae	Napier's formulae
	Sine formulae	Sine formulae
SSA	Sine formulae	Sine formulae
SAA	Sine formulae	Sine formulae

A problem on solution of triangles should be taken as a challenge and not merely as a routine problem where the rules set out in the book are to be applied. An ingenuity on the part of the reader may often reduce the labour considerably. For example, (*i*) in the SSS case, the triangle may be solved without the use of any special formulae (see example 10, page 513) ; (*ii*) one may compute one part without the aid of logarithms and another part with the aid of logarithms, (*iii*) when four parts are known, one may like to use the sine formula in the SSS or SAS case.

15·12. APPLICATIONS TO PROBLEMS ON HEIGHTS AND DISTANCES

Solution of triangles has enormous applications to surveying, navigation, etc. We shall now consider some simple ones from among them. For this purpose, we need to explain certain terms that are generally used in practical problems.

Let O denote the position of the observer's eye and let P denote a certain point on some object which the observer is watching. The straight line from O to P is called the observer's line of sight. Let OH be the horizontal line through O lying in the vertical plane containing OP. Then, if OP be above OH, the angle HOP is called the **angle of elevation** of P at/from O. If OP be below OH, the angle HOP is called the **angle of depression** of P at/from O.

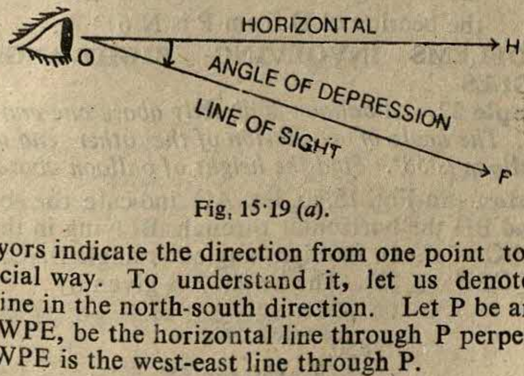


Fig. 15·19 (*a*).

Surveyors indicate the direction from one point to another in a rather special way. To understand it, let us denote by NS a horizontal line in the north-south direction. Let P be any point on NS and let WPE, be the horizontal line through P perpendicular to NE. Then WPE is the west-east line through P.

The direction from P to any point in the horizontal plane through P can be specified if we know (*i*) the positive angle which the line makes with NS (*ii*) whether the angle is measured from PN

or from PS (*iii*) whether the angle is measured to the west or the east.

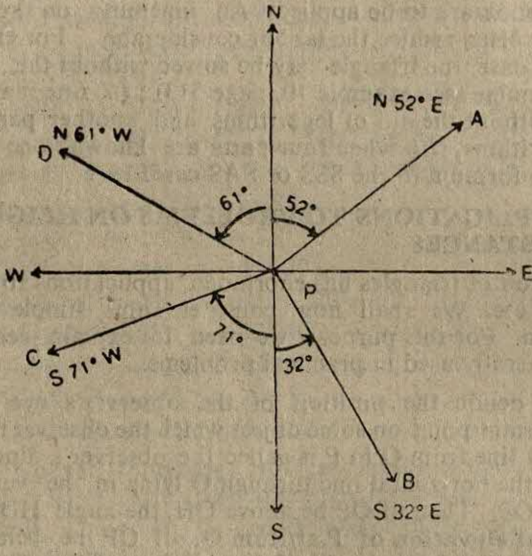


Fig. 15·19 (*b*).

Thus for example, in Fig. 15.19 (*b*) the direction PA is specified by saying that PA makes with PN an angle of 52° measured eastwards. We can express this more simply by saying that the *bearing* of A from P is N52°E. Similarly, for the points B, C and D in the same figure we would say :

the bearing of B from P is S 32° E ;
the bearing of C from P is S 71° W ;
the bearing of D from P is N 61° W.

15·13. PROBLEMS INVOLVING RIGHT-ANGLED TRIANGLES

Example 22. *A balloon is directly above one end of a bridge 850 m long. The angle of depression of the other end of the bridge from the balloon is 58°. Find the height of balloon above the bridge.*

Solution. In Fig. 15.20, let AC indicate the bridge, B the balloon, and BH the horizontal through B lying in the plane BAC. Then $\angle HBC = 58°$. In the right-angled triangle BAC, $AC = 850$ m, $\angle ACB = \angle CBH = \angle 58°$. The height h is given by

$$\frac{h}{850} = \tan 58°,$$

so that

$$h = 850 \tan 58°,$$
$$= 850 \times 1{\cdot}600,$$
$$= 1360.$$

Hence the required height is 1360 metres.

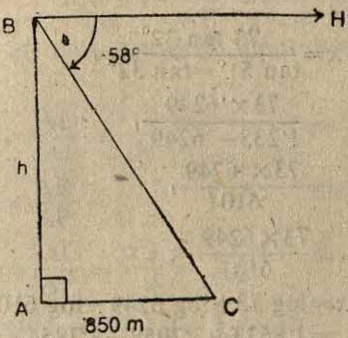


Fig. 15·20.

Example 23. *The angular elevation of the top of a vertical tower from a point A in the same horizontal plane as the foot of the tower is 51° and from a point B in the same horizontal line with the foot of the tower as A and 73 metres further away from it is 32°. Find the height of the tower and the distance of A from the foot of the tower.*

Solution. In the figure below, let F and T denote the foot and the top (respectively) of the tower. Also let h metres be the height of the tower, and x metres the distance of A from the foot of the tower.

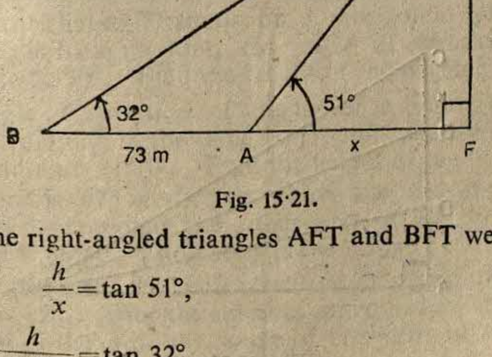


Fig. 15·21.

From the right-angled triangles AFT and BFT we have

$$\frac{h}{x} = \tan 51°, \qquad \ldots(i)$$

$$\frac{h}{x+73} = \tan 32°. \qquad \ldots(ii)$$

7. The angular elevation of the top of a vertical tower from a point A in the same horizontal plane as the foot of the tower is 60° and from a point B in the same horizontal line with the foot of the tower as A and 40 *m* further away is 30°. Find the height of the tower and also the distance of A from the foot of the tower.

8. AB is a straight horizontal road leading to F, the foot of a vertical tower, A being at a distance of 160 *m* from C, and B 100 *m* nearer. If the angle of elevation of the top of the tower at B be double the angle of elevation at A, find the height of the tower.

9. From a point A in the same horizontal plane as the foot of a vertical tower, the angles of elevation of two points B and C on the tower are 60° and 30° respectively. If BC=40 *m*, find the height of B from the foot of the tower, and the distance of A from the foot of the tower.

10. Two vertical pillars of equal height stand on either side of a roadway which is 50 *m* wide. At a point in the roadway between the pillars, the elevations of the tops of the pillars are 60° and 30°. Find the height of each pillar and the position of the point.

11. A ladder 8 *m* long reaches to a distance 8 *m* from the top of a vertical flagstaff. At the foot of the ladder the elevation of the top of the flagstaff is 60°. Find the height of the flagstaff.

12. A man on the top of a vertical lighthouse observes a boat coming directly towards it. If it takes 10 minutes for the angle of depression to change from 30° to 60°, how soon will it reach the lighthouse ?

13. A statue 6·8 *m* high standing on the top of a column 54·4 *m* high, on the bank of a river, subtends at a point on the opposite bank directly facing the column, the same angle as that subtended at the same point by a man 170 *cm* high at the base of the column. Find the breadth of the river.

14. A 9 *m* flag-pole stands on the top of a 4·5 *m* high building. How far from the base of the building should a man stand if the flag-pole and the building are to subtend equal angles at his eye which is 1·5 *m* above the ground.

15. A vertical telegraph pole is supported by two wires, each running from the top of the pole to the ground. One wire is 23 *m* long and makes an angle of 52° with the ground. Find the angle that the second wire makes with the ground, given that it is 20 *m* long.

16. A surveyor draws a line due west from A to B but cannot continue the line in the westerly direction because of an

obstacle So he draws a line 200 m long from B to C, in a direction 25° west of south and then draws another line in the direction 46° west of north. How long should CD be, if D is to be due west of B ?

17. From an observation tower a m. above the level of a river, the angle of depression of a point on the near shore is α and that of a point directly beyond on the far shore is β. Show that the width of the river is $a(\cot\beta-\cot\alpha)$ m.

18. From the top of a building a m. in height, on level ground, the angle of depression of the bottom of a vertical tower is α. From the bottom of the building the angle of elevation of the top of the tower is β. Show that the height of the tower is $a\tan\beta\cot\alpha$ m.

19. A vertical tower stands on a horizontal plane and is surmounted by a vertical flagstaff of height h. At a point on the plane, the angles of elevation of the bottom and the top of the flagstaff are respectively α and β. Prove that the height of the tower is

$$\frac{h\tan\alpha}{\tan\beta-\tan\alpha}\text{ m.}$$

20. The angles of elevation of the top of a vertical tower from two points distant a and b from the base and in the same straight line with B, are complementary. Prove that the height of the tower is $\sqrt{ab}$, and if θ be the angle subtended at the top of the tower by the line joining these points, then $\sin\theta=\dfrac{a-b}{a+b}$.

15·14. PROBLEMS INVOLVING OBLIQUE TRIANGLES

Example 25. *A tree stands vertically on a hillside which makes an angle of 20° with the horizontal. From a point 35 m directly down the hill from the base of the tree, the angle of elevation of the top of the tree is 43°. Find the height of the tree.*

Solution. In the adjoining figure, BC represents the tree, A is the point 35 m directly down the hill from the base of the tree, and a is the height of the tree.

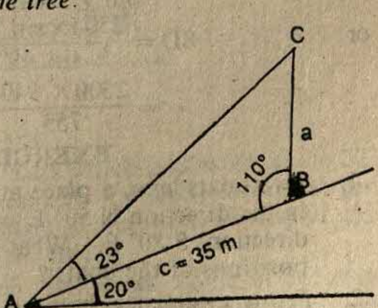


Fig. 15·23.

In the triangle ABC, we have

$\angle BAC=43°-20°=23°$,
$\angle ABC=180°-(90-20)°=110°$,
$\angle ACB=180°-(23°+110°)=47°$,
$c=35\ m$.

By the law of sines,

$$\frac{a}{\sin 23^\circ}=\frac{c}{\sin 47^\circ}$$

so that $$a=\frac{35 \sin 23^\circ}{\sin 47^\circ}=\frac{35\times \cdot 391}{\cdot 731}=19\ m.$$

Example 26. *A surveyor laying a road due west from A encounters a swamp at B. He changes his direction to N 29° W for 2300 m to C and then turns S 41° W. How far must he continue in this direction to reach a point D on the east-west line through A ?*

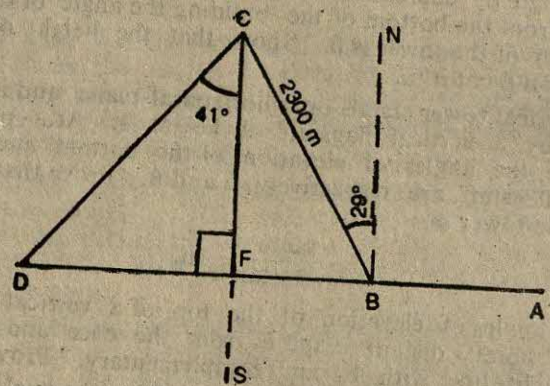


Fig. 15·24.

Solution. In Fig. 15·24, BC=2300 m,

$\angle CBD=90^\circ-29^\circ=61^\circ$,

$\angle CDB=90^\circ-41^\circ=49^\circ$,

$\angle DCB=180^\circ-(61^\circ+49^\circ)=70^\circ$.

By the sine law,

$$\frac{BD}{\sin \angle DCB}=\frac{BC}{\sin \angle CDB},$$

or $$BD=\frac{2300\times \sin 70^\circ}{\sin 49^\circ}\ m$$

$$=\frac{2300\times 940}{\cdot 755}=3100\ m.$$

EXERCISE 15 (g)

1. Two boats leave a place at the same time. One travels 56 *km* in the direction N 50° E while the other travels 48 *km* in the direction S 80° E. What is the distance between the new positions of the boats ?
2. From a defence battery P an enemy ship is sighted in the direction N 57°20′ E. From a second defence battery Q, 11·4 *km* due north of P, the ship is seen in the direction

S 40° 50′ E. How far is the ship from P ? How far is it from Q ?

3. Two ships leave a port at noon. One ship travels at 16 km per hour in the direction N 13° W. The other ship travels at 21 km per hour in the direction N 57° E. How far apart are they at 2 p.m. ?

4. To find the distance between two points A and B on the opposite sides of a river, we measure the distance from A to C to be 75 m, the angle CAB to be 70° and the angle ACB to be 42°. Find the distance AB.

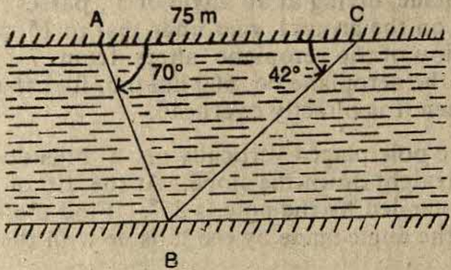


Fig. 15·25.

5. Two men, 750 *m* apart, sight a balloon which is between them and in their vertical plane. The angle of elevation of the balloon as measured by one of the men is 36° and by the other is 56°. Find the distance of the balloon from each observer and the height of the balloon.

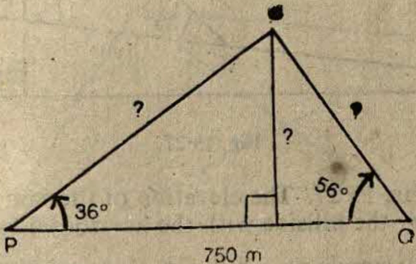


Fig. 15·26.

6. A tree stands vertically on a hillside which makes an angle 22° with the horizontal. From a point 20 *m* directly down the hill from the base of the tree, the angle of elevation of top of the tree is 40°. Find the height of the tree.

7. A tree stands on a hillside which makes an angle of 20° with the horizontal. From a point 50 *m* up the hill from the tree, the angle of depression of the top of the tree is 10°. Find the height of the tree.

8. A hill slopes at an angle of 18° to the horizontal. A vertical tower stands on the hill. A man walks 75 m directly down the hill from the foot of the tower and then observes that the angle of elevation of the top of the tower is 50°. Find the height of the tower.

9. Two points A and B are on opposite sides of a pond. The distance from A to a third point C is measured to be 10·5 m. and the distance from C to B is measured to be 13·8 m. Find the distance from A to B if the angle ACB is 51° 40′.

10. An aeroplane, diving at an angle of 5°, passes directly over an observer on the ground, who finds that in 15 seconds the angle of elevation of the aeroplane alters from 71° to 20°. If the speed of the aeroplane be 660 km p.h, find the height of aeroplane when it is directly overhead.

11. A 5-metre pole placed vertically on a hillside casts a shadow of 7 m straight down the slope. At the tip of the shadow, the angle subtended by the pole is 35°. Find the elevation of the sun and the angle made by the hillside with the horizontal.

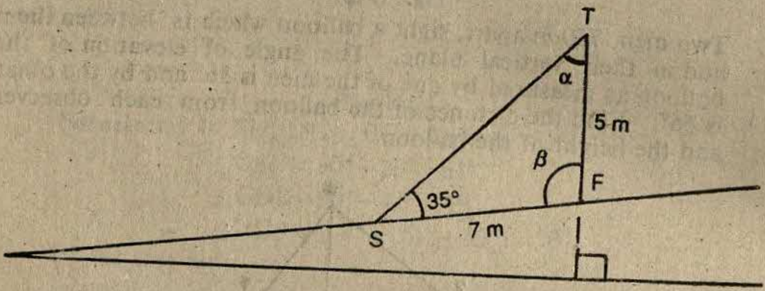


Fig. 15·27.

[**Hint.** See Fig. 15·27. The elevation of the sun is $90°-\alpha$; the angle made by the hillside with the horizontal is $\beta-90°$].

12. A vertical tree stands on a hill side that makes an angle α with the horizontal. From a point directly up the hill, the angle of elevation of the top of the tree is β. From a point l m. further up the hill, the angle of depression of the top of the tree is γ. If the tree be h m. high, express h in terms of l, α, β, γ.

13. P, Q, R are three towns. P is 270 km from Q and 180 km from R. The bearing of P from Q is N 18° W. The bearing of R from Q is N 55° W. What can we say about the distance between Q and R ?

14. On a hillside that makes an angle of 16° with the horizontal, a 10 metre tree leans downhill. When the elevation of the sun is 50°, the shadow of the tree is 5 m long and falls straight up the slope. Find the angle made by the tree with the vertical.

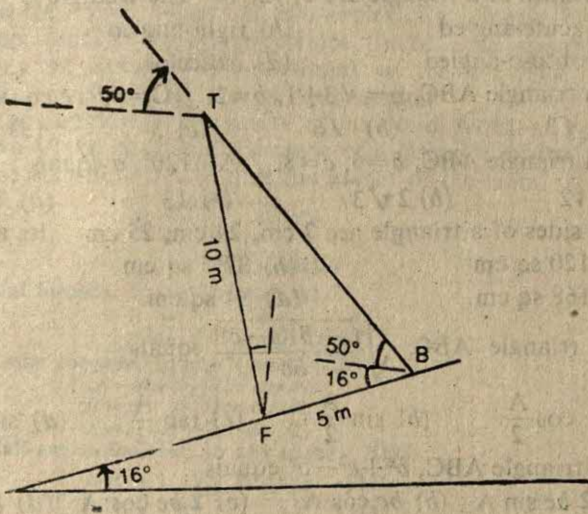


Fig. 15·28.

[**Hint.** See Fig. 15·28. The desired angle is $\alpha - 74°$.]

15. Find the length x in the following figure :

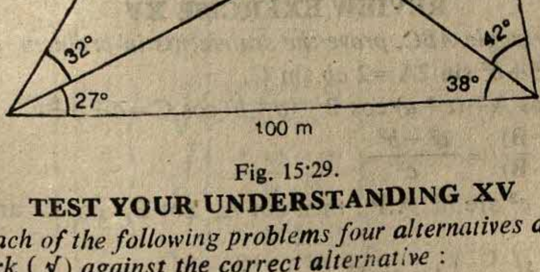


Fig. 15·29.

TEST YOUR UNDERSTANDING XV

In each of the following problems four alternatives are given. Put a tick mark (√) against the correct alternative :

1. In a triangle ABC, $(a \sin B)/\sin A$ equals

(*a*) b (*b*) $\frac{1}{b}$ (*c*) c (*d*) $\sin^2 B$.

2. The sides of a triangle are 1, $\sqrt{3}$, 2. The smallest angle of the triangle is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $\frac{2\pi}{5}$.

3. The sides of a triangle are 5, 12, 14. The triangle is

(a) acute-angled (b) right-angled

(c) obtuse-angled (d) isosceles.

4. In a triangle ABC, $a=\sqrt{3}+1$, $b=2$, $\angle C=60^\circ$. c equals

(a) $\sqrt{3}-1$ (b) $\sqrt{6}$ (c) 3 (d) $\sqrt{5}$.

5. In a triangle ABC, $b=6$, $c=8$, $\angle A=120^\circ$, a equals

(a) 12 (b) $2\sqrt{37}$ (c) 13 (d) 10.

6. The sides of a triangle are 7 cm, 24 cm, 25 cm Its area is

(a) 120 sq cm (b) 87·5 sq cm

(c) 168 sq cm (d) 84 sq cm.

7. In a triangle ABC, $\sqrt{\frac{(s-b)(s-c)}{bc}}$ equals

(a) $\cos\frac{A}{2}$ (b) $\sin\frac{A}{2}$ (c) $\tan\frac{A}{2}$ (d) $\sin A$.

8. In a triangle ABC, $b^2+c^2-a^2$ equals

(a) $2\,bc\sin A$ (b) $bc\cos A$ (c) $2\,bc\cos A$ (d) $bc\sin A$.

9. In a triangle ABC, $b:c=3:2$, and $\tan\frac{A}{2}=\frac{1}{10}$. The value of $\tan\left\{\frac{1}{2}(B-C)\right\}$ is

(a) 2 (b) $\frac{1}{2}$ (c) 5 (d) $\frac{1}{4}$.

10. The angle of elevation of the top of a tower from a point on the ground at a distance 100 m from the foot of the tower is 30°. The height of the tower is

(a) $100\sqrt{3}$ m (b) $50\sqrt{3}$ m (c) $100\sqrt{3}/3$ m (d) 150 m.

REVIEW EXERCISE XV

In any triangle ABC, prove the statements in problem 1—5 :

1. $a^2\sin 2B+b^2\sin 2A=2\,ab\sin C$.
2. $(b+c)\cos A+(c+a)\cos B+(a+b)\cos C=a+b+c$.
3. $\frac{\sin(A-B)}{\sin(A+B)}=\frac{a^2-b^2}{c^2}$.
4. If a^2, b^2, c^2 are in A.P., then cot A, cot B, cot C are also in A.P.
5. If $a\cos^2\left(\frac{C}{2}\right)+c\cos^2\left(\frac{A}{2}\right)=\frac{3b}{2}$, then a, b, c are in A.P.
6. The sides of a triangle are $4\sqrt{3}$, 7, $\sqrt{13}$ cm respectively. Find its smallest angle.

7. Show that $a, b, \sqrt{\{a^2+b^2+ab\}}$ determine a triangle for each pair of positive values of a and b. Find the greatest angle of any such triangle.
8. The three sides of a triangle are in A.P. and the greatest angle exceeds the least by a right angle. Prove that the sides are in the ratios $\sqrt{7}+1 : \sqrt{7} : \sqrt{7}-1$.
9. Show that if the cosines of two angles of a triangle be directly proportional to the opposite sides, the triangle is isosceles ; but if they are inversely proportional to the opposite sides, the triangle is either isosceles or right-angled.
10. If the angle of elevation of a cloud from a point h m above a lake be β, and the angle of depression of its reflection in the lake be α, prove that the height of the cloud is

$$h\,\frac{\sin(\alpha-\beta)}{\sin(\alpha+\beta)}\,m.$$

SUMMARY

1. **Sine formula.** In any triangle ABC

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}.$$

2. **Cosine formula.** In any triangle ABC

$$a^2=b^2+c^2-2bc\cos A,$$
$$b^2=c^2+a^2-2ca\cos B,$$
$$c^2=a^2+b^2-2ab\cos C.$$

3. **Half-anglae formula.** In any triangle ABC

$$\sin\left(\frac{A}{2}\right)=\sqrt{\left\{\frac{(s-b)(s-c)}{bc}\right\}},$$

$$\sin\left(\frac{B}{2}\right)=\sqrt{\left\{\frac{(s-c)(s-a)}{ca}\right\}},$$

$$\sin\left(\frac{C}{2}\right)=\sqrt{\left\{\frac{(s-a)(s-b)}{ab}\right\}};$$

$$\cos\left(\frac{A}{2}\right)=\sqrt{\left\{\frac{s(s-a)}{bc}\right\}},$$

$$\cos\left(\frac{B}{2}\right)=\sqrt{\left\{\frac{s(s-b)}{ca}\right\}},$$

$$\cos\left(\frac{C}{2}\right)=\sqrt{\left\{\frac{s(s-c)}{ab}\right\}},$$

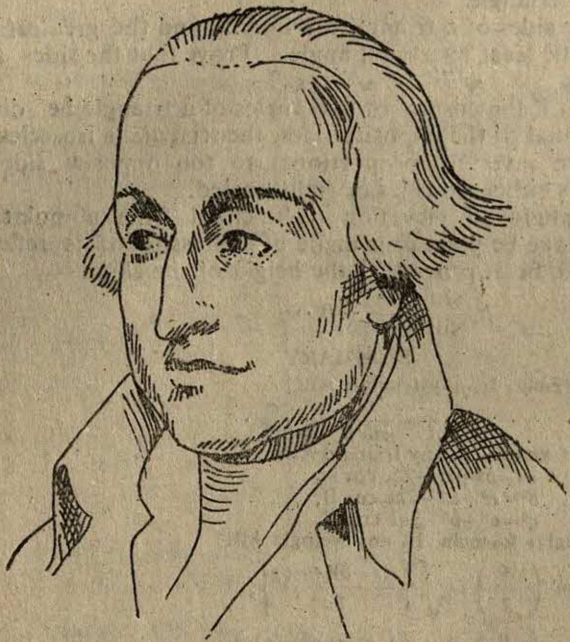
$$\tan\left(\frac{A}{2}\right)=\sqrt{\left\{\frac{(s-b)(s-c)}{s(s-a)}\right\}},$$

$$\tan\left(\frac{B}{2}\right)=\sqrt{\left\{\frac{(s-c)(s-a)}{s(s-b)}\right\}},$$

$$\tan\left(\frac{C}{2}\right)=\sqrt{\left\{\frac{(s-a)(s-b)}{s(s-c)}\right\}}.$$

4. Area of a triangle $=\frac{1}{2}bc\sin A=\frac{1}{2}ca\sin B=\frac{1}{2}ab\sin C,$

$$=\sqrt{s(s-a)(s-b)(s-c)}.$$



JOSEPH LOUIS LAGRANGE (1736—1813)

Joseph Louis Lagrange was born in 1736 in Turin, Italy. He was the greatest and the most modest mathematician of the eighteenth century. At the age of sixteen, Lagrange became professor of mathematics at the Royal Artillery School in Turin. At the age of twenty-three he was elected a foreign member of Berlin Academy. At the age of thirty he was invited to join the court of Frederick the Great at Berlin, a post which he held for twenty years. A few years after leaving Berlin, he became professor at the prestigeous Ecole Polytechnic.

Lagrange gave the first published proof of the theorem that every positive integer can be expressed as the sum of not more than four squares.

In the words of Napoleon Bonaparte, Lagrange was the *lofty pyramid of mathematical sciences.*

CHAPTER 16

Inverse Trigonometric Functions and Trigonometric Equations

16·1. INTRODUCTION

We shall first of all recall some basic facts about functions.

Let f be a function with domain X and range Y. If we can find a function g with domain Y and range X such that the composite function gof is the identity function on X, then we say that the function g is an inverse of f. Since a function can have at the most one inverse, therefore, we shall henceforth talk of *the* inverse of a function rather than *an* inverse.

If f be a function with domain X and range Y, and g be the inverse of f, then it can be shown that f is the inverse of g, and therefore we can talk of a pair of functions as *inverse functions* rather than talking of one of them as being the inverse of the other.

It is not necessary for every function to possess an inverse. If f and g be a pair of inverse functions such that domain of f (range of g) is X and the range of f (domain of g) is Y, then f and g must be both one-to-one. Conversely, if f be a one-to-one function, then it must possess an inverse. A function which possesses an inverse is said to be *invertible*. In view of the preceding discussion, a function is invertible iff it is one-to-one. An important and useful criterion for the invertibility of a function is the following:

A function f with domain X is invertible iff the relation

$$\{(f(x), x) : x \in X)\}$$

is a function. Also, the inverse of f is then this relation

$$\{(f(x), x)\ x \in X\}.$$

For any function f with domain X, we call $\{(f(x), x) : x \in X\}$ the *inverse relation of f*. The above criterion can then be stated in the following form:

A function f is invertible iff its inverse relation is a function. Also, if this be the case, then this function is the inverse of f.

In the present chapter we shall consider the inverse relations of the trigonometric functions. We shall see that while the trigonometric functions are not invertible, it is possible to define certain functions, to be called inverse trigonometric functions (inverse circular functions) which are subsets of the inverse relations of the trigonometric functions. Corresponding to the six trigonometric functions, we have the six inverse trigonometric functions, $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$, $\cot^{-1}$, $\sec^{-1}$ and $\csc^{-1}$. Of course, only the first three are more important (because the last three can be defined in terms of the first three as we shall when we study them).

We shall also consider the applications of inverse trigonometric functions to solution of trigonometric equations.

16·2. THE INVERSE SINE AND THE INVERSE COSINE FUNCTIONS

16·2·1. The Inverse Sine Function

Let us consider the sine function. We know that

$$\sin=\{(x, y) : y=\sin x,\ x \in \mathbf{R}\}.$$

The inverse relation of the sine is the relation

$$\{(y, x) : y=\sin x, x \in \mathbf{R}\},$$

i.e., $$\{(x, y) : x=\sin y, y \in \mathbf{R}\}. \qquad ...(1)$$

This relation is called the **inverse sine relation** or **arcsine relation.**

The inverse sine relation is *not* a function because for a given value of x there may exist distinct numbers y_1, y_2 such that (x, y_1) and (x, y_2) both belong to inverse sine. For example, if we consider $x=0$, we find that ...... $(0, -\pi)$, $(0, 0)$, $(0, \pi)$, $(0, 2\pi)$, $(0, 3\pi)$ ...... all belong to inverse sine, *i.e.*, $(0, n\pi)$ belongs to inverse sine for each $n \in \mathbf{Z}$ (Fig. 16.1). We can, however, obtain a function from the inverse sine relation by placing a suitable restriction on the range of the relation (1) so that for each x in the domain of the relation there exists only one member of the range. Let us see how it can be done.

Let us look at the graph of the sine function (Fig. 16·2) $\{(x, y) : y=\sin x, x \in \mathbf{R}\}$. For each real number a such that $-1 \leqslant a \leqslant 1$, there is one and only one real number x such that $\sin x=a$ and $-\pi/2 \leqslant x \leqslant \pi/2$. We denote this real number by *$\sin^{-1} a$* or *arcsin a*. Thus $x=\sin^{-1} a$ iff $\sin x=a$, and $-\pi/2 \leqslant x \leqslant \pi/2$. For example, $\sin^{-1} \frac{1}{2}=\pi/6$, $\sin^{-1} (-\sqrt{3}/2)=-\pi/3$. The set of all ordered pairs $(x, \sin^{-1} x)$ where $-1 \leqslant x \leqslant 1$ is called the **inverse sine function** or **arcsine function** and is written as arcsin or $\sin^{-1}$. Thus

$$\sin^{-1}=\{(x, \sin^{-1} x) : -1 \leqslant x \leqslant 1\},$$
$$=\{(x, y) : y=\sin^{-1} x, -1 \leqslant x \leqslant 1\}.$$

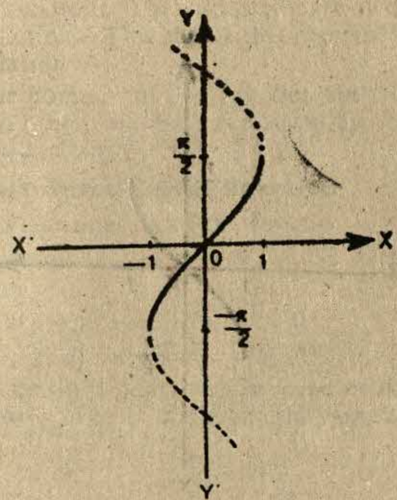


Fig. 16.1. Graph of the inverse sine relation (incomplete graph)

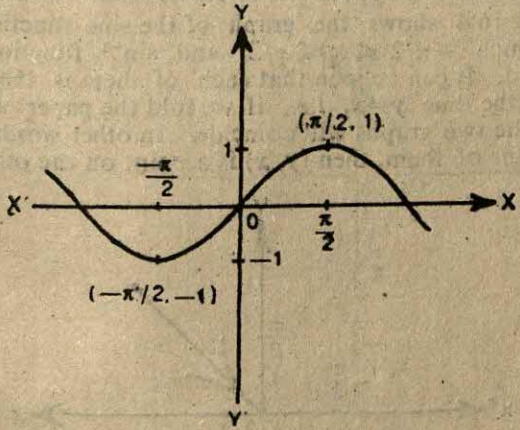


Fig. 16·2. Graph of the sine function (incomplete graph)

It can be easily seen that $\sin^{-1}$ is in fact a function. (Seeing this amounts to verifying that $\sin^{-1} x_1 \neq \sin^{-1} x_2 \Rightarrow x_1 \neq x_2$; and this is true because the sine function is strictly increasing in $[-\pi/2, \pi/2]$, so that if we write $y_1 = \sin^{-1} x_1$, $y_2 = \sin^{-1} x_2$, then $x_1 = \sin y_1$, $x_2 = \sin y_2$ and $\sin^{-1} x_1 \neq \sin^{-1} x_2 \Rightarrow y_1 \neq y_2 \Rightarrow \sin y_1 \neq \sin y_2 \Rightarrow x_1 \neq x_2$).

Fig. 16·3 shows the graph of the inverse sine function.

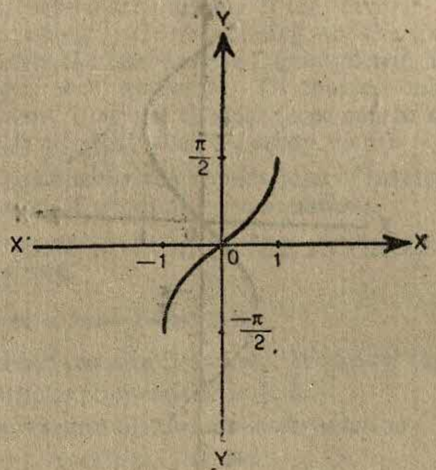


Fig. 16·3. Graph of the inverse sine function

Remarks. 1. Notice that $\sin^{-1}$ is an increasing function. As x increases from -1 to 1, $\sin^{-1} x$ increases from $-\pi/2$ to $\pi/2$.

2. Fig 16·4 shows the graph of the sine function (only that part for which $-\pi/2 \leqslant x \leqslant \pi/2$) and $\sin^{-1}$ function, the latter shown dotted. It can be seen that each of them is the reflection of the other in the line $y=x$, *i.e.*, if we fold the paper along the line $y=x$, then the two graphs will coincide. In other words, if (x, y) is a point on one of them, then (y, x) is a point on the other.

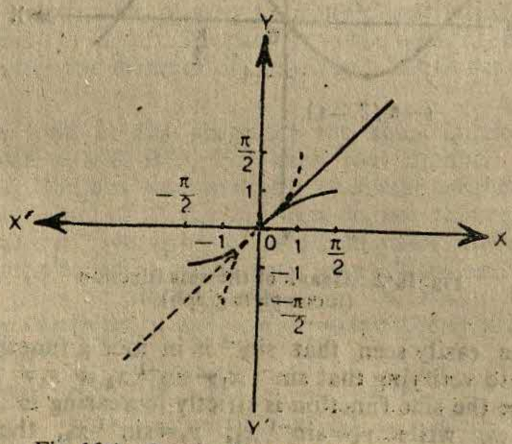


Fig. 16·4. Graph of the sine (incomplete) and inverse sine functions

3. Let us once again consider Fig. 16·1. It shows the graph of the $\sin^{-1}$ relation. The undotted part of the graph is the graph of the $\sin^{-1}$ function. This shows that the $\sin^{-1}$ function is a subset of the $\sin^{-1}$ relation.

4. The domain of the function $\sin^{-1}$ is $[-1, 1]$ and its range is $[-\pi/2, \pi/2]$, *i.e.*, $\sin^{-1} x$ has a meaning only when $-1 \leqslant x \leqslant 1$, and then $-\pi/2 \leqslant \sin^{-1} x \leqslant \pi/2$.

16·2·2. The Inverse Cosine Function

The inverse relation of the function

$$\cos = \{(x, y) : y = \cos x,\ x \in \mathbf{R}\}$$

is the relation

$$\{(y, x) : y = \cos x,\ x \in \mathbf{R}\},$$

i.e., $$\{(x, y) : x = \cos y,\ y \in \mathbf{R}\}.$$

This relation is called the **inverse cosine relation** or **arccosine relation**. Fig 16·5 shows the graph of the inverse cosine relation.

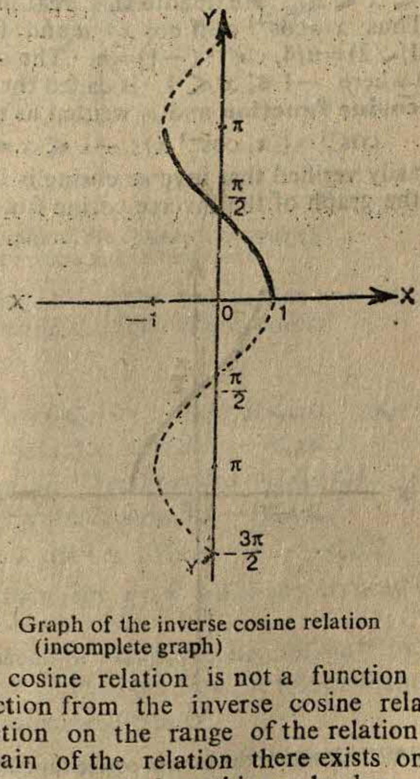


Fig. 16·5. Graph of the inverse cosine relation (incomplete graph)

The inverse cosine relation is not a function (why ?), but we can obtain a function from the inverse cosine relation by placing a suitable restriction on the range of the relation (2) so that for each x in the domain of the relation there exists only one member of the range. Let us see as to how this can be done.

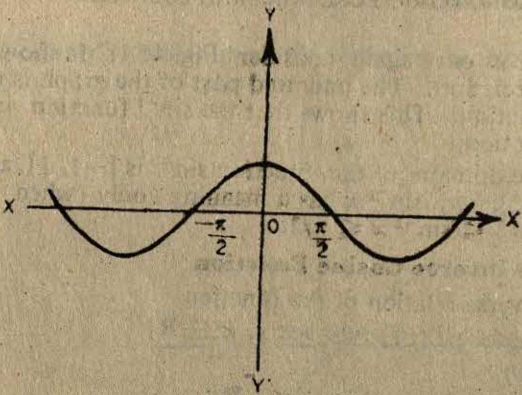


Fig. 16·6. Graph of the cosine function (incomplete graph)

Let us look at the graph of the cosine function (Fig. 16·6) $\{(x, y) : y = \cos x,\ x \in \mathbf{R}\}$. For each real number a such that $-1 \leqslant a \leqslant 1$ there is one and only one real number x such that $\cos x = a$ and $0 \leqslant x \leqslant \pi$. We denote this real number by $\cos^{-1} a$ or *arccos a*. Thus $x = \cos^{-1} a$ iff $\cos x = a$ and $0 \leqslant x \leqslant \pi$. For example, $\cos^{-1}(1/\sqrt{2}) = \pi/4$, $\cos^{-1}(-1) = \pi$. The set of all ordered pairs $(x, \cos^{-1} x)$ where $-1 \leqslant x \leqslant 1$, is called the **inverse cosine function** or **arccosine function** and is written as $\cos^{-1}$ or arccos.

Thus $\qquad \cos^{-1} = \{(x, \cos^{-1} x) : -1 \leqslant x \leqslant 1\}.$

It can be easily verified that inverse cosine is in fact a function. Fig. 16·7 shows the graph of the inverse cosine function.

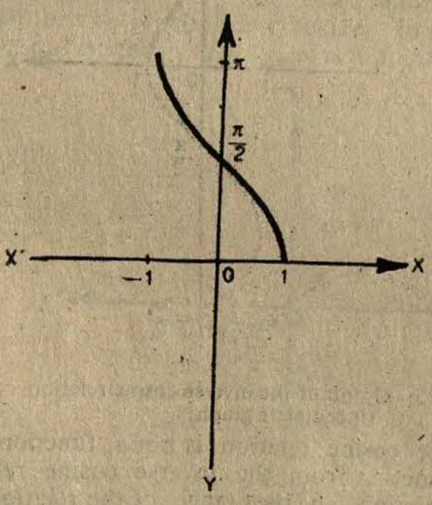


Fig. 16·7. Graph of the inverse cosine function

Remarks. 1. Note that $\cos^{-1}$ is a decreasing function. As x increases from -1 to 1, $\cos^{-1} x$ decreases from x to 0.

2. Fig. 16·8 shows the graph of that part of the cosine function which corresponds to $0 < x \leqslant \pi$, shown dotted, and that of the inverse cosine function. It can be easily seen that each of them is the reflection of the other in the line $y=x$, *i.e.*, if (x, y) is a point on one of them, then (y, x) is a point on the other.

3. The domain of the function $\cos^{-1}$ is $[-1, 1]$ and its range is $[0, \pi]$, *i.e.*, $\cos^{-1} x$ has a meaning only when $-1 \leqslant x < 1$, and then $0 < \cos^{-1} x \leqslant \pi$.

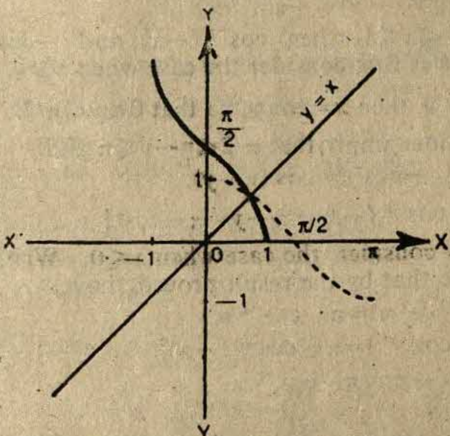


Fig. 16·8. Graphs of the cosine (incomplete) and inverse cosine functions

Some important facts about the inverse sine and inverse cosine functions are summarised in the following theorem :

Theorem 16·1.

(*a*) *If* $0 \leqslant x \leqslant 1$, *then* $0 \leqslant \sin^{-1} x \leqslant \pi/2$, $0 < \cos^{-1} x \leqslant \pi/2$.

(*b*) $\sin^{-1}(-x) = -\sin^{-1} x$, *provided* $-1 \leqslant x \leqslant 1$.

(*c*) $\cos^{-1}(-x) = \pi - \cos^{-1} x$, *provided* $-1 \leqslant x \leqslant 1$.

(*d*) $\cos^{-1} x + \sin^{-1} x = \pi/2$, *provided* $-1 \leqslant x \leqslant 1$.

Proof. (*a*) Let $y = \sin^{-1} x$. Then $-\pi/2 < y \leqslant \pi/2$.

If $-d/2 \leqslant y < 0$, then $\sin y \leqslant 0$ which is a contradiction since $y = \sin^{-1} x \Rightarrow x = \sin y \Rightarrow \sin y \geqslant 0$.

Therefore, we must have $0 < y \leqslant \pi/2$, *i.e.*, $0 \leqslant \sin^{-1} x \leqslant \pi/2$.

Again, let $z = \cos^{-1} x$. Then $0 < z < \pi$. If $p/2 < z \leqslant \pi$, then $\cos z < 0$ which is a contradiction.

Therefore, we must have $0 \leqslant z < \pi/2$, *i.e.*, $0 \leqslant \cos^{-1} x < \pi/2$.

(b) If $-1 \leqslant x \leqslant 1$, then $\sin^{-1}(-x)$ and $-\sin^{-1}(x)$ are both meaningful. Let us first consider the case when $x \geqslant 0$.

If $\sin^{-1} x = y$, then $\sin y = x \geqslant 0$, which means $0 \leqslant y \leqslant \pi/2$.

These relations imply that $-\pi/2 \leqslant -y \leqslant 0$ and $-x = \sin(-y)$.

Therefore, $\sin^{-1}(-x) = -y = -\sin^{-1} x$.

Let us now consider the case when $x < 0$. Writing $-x = u$, we find that $u \geqslant 0$, so that by the result proved above.

$\sin^{-1}(-u) = -\sin^{-1} u$, *i.e.*, $\sin^{-1} x = -\sin^{-1}(-x)$, whence we again have $\sin^{-1}(-x) = -\sin^{-1} x$.

(c) If $-1 \leqslant x \leqslant 1$, then $\cos^{-1}(-x)$ and $-\cos^{-1} x$ are both meaningful. Let us first consider the case when $x \geqslant 0$.

If $\cos^{-1} x = y$, then $x = \cos y$, so that $0 \leqslant y \leqslant \pi/2$.

These relations imply that $\pi/2 \leqslant \pi - y \leqslant \pi$ and

$$-x = -\cos y = \cos(\pi - y).$$

Therefore, $\cos^{-1}(-x) = \pi - y = \pi - \cos^{-1} x$.

Let us now consider the case when $x \leqslant 0$. Writing $-x = u$ we find that $u \geqslant 0$, so that by the result proved above,

$$\cos^{-1}(-u) = \pi - \cos^{-1} u,$$

or

$$\cos^{-1} x = \pi - \cos^{-1}(-x),$$

or

$$\cos^{-1}(-x) = \pi - \cos^{-1} x$$

as before.

(d) Since $-1 \leqslant x \leqslant 1$, therefore, each term on the left-hand side is meaningful.

Let $x \geqslant 0$ and let $\cos^{-1} x = y$. Then $\cos y = x$ and $0 \leqslant y \leqslant \pi/2$.

Therefore, $\sin(\pi/2 - y) = x$ and $0 \leqslant \pi/2 - y \leqslant \pi/2$.

Therefore, $\sin^{-1} x = \pi/2 - y = \pi/2 - \cos^{-1} x$,

so that
$$\cos^{-1} x + \sin^{-1} x = \pi/2. \qquad ...(i)$$

If $x < 0$, then $-x > 0$, and (*i*) gives

$$\cos^{-1}(-x) + \sin^{-1}(-x) = \pi/2.$$

But $\cos^{-1}(-x) = \pi - \cos^{-1} x$, and $\sin^{-1}(-x) = -\sin^{-1} x$.

Therefore, $(\pi - \cos^{-1} x) + (-\sin^{-1} x) = \pi/2$,

so that
$$\cos^{-1} x + \sin^{-1} x = \pi/2. \qquad ...(ii)$$

From (*i*) and (*ii*), we find that

$$\boxed{\begin{array}{c}\text{If } -1 \leqslant x \leqslant 1, \text{ then} \\ \cos^{-1} x + \sin^{-1} x = \pi/2.\end{array}}$$

Remark. The above theorem enables us to express $\sin^{-1} x$ in terms of $\sin^{-1}(|x|)$ and $\cos^{-1} x$ in terms of $\cos^{-1}(|x|)$. Also,

for any given positive real number x, the values of expressions of the form $\cos(\sin^{-1} x)$ etc., can be computed by the use of right-angled triangles. Thus we can now compute $\sin(\cos^{-1} x)$, $\cos(\sin^{-1} x)$, etc., whenever such an expression has a meaning.

Example 1. *Evaluate* $\cos\left(\sin^{-1} \frac{3}{5}\right)$.

Solution. Let $\sin^{-1}(3/5)=y$, so that $\sin y=3/5$,

and $$0 \leqslant y < \frac{\pi}{2}.$$

Construct a right-angled triangle OAB in which

AB=3 units, OB=5 units,

Then $\sin \angle AOB=3/5$, so that the measures of $\angle AOB$ in radians is $\sin^{-1}(3/5)$.

In the right-angled triangle OAB.

$OA^2=OB^2-AB^2=5^2-3^2=16$,

so that OA=4 units.

Then $\cos\left(\sin^{-1}\frac{3}{5}\right)=\cos y$

$$=\cos \angle AOB=\frac{OA}{OB}=\frac{4}{5}.$$

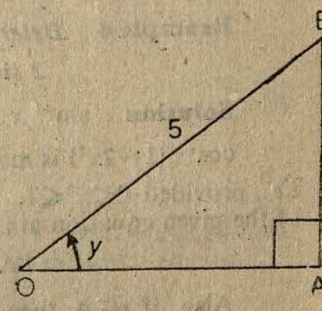


Fig. 16·9.

Example 2. *Evaluate* $\cos\left[\sin^{-1}\left(-\frac{3}{5}\right)\right]$.

Solution. Since $\sin^{-1}(-x)=-\sin^{-1} x$,

therefore, $\sin^{-1}\left(-\frac{3}{5}\right)=\left(-\sin^{-1}\frac{3}{5}\right)$ and hence

since $\cos(-x)=\cos x$,

$$\cos\left[\sin^{-1}\left(-\frac{3}{5}\right)\right]=\cos\left(\sin^{-1}\frac{3}{5}\right)$$

$$=\frac{4}{5}, \text{ as in Example 1.}$$

Example 3. *Evaluate* $\tan\left(\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{5}{13}\right)$.

Solution. Let $\sin^{-1}(3/5)=x$ and $\sin^{-1}(5/13)=y$,

so that

$$\sin x=3/5,\ 0\leqslant x<\pi/2,$$
$$\sin y=5/13,\ 0<y\leqslant\pi/2.$$

$\therefore$

$$\tan x=\frac{\sin x}{\sqrt{(1-\sin^2 x)}}=\frac{3/5}{\sqrt{[1-(3/5)^2]}},$$
$$=3/4,$$

$$\tan y=\frac{\sin y}{\sqrt{(1-\sin y^2)}}=\frac{5/13}{\sqrt{[1-(5/13)^2]}},$$
$$=5/12.$$

$$\therefore\quad \tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}=\frac{\frac{3}{4}+\frac{5}{12}}{1-\frac{3}{4}\cdot\frac{5}{12}}=\frac{56}{33}.$$

Example 4. *Determine the solution set of the equation*

$$2\sin^{-1}x=\cos^{-1}(1-2x^2).$$

Solution. $\sin^{-1}x$ is meaningful provided $-1\leqslant x\leqslant 1$.

$\cos^{-1}(1-2x^2)$ is meaningful provided $-1\leqslant 1-2x^2\leqslant 1$, *i.e.*, provided $0\leqslant x^2\leqslant 1$, *i.e.*, provided $|x|\leqslant 1$. Thus both sides of the given equation are meaningful when x belongs to the set

$$A=\{a:-1\leqslant a\leqslant 1\}.$$

Also, if $x\in A$, then $\cos^{-1}(1-2x^2)$ lies between 0 and π.

Therefore, if the equality has to hold, then we must have

$$0<2\sin^{-1}x<\pi,$$

i.e., $$0\leqslant\sin^{-1}x\leqslant\frac{\pi}{2},$$

i.e., $$0\leqslant x\leqslant 1,$$

i.e., x belongs to the set

$$B=\{a:0<a\leqslant 1\}.$$

We shall now show that if

$$x\in A\cap B=\{a:0<a\leqslant 1\},$$

then the given equation is satisfied.

Let $\quad x\in A\cap B$ and let $\sin^{-1}x=y$.

Then $x=\sin y$, and $0<y\leqslant\pi/2$, since y must always lie between $-\pi/2$ and $\pi/2$, and since x is non-negative, therefore, y cannot be negative.

Therefore, $\cos 2y=1-2\sin^2 y=1-2x^2$.

and $$0<2y\leqslant\pi.$$

Thus $$2y=\cos^{-1}(1-2x^2),$$

or $$2\sin^{-1} x=\cos^{-1}(1-2x^2).$$

Hence the solution set of the given equation is

$$\{a : 0<a\leqslant 1\}.$$

Example 5. *Determine the solution-set of the equation*

$$\sin^{-1}(2\sqrt{x})=\cos^{-1} x.$$

Solution. Since $\sqrt{x}$ occurs in the left-hand side of the given equation, therefore, $x\geqslant 0$. Also, if $x\geqslant 0$, then $\cos^{-1} x$ lies between 0 and $\pi/2$.

Writing $\sin^{-1}\sqrt{(2x)}=\cos^{-1} x=y$, we have

$$2\sqrt{x}=\sin y,\; x=\cos y.$$

Since $\cos^2 y+\sin^2 y=1$,

therefore, we must have $x^2+4x=1$. ...(*i*)

Solving equation (*i*), we have

$$x=-2\pm\sqrt{5}.$$

Since $x\geqslant 0$, therefore, we must have $x=\sqrt{5}-2$. We shall now verify that $x=\sqrt{5}-2$ is actually a solution of the given equation.

Now $x=\sqrt{5}-2 \Rightarrow (x+2)^2=5,$

$\Rightarrow 4x=1-x^2,$

$\Rightarrow 2\sqrt{x}=\sqrt{1-x^2}$, since x is positive,

$\Rightarrow \sin^{-1}(2\sqrt{x})=\sin^{-1}\{\sqrt{(1-x^2)}\}$ ...(*ii*)

Also, $\sin^{-1}(\sqrt{1-x^2})=\cos^{-1} x$, whenever $0<x<1$. ...(*iii*)

From (*ii*) and (*iii*), we have

$\sin^{-1}(2\sqrt{x})=\cos^{-1} x$, when $x=\sqrt{5}-2$.

Therefore, $x=\sqrt{5}-2$ is the solution.

EXERCISE 16 (*a*)

1. Find the value of :

(*i*) $\sin^{-1}(-1)$ (*ii*) $\sin^{-1}\frac{1}{2}$

(*iii*) $\sin^{-1}(1/\sqrt{2})$ (*iv*) $\sin^{-1}(-\sqrt{3}/2)$

(*v*) $\cos^{-1}(-1)$ (*vi*) $\cos^{-1}(1/\sqrt{2})$

(*vii*) $\cos^{-1}(-\sqrt{3}/2)$ (*viii*) $\cos^{-1}(-\frac{1}{2})$

Evaluate :

2. $\cos\left(\sin^{-1}\frac{5}{13}\right)$. 3. $\cos\left(-\cos^{-1}\frac{4}{5}\right)$.

4. $\cos\left[\sin^{-1}\left(-\frac{4}{5}\right)\right]$. 5. $\tan\left(\cos^{-1}\frac{8}{17}\right)$.

6. $\cot\left[\sin^{-1}\left(-\frac{7}{25}\right)\right]$. 7. $\csc\left[\cos^{-1}\left(-\frac{12}{13}\right)\right]$.

8. $\cos\left\{\sin^{-1}\frac{5}{13}+\cos^{-1}\left(-\frac{4}{5}\right)\right\}$.

9. $\tan\left\{\sin^{-1}\left(-\frac{5}{13}\right)+\cos^{-1}\frac{8}{17}\right\}$.

10. $\tan\left\{\cos^{-1}\left(-\frac{1}{2}\right)+\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right\}$.

Verify each of the following statements :

11. $\cos^{-1}\frac{11}{14}+\sin^{-1}\frac{3\sqrt{3}}{14}=\frac{\pi}{3}$.

12. $\sin^{-1}\left(\frac{1}{\sqrt{82}}\right)+\sin^{-1}\left(\frac{4}{\sqrt{41}}\right)=\frac{\pi}{4}$.

13. $\sin^{-1}\frac{59}{91}+\sin^{-1}\frac{1}{7}=\cos^{-1}\frac{11}{13}$.

14. $\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{33}{65}=\sin^{-1}\frac{4}{5}$.

16·3. THE INVERSE TANGENT AND THE INVERSE COTANGENT FUNCTIONS

16·3·1. The Inverse Tangent Function

Recall that the domain of the tangent function is the set

$$\mathbf{R}^*=\mathbf{R}\sim\left\{(2n+1)\frac{\pi}{2}:n\in\mathbf{Z}\right\}.$$

The inverse relation of the function

$$\tan=\{(x,y):y=\tan x,\ x\in\mathbf{R}^*\},$$

is the relation

$$\{(y,x):y=\tan x,\ x\in\mathbf{R}^*\},$$

i.e., $$\{(x,y):x=\tan y,\ y\in\mathbf{R}^*\}. \quad ...(i)$$

This relation is called the **inverse tangent relation** or **arctangent relation.** The inverse tangent relation is not a function (why ?), but we can obtain a function from it by placing a suitable restriction on the range of (*i*) so that for each x in the domain of the relation there exists only one member of the range. Let us see how this can be done.

Let us look at the graph of the tangent function (Fig. 16·10) $\{(x, y): y=\tan x, \ x \in \mathbf{R}^*\}$. The graph indicates that for each real number a, there is a unique real number x such that

$$\tan x=a, -\frac{\pi}{2}<x<\frac{\pi}{2}$$

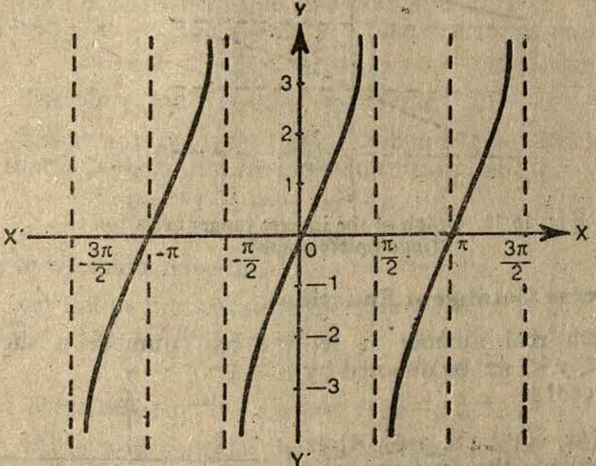


Fig. 16·10. Graph of the tangent function (incomplete graph)

We denote this number by tan^{-1} *a* or *arctan a*. Thus $x=\tan^{-1} a$ iff $\tan x=a$ and $-\pi/2<x<\pi/2$.

For example,

$$\tan^{-1} \sqrt{3}=\frac{\pi}{3}, \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)=-\frac{\pi}{6}.$$

The set of all ordered pairs $(x, \tan^{-1} x)$, where $-\pi/2<x<\pi/2$ is called the **inverse tangent function** or **arctangent function** and is written as $\tan^{-1}$ or arctan.

Thus $\tan^{-1}=\{(x, \tan^{-1} x): x \in \mathbf{R}\}$.

Fig. 16·11 shows the graph of the inverse tangent relation. The undotted part of the graph is the graph of the inverse tangent function. Note that $\tan^{-1}$ is an increasing function.

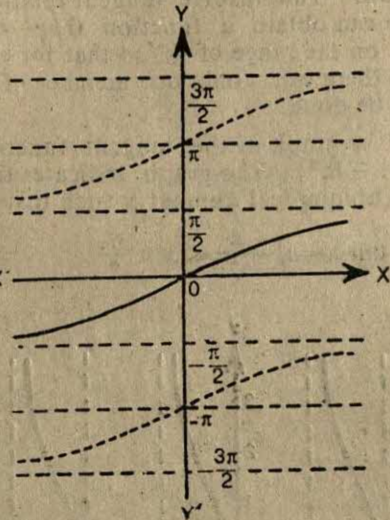


Fig. 16·11. Graph of the inverse tangent relation (incomplete graph)

16·3·2. Inverse Cotangent Function

For each real number x, let the real number y such that $\cot y = x$, $0 < y < \pi$, be denoted by $\cot^{-1} a$ or arccot x.

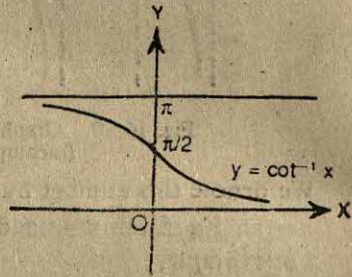


Fig. 16.12. Graph of the inverse cotangent function.

Then $\{(x, \cot^{-1} x) : x \in \mathbf{R}\}$ is a function, called the **inverse cotangent function** or **arccotangent function** and is denoted by $\cot^{-1}$ or arccot. The domain of this function is $\mathbf{R}$, and its range is $]0, \pi[$. If $x \geqslant 0$, $\cot^{-1} x$ lies between $\pi/2$ and 0 as will be shown presently. If $x \leqslant 0$, $\cot^{-1} x$ lies between $\pi/2$ and π. The function is monotonically decreasing throughout. The graph of the function $\cot^{-1}$ is as shown in Fig. 16·12. Some useful facts about $\tan^{-1} x$ and $\cot^{-1} x$ are summarised in the following theorem :

Theorem 16·2.

(*a*) *If* $x \geqslant 0$, *then* $0 \leqslant \tan^{-1} x < \pi/2$, $0 < \cot^{-1} x \leqslant \pi/2$,

(*b*) $\tan^{-1} (-x) = -\tan^{-1} x$.

(*c*) $\cot^{-1} (-x) = \pi - \cot^{-1} x$.

(*d*) *If* $-1 \leqslant x \leqslant 1$, *then* $\tan^{-1} x + \cot^{-1} x = \pi/2$.

Proof. (*a*) Let $\tan^{-1} x = u$.

Then $x=\tan u$ and $-\pi/2 < u < \pi/2$.

Since $x \geqslant 0$, therefore, u cannot lie in $]-\pi/2, 0[$.

Hence u lies in $[0, \pi/2[$,

i.e., $\quad 0 \leqslant \tan^{-1} x < \pi/2$.

Again, let $\cot^{-1} x=v$.

Then $x=\cot v$ and v lies either in $[-\pi/2, [0$ or in $]0, \pi/2]$.

Since $x \geqslant 0$, therefore, v cannot lie in $[-\pi/2, 0[$.

Therefore, v lies in $]0, \pi/2]$.

(*b*) Let us first consider the case when $x \geqslant 0$. If $\tan^{-1} x=y$, then $0 \leqslant y < \pi/2$ and $x=\tan y$. These relations imply that

$$-\pi/2 < -y \leqslant 0 \text{ and } -x=-\tan y=\tan(-y).$$

Therefore, $\tan^{-1}(-x)=-y=-\tan^{-1} x$.

Let us now consider the case when $x \leqslant 0$. Writing $-x=u$ we find that $u \geqslant 0$, so that by the result proved above,

$$\tan^{-1}(-u)=-\tan^{-1} u,$$

or

$$\tan^{-1} x=-\tan^{-1}(-x),$$

whence we again have $\tan^{-1}(-x)=-\tan^{-1} x$.

(*c*) Let us first consider the case when $x \geqslant 0$. If $\cot^{-1} x=y$, then $0 < y \leqslant \pi/2$ and $x=\cot y$. These relations imply that

$$\pi/2 \leqslant \pi-y < \pi \text{ and } -x=\cot(\pi-y).$$

Therefore, $\cot^{-1}(-x)=\pi-y=\pi-\cot^{-1} x$.

Let us now consider the case when $x \leqslant 0$. Writing $-x=u$, we find that $u \geqslant 0$, so that by the result proved above,

$$\cot^{-1}(-u)=\pi-\cot^{-1} u,$$

or

$$\cot^{-1} x=\pi-\cot^{-1}(-x),$$

or

$$\cot^{-1}(-x)=\pi-\cot^{-1} x.$$

(*d*) Let us first consider the case when $x \geqslant 0$.

If $\tan^{-1} x=y$, then $x=\tan y$ and $0 \leqslant y < \pi/2$,

i.e., $x=\tan y=\cot(\pi/2-y)$, $0 < \pi/2-y \leqslant \pi/2$.

Therefore, $\cot^{-1} x=\pi/2-y=\pi/2-\tan^{-1} x$,

so that $\quad \cot^{-1} x+\tan^{-1} x=\pi/2$, provided $x \geqslant 0 \qquad \ldots(i)$

Let us now consider the case when $x \leqslant 0$. Writing $x=-u$, we find that $u \geqslant 0$, so that by the result proved above we have

$$\tan^{-1} u+\cot^{-1} u=\pi/2,$$

or

$$\tan^{-1}(-x)+\cot^{-1}(-x)=\pi/2.$$

But $\quad \tan^{-1}(-x)=-\tan^{-1} x,$

and $\quad \cot^{-1}(-x)=\pi-\cot^{-1} x.$

Therefore, $-(\tan^{-1} x)+(\pi-\cot^{-1} x)=\pi/2$,

so that $$\tan^{-1} x+\cot^{-1} x=\pi/2, \text{ provided } x \leqslant 0. \quad ...(ii)$$

From (*i*) and (*ii*), we find that for all $x \in \mathbf{R}$

$$\tan^{-1} x+\cot^{-1} x=\pi/2.$$

Example 6. *Show that* $\cos^{-1} (12/13)=\tan^{-1} (5/12)$.

Solution. Let $\cos^{-1} (12/13)=y$. Then we have to show that

$$\tan y=5/12 \text{ and } -\pi/2 < y < \pi/2.$$

Since 12/13 lies between 0 and 1, therefore, y lies between 0 and $\pi/2$. The second condition is thus satisfied and we have only to prove that $\tan y=5/12$. Construct a right-angled triangle OAB with OA=12 unit, OB=13 units.

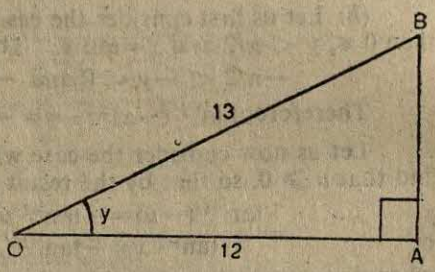


Fig. 16·13.

Now $AB^2=OB^2-OA^2$,
$=(13)^2-(12)^2$,
$=5^2$,

so that $AB=5$ units.

Then $\cos \angle AOB=12/13$,

so that the measure of AOB in radians is y.

Now $\tan y=\tan \angle AOB=AB/OA=5/12$.

Hence $\cos^{-1} (12/13)=y=\tan^{-1} (5/12)$.

Aliter. Let $\cos^{-1} (12/13)=y$. Then we have to show that

$$\tan y=5/12 \text{ and } -\pi/2 < y < \pi/2.$$

Since 12/13 lies between 0 and 1, therefore, y lies between 0 and $\pi/2$. The second condition is thus satisfied and we have only to prove that $\tan y=5/12$.

$$\text{Now } \tan^2 y=\frac{1-\cos^2 y}{\cos^2 y}=\frac{1-(12/13)^2}{(12/13)^2}=\frac{25}{144},$$

so that $\tan y=\pm 5/12$.

Since y lies between 0 and $\pi/2$, therefore, $\tan y$ is positive, and consequently $\tan y=5/12$.

Hence $\cos^{-1} (12/13)=\tan^{-1} (5/12)$.

Remark. We have solved the above example by two different methods. In the first method we have used a right-angled triangle. The second method illustrates the fact that it is not necessary to base the solution on a right angled triangle.

Example 7. *Show that* $\cos^{-1} (-12/13)=\pi+\tan^{-1} (-5/12)$.

Solution. Since $\cos^{-1}(-x)=\pi-\cos^{-1}x$, therefore,

$$\cos^{-1}(-12/13)=\pi-\cos^{-1}(12/13),$$
$$=\pi-\tan^{-1}(5/12), \text{ as in Example 6},$$
$$=\pi+[-\tan^{-1}(5/12)],$$
$$=\pi+\tan^{-1}(-5/12),$$

since $\quad \tan^{-1}(-x)=-\tan^{-1}x$.

Example 8. *Evaluate the expression*

$$\sin[\cos^{-1}(3/5)+\tan^{-1}(-2)].$$

Solution. Let $\cos^{-1}(3/5)=x$ and $\tan^{-1}(-2)=y$.

Then $\cos x=3/5$, $0\leqslant x\leqslant\pi/2$ and $\tan y=-2$, $-\pi/2<y\leqslant 0$.

Since $\sin^2 x=1-\cos^2 x=1-(3/5)^2=(4/5)^2$, therefore,

$$\sin x=\pm 4/5.$$

Since $0\leqslant x\leqslant\pi/2$, $\sin x$ must be non-negative. Consequently,

$$\sin x=4/5.$$

Now $\qquad \cos^2 y=\dfrac{1}{1+\tan^2 y}=\dfrac{1}{1+(-2)^2}=\dfrac{1}{5}.$

Since $-\pi/2<y\leqslant 0$, therefore, $\cos y$ must be non-negative.

Consequently, $\cos y=1/\sqrt{5}$.

Also, $\sin y=\tan y\cos y=(-2).\dfrac{1}{\sqrt{5}}=-\dfrac{2}{\sqrt{5}}.$

Now $\sin\{\cos^{-1}(3/5)+\tan^{-1}(-2)\}=\sin(x+y)$,

$$=\sin x\cos y+\cos x\sin y,$$
$$=\frac{4}{5}\cdot\frac{1}{\sqrt{5}}+\frac{3}{5}\cdot\left(\frac{-2}{\sqrt{5}}\right),$$
$$=-\frac{2}{5\sqrt{5}}.$$

Example 9. *Evaluate the expression*

$$\tan[\sin^{-1}(-3/5)+\cot^{-1}3].$$

Solution. Let $\sin^{-1}(-3/5)=x$ and $\cot^{-1}3=y$.

Then $\sin x=-3/5$, $-\pi/2\leqslant x\leqslant 0$, and $\cot y=3$, $0<y\leqslant\pi/2$.

Since $-\pi/2\leqslant x\leqslant 0$, therefore, $\tan x\leqslant 0$. Also,

$$\tan^2 x=\frac{\sin^2 x}{1-\sin^2 x}=\frac{(-3/5)^2}{1-(-3/5)^2}=\frac{9}{16},$$

so that $\qquad \tan x=\dfrac{-3}{4}.$

Since $\cot y=3$, therefore, $\tan y=1/3$.

Now $\tan\{\sin^{-1}(-3/5)+\cot^{-1}5\}=\tan(x+y)$,

$$=\frac{\tan x+\tan y}{1-\tan x\tan y},$$

$$=\frac{-3/4+1/3}{1-(-3/4).1/3}=\frac{-1}{3}.$$

Example 10. *Verify that*

$$\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{3}\right)=\frac{\pi}{4}.$$

Solution. Let $\tan^{-1}\left(\frac{1}{2}\right)=x$ and $\tan^{-1}\left(\frac{1}{3}\right)=y$.

Since $0<\frac{1}{2}<1$, therefore, $0<\tan^{-1}\left(\frac{1}{2}\right)<\frac{\pi}{4}$, *i.e.*, $0<x<\frac{\pi}{4}$.

Similarly $0<y<\frac{\pi}{4}$.

Therefore, $0<x+y<\frac{\pi}{2}$. ...(*i*)

Also, $\tan x=\frac{1}{2}$, $\tan y=\frac{1}{3}$.

Now $\tan\left\{\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{3}\right)\right\}=\tan(x+y)$,

$$=\frac{\tan x+\tan y}{1-\tan x\tan y},$$

$$=\frac{1/2+1/3}{1-1/2.1/3},$$

$$=1. \qquad ...(ii)$$

Since $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}$ lies between 0 and $\frac{\pi}{2}$, therefore, we have from (*ii*),

$$\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{3}\right)=\tan^{-1}1=\frac{\pi}{4}.$$

16·3·3. A Sum Theorem for Inverse Tangent Function

Theorem 16·3.

(*a*) *If* $0\leqslant xy<1$, *then*

$$\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right).$$

(*b*) *If* $xy=1$, *then*

$$\tan^{-1}x+\tan^{-1}y=\begin{cases}\pi/2, & \text{if } x>0,\ y>0;\\ -\pi/2, & \text{if } x<0,\ y<0.\end{cases}$$

(*c*) *If* $xy > 1$, *then*

$$\tan^{-1} x+\tan^{-1} y=\begin{cases} \pi+\tan^{-1}\left(\dfrac{x+y}{1-xy}\right), & \text{if } x>0,\ y>0; \\ -\pi+\tan^{-1}\left(\dfrac{x+y}{1-xy}\right), & \text{if } x<0,\ y<0. \end{cases}$$

(*d*) *If* $x \geqslant 0$, $y \geqslant 0$, *then*

$$\tan^{-1} x-\tan^{-1} y=\tan^{-1}\left(\frac{x-y}{1+xy}\right).$$

Proof. (*a*) Let $u=\tan^{-1} x$ and $v=\tan^{-1} y$.

If $xy=0$, the proof is trivial. Let us, therefore, consider the case $0<xy<1$. Two different cases arise according as x and y are both positive or both negative.

Let us first assume that $x>0$ and $y>0$.

Then $0<u<\frac{\pi}{2}$, $0<v<\frac{\pi}{2}$, $x=\tan u$, $y=\tan v$.

Since $xy<1$, therefore, $x<\frac{1}{y}$, *i.e.*, $\tan u<\frac{1}{\tan v}$, *i.e.*,

$$\tan u<\tan\left(\frac{\pi}{2}-v\right). \qquad ...(i)$$

Since tan is strictly increasing in $\left]0, \frac{\pi}{2}\right[$, and u and $\frac{\pi}{2}-v$ both lie between 0 and $\frac{\pi}{2}$, therefore, we have from (*i*)

$$u<\frac{\pi}{2}-v,$$

so that

$$0<u+v<\frac{\pi}{2}. \qquad ...(ii)$$

Therefore, $\tan(u+v)$ is defined.

Now $\tan(u+v)=\dfrac{\tan u+\tan v}{1-\tan u \tan v}$,

$$=\frac{x+y}{1-xy}. \qquad ...(iii)$$

From (*ii*) and (*iii*) we have

$$u+v=\tan^{-1}\frac{x+y}{1-xy}.$$

Thus $\tan^{-1} x+\tan^{-1} y=\tan^{-1}\dfrac{x+y}{1-xy}$, provided $x>0$, $y>0$ and $xy<1$. ...(*iv*)

Let now $x<0$, $y<0$. Then $-x>0$, $-y>0$, $(-x)(-y)<1$. From (*iv*) we have

$$\tan^{-1}(-x)+\tan^{-1}(-y)=\tan^{-1}\frac{(-x)+(-y)}{1-(-x)(-y)},$$

or $$\tan^{-1}(-x)+\tan^{-1}(-y)=\tan^{-1}\left\{-\left(\frac{x+y}{1-xy}\right)\right\}. \qquad ...(v)$$

Since $\tan^{-1}(-x)=-\tan^{-1}x$, therefore, we may write (*v*) as

$$(-\tan^{-1}x)+(-\tan^{-1}y)=-\tan^{-1}\left(\frac{x+y}{1-xy}\right),$$

or $$\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right).$$

(*b*) Let $xy=1$, $x>0$, $y>0$ and $\tan^{-1}x=u$.

Then $0<u<\frac{\pi}{2}$ and $\tan u=x$. Therefore, $\frac{1}{x}=\cot u$. Since $0<u<\frac{\pi}{2}$, therefore,

$$\cot^{-1}\frac{1}{x}=u=\tan^{-1}x.$$

Thus $$\tan^{-1}x+\tan^{-1}y=\cot^{-1}\frac{1}{x}+\tan^{-1}y,$$
$$=\cot^{-1}y+\tan^{-1}y,$$
$$=\frac{\pi}{2}, \text{ by Theorem 16·2.} \qquad ...(i)$$

If $xy=1$, $x<0$, $y<0$, then $-x>0$, $-y>0$, $(-x)(-y)=1$, so that we have from (*i*),

$$\tan^{-1}(-x)+\tan^{-1}(-y)=\frac{\pi}{2}. \qquad ...(ii)$$

Since $\tan^{-1}(-x)=-\tan^{-1}x$, therefore, (*ii*) may be written as

$$(-\tan^{-1}x)+(-\tan^{-1}y)=\frac{\pi}{2},$$

or $$\tan^{-1}x+\tan^{-1}y=-\frac{\pi}{2}$$

(*c*) Let us first consider the case when $x>0$, $y>0$.

Let $\tan^{-1}x=u$ and $\tan^{-1}y=v$. Then

$$x=\tan u,\ y=\tan v,\ 0<u<\frac{\pi}{2},\ 0<v<\frac{\pi}{2}.$$

Since $xy>1$ and $y>0$, therefore, $x>1/y$, *i.e.*, $\tan u>\cot v$, *i.e.*, $\tan u>\tan(\pi/2-v)$. Since $0<u<\pi/2$, $0<\pi/2-v<\pi/2$, and since 'tan' is strictly increasing in $[0, \pi/2[$, therefore, $u>\pi/2-v$ or $u+v>\pi/2$. Also from $0<u<\pi/2$, $0<v<\pi/2$, we have $0<u+v<\pi$.

Thus $\frac{\pi}{2} < u+v < \pi,$

or $-\frac{\pi}{2} < u+v-\pi < 0.$ ...(*i*)

Now $\tan(u+v-\pi) = -\tan[\pi-(u+v)],$

$$=\tan(u+v),$$

$$=\frac{\tan u+\tan v}{1-\tan u \tan v},$$

$$=\frac{x+y}{1-xy}. \quad ...(ii)$$

From (*i*) and (*ii*) it follows that

$$\tan^{-1}\left(\frac{x+y}{1-xy}\right)=u+v-\pi=\tan^{-1}x+\tan^{-1}y-\pi,$$

or $\tan^{-1}x+\tan^{-1}y=\pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right).$ ...(*iii*)

Let us now consider the case $x<0$, $y<0$.

Since $x<0$, $y<0$, $xy>1$, therefore, $-x>0$, $-y>0$, $(-x)(-y)>1$.

From (*iii*) we then have

$$\tan^{-1}(-x)+\tan^{-1}(-y)=\pi+\tan^{-1}\frac{(-x)+(-y)}{1-(-x)(-y)}. \quad ...(iv)$$

Since $\tan^{-1}(-x)=-\tan^{-1}x$, we may rewrite (*iv*) as

$$-\tan^{-1}x+(-\tan^{-1}y)=\pi-\tan^{1}\left(\frac{x+y}{1-xy}\right),$$

i.e., $\tan^{-1}x+\tan^{-1}y=-\pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right).$

(*d*) Let $\tan^{-1}x=u$ and $\tan^{-1}y=v$. Since $x \geqslant 0$, $y \geqslant 0$, therefore, $0 \leqslant u < \frac{\pi}{2}$ and $0 \leqslant v < \frac{\pi}{2}$, so that

$$-\frac{\pi}{2} < u-v < \frac{\pi}{2}. \quad ...(i)$$

Also, $x=\tan u$, $y=\tan v$, so that

$$\tan(u-v)=\frac{x-y}{1+xy}. \quad ...(ii)$$

From (*i*) and (*ii*) it follows that

$$u-v=\tan^{-1}\left(\frac{x-y}{1+xy}\right),$$

or $\tan^{-1}x-\tan^{-1}y=\tan^{-1}\left(\frac{x-y}{1+xy}\right).$

Corollary. If $-1<x<1$, then

$$2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}.$$

Proof. Put $y=x$ in (a).

Example 11. *Prove that*

(a) $\tan^{-1}(1/2)+\tan^{-1}(1/3)=\pi/4$;

(b) $\tan^{-1}(1/2)+\tan^{-1}2=\pi/2$;

(c) $\tan^{-1}1+\tan^{-1}2+\tan^{-1}3=\pi$.

Solution. (a) Let $x=1/2$, $y=1/3$. Then $xy<1$, so that by Theorem 16·3 (a),

$$\begin{aligned}\tan^{-1}(1/2)+\tan^{-1}(1/3)&=\tan^{-1}x+\tan^{-1}y,\\ &=\tan^{-1}\left(\frac{x+y}{1-xy}\right),\\ &=\tan^{-1}\left(\frac{1/2+1/3}{1-1/2.1/3}\right),\\ &=\tan^{-1}(1),\\ &=\frac{\pi}{4}\end{aligned}$$

(b) Let $x=1/2$, $y=2$. Then $x>0$, $y>0$, $xy=1$, so that by Theorem 16 3 (b),

$$\tan^{-1}(1/2)+\tan^{-1}2=\frac{\pi}{2}.$$

(c) Let $x=2$, $y=3$. Then $x>0$, $y>0$, $xy>1$, so that by Theorem 16·3 (c),

$$\begin{aligned}\tan^{-1}2+\tan^{-1}3&=\pi+\tan^{-1}\left(\frac{2+3}{1-2.3}\right),\\ &=\pi+\tan^{-1}(-1),\\ &=\pi-\tan^{-1}1,\end{aligned}$$

whence $\quad \tan^{-1}1+\tan^{-1}2+\tan^{-1}3=\pi.$

Example 12. *Prove that*

$$4\tan^{-1}(1/5)-\tan^{-1}(1/239)=\frac{\pi}{4}.$$

Solution. By corollary to Theorem 16·3,

$$2\tan^{-1}\left(\frac{1}{5}\right)=\tan^{-1}\left(\frac{2.1/5}{1-(1/5)^2}\right)=\tan^{-1}\left(\frac{5}{12}\right).$$

Again by the same corollary,

$$2\tan^{-1}\left(\frac{5}{12}\right)=\tan^{-1}\left[\frac{2\cdot\frac{5}{12}}{1-\left(\frac{5}{12}\right)^2}\right]=\tan^{-1}\left(\frac{120}{119}\right).$$

Now $\quad 4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239}$

$$=2\left(2 \tan^{-1} \frac{1}{5}\right) - \tan^{-1} \frac{1}{239},$$

$$=2 \tan^{-1} \frac{5}{12} - \tan^{-1} \frac{1}{239},$$

$$=\tan^{-1} \frac{120}{119} - \tan^{-1} \frac{1}{239}. \qquad ...(i)$$

Since $\frac{120}{119} > 0, \frac{1}{239} > 0$, therefore, by Theorem 16·3 (d), the right-hand member of (i) becomes

$$\tan^{-1}\left(\frac{\frac{120}{119} - \frac{1}{239}}{1 + \frac{120}{119} \cdot \frac{1}{239}}\right) = \tan^{-1} 1 = \frac{\pi}{4}.$$

Hence $\quad 4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239} = \frac{\pi}{4}.$

Example 13. *Solve* :

$$\cos^{-1} x + \cos^{-1} y = \frac{\pi}{2}$$

and
$$\tan^{-1} x - \tan^{-1} y = 0.$$

Solution. From the second of the given equations, we have

$$x = y. \qquad ...(i)$$

Substituting $x=y$ in the first of the given equations, we have

$$2 \cos^{-1} x = \pi/2,$$

or
$$\cos^{-1} x = \pi/4,$$

or
$$x = \cos(\pi/4) = 1/\sqrt{2} \qquad ...(ii)$$

From (i) and (ii) we have $x=y=1/\sqrt{2}$. We can easily see that these values actually satisfy the given equations. Hence the solution-set of the given equations is

$$\left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\}.$$

EXERCISE 16 (b)

1. Find the value of :

(i) $\cot^{-1}(-1)$, (ii) $\cot^{-1}(-1/\sqrt{3})$,

(iii) $\cot^{-1}(\sqrt{3})$, (iv) $\cot^{-1} 1$.

Evaluate :

2. $\sin\left(\tan^{-1} \frac{24}{7}\right)$.
3. $\tan\left(\sin^{-1} \frac{5}{6}\right)$.

4. $\sin\left[\tan^{-1}\left(-\frac{5}{7}\right)\right]$.

5. $\tan\left[\cos^{-1}\left(-\frac{5}{11}\right)\right]$.

6. $\cot\left[\sin^{-1}\left(-\frac{24}{25}\right)\right]$.

7. $\cot\left[\cos^{-1}\left(-\frac{8}{17}\right)\right]$.

8. Show that $\sin^{-1}\left(-\frac{4}{5}\right)=\tan^{-1}\left(-\frac{4}{3}\right)=\cos^{-1}\left(-\frac{3}{5}\right)-\pi$.

Prove each of the following identities :

9. $\tan^{-1}\frac{2}{3}+\tan^{-1}\frac{1}{5}=\frac{\pi}{4}$.

10. $\tan^{-1}\frac{4}{3}+\tan^{-1} 7=\frac{3\pi}{4}$.

11. $\tan^{-1}\frac{1}{7}+2\tan^{-1}\frac{1}{3}=\frac{\pi}{4}$.

12. $\tan^{-1}\frac{1}{8}+\tan^{-1}\frac{1}{5}=\tan^{-1}\frac{1}{3}$.

Evaluate each of the following expressions :

13. $\cos\left(\sin^{-1}\frac{8}{17}+\cot^{-1}\frac{4}{3}\right)$.

14. $\tan\left(\sin^{-1}\frac{15}{17}+\cot^{-1}\frac{8}{15}\right)$

15. $\sin\left(\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{12}{5}\right)$.

16·4. THE INVERSE SECANT FUNCTION

For each real number x such that $|x|\geqslant 1$, we define $\sec^{-1} x$ (also written as arcsec x) by setting

$$\sec^{-1} x=\cos^{-1}(1/x).$$

The relation $\{(x, \sec^{-1} x) : x\in\mathbf{R},\ |x|\geqslant 1\}$ is a function called the **inverse secant function (or arcsecant function)**.

Thus $\sec^{-1} x=\{(x, \sec^{-1} x) : x\in\mathbf{R},\ |x|\geqslant 1\}$.

Since $\cos^{-1} x$ lies in $[0, \pi]$, and $\cos^{-1} 0=\pi/2$, therefore, from the equality $\sec^{-1} x=\cos^{-1}(1/x)$ it follows that $\sec^{-1} x$ lies in $[0, \pi]$ but is never equal to $\pi/2$.

In the interval $[1, \infty[$, $(1/x)$ decreases, as x increases. Therefore, $\cos^{-1} x$ being a decreasing function, as x increases and hence $1/x$ decreases, $\cos^{-1}(1/x)$ increases. Thus $\sec^{-1}$ is monotonically increasing in $[1, \infty[$. Similarly $\sec^{-1}$ is monotonically increasing in $]-\infty, -1]$ as well.

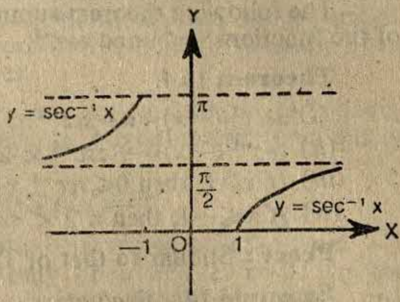


Fig. 16·14. Graph of the inverse secant function

16·5. THE INVERSE COSECANT FUNCTION

For each real number x such that $|x| \geqslant 1$, we define $\csc^{-1} x$ (also written as arccsc x) by setting

$$\csc^{-1} x = \sin^{-1} (1/x).$$

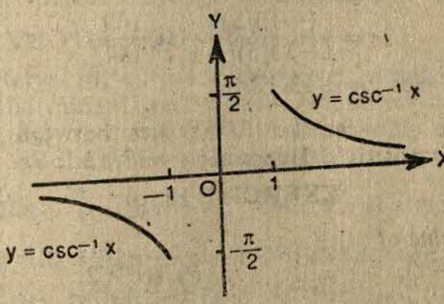


Fig. 16·15. Graph of the inverse cosecant function

The relation

$$\{(x, \csc^{-1} x) : x \in \mathbf{R}, |x| \geqslant 1\}$$

is a function called the **inverse cosecant function or arccosecant function**).

Thus $\csc^{-1} = \{(x, \csc^{-1} x) : x \in \mathbf{R}, |x| \geqslant 1\}$.

Since $\sin^{-1} x$ lies in $[-\pi/2, \pi/2]$ and $\sin^{-1} 0 = 0$, therefore, from the equality $\csc^{-1} x = \sin^{-1} (1/x)$, it follows that $\csc^{-1} x$ lies in $[-\pi/2, \pi/2]$ but is never equal to 0.

In the interval $[1, \infty[$, $1/x$ decreases as x increases and therefore, $\sin^{-1}$ being an increasing function, $\sin^{-1} (1/x)$ decreases as x increases. Thus $\csc^{-1}$ is monotonically decreasing in $[1, \infty[$. Similarly $\csc^{-1}$ is monotonically decreasing in $]-\infty, -1]$ as well.

The following theorem summarizes some important properties of the functions $\sec^{-1}$ and $\csc^{-1}$.

Theorem 16.4.

(*i*) $\sec^{-1}(-x)=\pi-\sec^{-1}x$, *whenever* $|x|\geqslant 1$.
(*ii*) $\csc^{-1}(-x)=-\csc^{-1}x$, *whenever* $|x|\geqslant 1$.
(*iii*) *If* $x\geqslant 1$, then $0\leqslant \sec^{-1}x<\pi/2,\ 0<\csc^{-1}x\leqslant\pi/2$.
(*iv*) *If* $x\leqslant -1$, then $\pi/2<\sec^{-1}x\leqslant\pi,\ -\pi/2\leqslant\csc^{-1}x<0$.

Proof : Similar to that of Theorem 16·1.

Example 14. *Evaluate :*

$$\sec(\csc^{-1}(-13/5)).$$

Solution. Let $\csc^{-1}(-13/5)=y$. Then

$$\csc y=-13/5.$$

$$\sec^2 y=1+\tan^2 y=1+\frac{1}{\csc^2 y-1},$$

$$=\frac{\csc^2 y}{\csc^2 y-1},$$

$$=\frac{(-13/5)^2}{(-13/5)^2-1}=\frac{169}{144}.$$

$$\therefore \quad \sec y=\pm 13/12.$$

Also, since $y\ (=\csc^{-1}(-13/15))$ lies between $-\pi/2$ and 0, therefore, $\sec y$ is positive. Hence $\sec y=13/12$.

EXERCISE 16 (*c*)

Find the value of :

1. (*i*) $\sec^{-1}(-1)$, (*ii*) $\sec^{-1}2$,
(*iii*) $\sec^{-1}(-\sqrt{2})$, (*iv*) $\sec^{-1}(2/\sqrt{3})$,
(*v*) $\csc^{-1}1$, (*vi*) $\csc^{-1}(-2)$
(*vii*) $\csc^{-1}(-2/\sqrt{3})$, (*viii*) $\csc^{-1}(\sqrt{2})$.
2. $\sec(\csc^{-1}(5/3))$.
3. $\csc(\sec^{-1}(17/15))$.
4. $\sec(\cot^{-1}(-12/5))$.
5. $\csc(\cos^{-1}(-5/13))$.
6. $\sec(\tan^{-1}(18/15))$.
7. $\csc(\tan^{-1}(24/7))$.
8. $\sec(\sin^{-1}(-7/25))$.
9. $\csc(\cot^{-1}(-15/8))$.
10. $\sin\{\tan^{-1}(-4/3)+\sec^{-1}(13/5)\}$.

16·6. TRIGONOMETRIC EQUATIONS

In the preceding sections, we were primarily concerned with problems of the following type :

To determine x such that $-\pi/2\leqslant x\leqslant\pi/2$ and $\sin x=k$, k being a given real number : or equivalently, to find that solution of $\sin x=k$ which lies in the interval $[-\pi/2, \pi/2]$.

We shall now devote ourselves to the more general problem of finding *all* solutions of the equation $\sin x=k$ (and other similar equations of course !), k being a given real number. We shall, however, first take up some special cases.

The results of the following sections will apply to trigonometric functions of numbers as also to trigonometric functions of angles.

16·7. TO SOLVE THE EQUATION sin x=0

In the interval $[-\pi/2, \pi/2]$, $\sin x$ increases strictly from -1 to $+1$ and assumes the value 0 only once, *viz.*, at $x=0$. In the interval $[\pi/2, 3\pi/2]$, $\sin x$ decreases strictly from $+1$ to -1 and assumes the value 0 only once at $x=\pi$. By periodicty, all the values of x for which $\sin x=0$, are given by $2p\pi$ and $2p\pi+\pi$ where p is any integer, so that

$$\{x : \sin x=0\}=\{2p\pi : p\in \mathbf{Z}\}\cup\{(2p+1)\pi : p\in \mathbf{Z}\},$$
$$=\{n\pi : n\in \mathbf{Z}\}.$$

Hence $\boxed{\{x : \sin x=0\}=\{n\pi : n\in \mathbf{Z}\}.}$

16·8. TO SOLVE THE EQUATION cos x=0

In the interval $[0, \pi]$, $\cos x$ decreases strictly from $+1$ to -1 and assumes the value 0 exactly once, *viz.*, at $x=\pi/2$. Similarly in the interval $[\pi, 2\pi[$ it assumes the value 0 only once at $x=3\pi/2$. By periodicty, all the values of the variable x, for which $\cos x=0$, are given by $2k\pi+\dfrac{\pi}{2}$, $2k\pi+\dfrac{3\pi}{2}$, where k is any integer, so that

$$\{x : \cos x=0\}=\{2k\pi+\pi/2 : k\in \mathbf{Z}\}\cup\{2k\pi+3\pi/2 : k\in \mathbf{Z}\},$$
$$=\{(2n+1)(\pi/2) : n\in \mathbf{Z}\}.$$

Hence $\boxed{\{x : \cos x=0\}=\{(2n+1)\pi/2 : n\in \mathbf{Z}\}.}$

EXERCISE 16 (*d*)

1. Solve the equation $\tan x=0$.
2. Solve the equation $\cot x=0$.
3. Solve the following equations :

   (*i*) $\sin x=1/2$. (*ii*) $\sin x=-\sqrt{3}/2$.

   (*iii*) $\cos x=1$. (*iv*) $\cos x=-1/2$.

16·9. TO SOLVE THE EQUATION sin x=k, k BEING A FIXED REAL NUMBER IN [−1, 1]

Let $\alpha=\sin^{-1} k$. In the interval $[-\pi/2, \pi/2]$, $\sin x$ increases strictly from -1 to 1 and assumes the value k only once, *viz.*, at α.

In the interval $[\pi/2, 3\pi/2]$, $\sin x$ decreases strictly from 1 to -1 and assumes each value only once. In particular, it assumes the value k only once, namely at $x=\pi-\alpha$. In fact, $\sin(\pi-\alpha)=\sin\alpha=k$; also since $-\pi/2 \leqslant \alpha \leqslant \pi/2$, therefore, $\pi/2 \leqslant \pi-\alpha \leqslant 3\pi/2$.

By periodicity of $\sin x$, all the values of x, for which $\sin x=k$, are given by $2p\pi+\alpha$, $2p\pi+(\pi-\alpha)$, where p is any integer, so that

$$\begin{aligned}\{x : \sin x=k\}&=\{2p\pi+\alpha : p\in\mathbf{Z}\} \cup \{2p\pi+(\pi-\alpha) : p\in\mathbf{Z}\},\\ &=\{n\pi+(-1)^n\alpha : n\in\mathbf{Z}\},\\ &=\{n\pi+(-1)^n \sin^{-1} k : n\in\mathbf{Z}\}.\end{aligned}$$

Alternatively, we may proceed as follows :

Let $\alpha=\sin^{-1} k$. Then the given equation can be written as

$$\sin x=\sin\alpha,$$

or

$$\sin x-\sin\alpha=0.$$

Now $\sin x-\sin\alpha=0$

$$\begin{aligned}&\Rightarrow 2\cos\{\tfrac{1}{2}(x+\alpha)\}\sin\{\tfrac{1}{2}(x-\alpha)\}=0,\\ &\Rightarrow \cos\{\tfrac{1}{2}(x+\alpha)\}=0 \vee \sin\{\tfrac{1}{2}(x-\alpha)\}=0,\\ &\Rightarrow \tfrac{1}{2}(x+\alpha)\in\{(2m+1)\pi/2 : m\in\mathbf{Z}\} \vee \tfrac{1}{2}(x-\alpha)\\ &\qquad\qquad\in\{m\pi : m\in\mathbf{Z}\},\\ &\Rightarrow x\in\{(2m+1)\pi-\alpha : m\in\mathbf{Z}\} \vee x\in\{2m\pi+\alpha : m\in\mathbf{Z}\},\\ &\Rightarrow x\in\{n\pi+(-1)^n\alpha : n\in\mathbf{Z}\}. \qquad \cdots(i)\end{aligned}$$

Also,

$$\begin{aligned}&x\in\{n\pi+(-1)^n\alpha : n\in\mathbf{Z}\}\\ &\Rightarrow \sin x=\sin(n\pi+(-1)^n\alpha), \text{ where } n\in\mathbf{Z},\\ &\Rightarrow \sin x=(-1)^n\sin\{(-1)^n\alpha\},\\ &\Rightarrow \sin x=\sin\alpha. \qquad \cdots(ii)\end{aligned}$$

From (i) and (ii), we find that

$$\sin x-\sin\alpha=0 \Leftrightarrow x\in\{n\pi+(-1)^n\alpha : n\in\mathbf{Z}\}.$$

Hence $\boxed{\{x : \sin x=k\}=\{n\pi+(-1)^n\sin^{-1} k : n\in\mathbf{Z}\}.}$

Remark. From the above working it can be easily seen that if α be any solution of the equation $\sin x=k$ (*i.e.*, if α is not necessarily $\sin^{-1} k$, but any number such that $\sin\alpha=k$), even then the solution-set of the equation $\sin x=k$ is $\{n\pi+(-1)^n\alpha : n\in\mathbf{Z}\}$. This observation would be found extremely useful in problems later on.

16·10. TO SOLVE THE EQUATION $\cos x=k$, k BEING A FIXED REAL NUMBER IN $[-1, 1]$

Let $\alpha=\cos^{-1} k$. In the interval $[0, \pi]$, $\cos x$ decreases strictly from $+1$ to -1 and assumes each value between 1 and -1 exactly once. In particular, it assumes the value k only at α. In the interval $[\pi, 2\pi]$, $\cos x$ increases strictly from -1 to 1 and assumes each value

between -1 and 1 exactly once. In particular, it assumes the value k only once, namely at $x=2\pi-\alpha$. In fact, $\cos(2\pi-\alpha)=\cos\alpha=k$. Also, since $0<\alpha<\pi$, therefore,

$$\pi < 2\pi-\alpha < 2\pi.$$

By periodicity of cos, all the values of x for which $\cos x=k$ are given by $2p\pi+a$, $2p\pi+(2\pi-\alpha)$, where p is any integer, so that

$$\{x : \cos x=k\}=2p\pi+a : p \in \mathbf{Z}\} \cup \{2p\pi+(2\pi-\alpha) : p \in \mathbf{Z}\},$$
$$=\{2n\pi\pm\alpha : \in \mathbf{Z}\},$$
$$=\{2n\pi\pm\cos^{-1} k : n \in \mathbf{Z}\}.$$

Hence $\boxed{\{x : \cos x=k\}=\{2n\pi\pm\cos^{-1} k : n \in \mathbf{Z}\}}$

Alternatively, we may proceed as follows :

Let $\alpha=\cos^{-1} k$. Then the given equation can be written as

$$\cos x=\cos\alpha,$$

or $$\cos x-\cos\alpha=0.$$

Now $\cos x-\cos\alpha=0$

$\Rightarrow -2\sin\{\frac{1}{2}(x+\alpha)\}\sin\{\frac{1}{2}(x-\alpha)\}=0,$

$\Rightarrow \sin\{\frac{1}{2}(x+\alpha)\}=0 \vee \sin\{\frac{1}{2}(x-\alpha)\}=0,$

$\Rightarrow \frac{1}{2}(x+\alpha) \in \{n\pi : \in \mathbf{Z}\} \vee \frac{1}{2}(x-\alpha) \in \{n\pi : n \in \mathbf{Z}\},$

$\Rightarrow \in \{2n\pi-\alpha : n \in \mathbf{Z}\} \vee x \in \{2n\pi+\alpha : n \in \mathbf{Z}),$

$\Rightarrow \in \{2n\pi\pm a : n \in \mathbf{Z}\}.$ ...(*i*)

Also, $x \in \{2n\pi\pm\alpha : n \in \mathbf{Z}\}$

$\Rightarrow \cos x=\cos(2n\pi\pm\alpha)$, where $n \in \mathbf{Z}$,

$\Rightarrow \cos x=\cos\alpha.$ ...(*ii*)

From (*i*) and (*ii*), we find that

$$\cos x-\cos\alpha=0 \Leftrightarrow x \in \{2n\pi\pm\alpha : n \in \mathbf{Z}).$$

Hence $\{x : \cos x=k\}=\{2n\pi\pm\cos^{-1} k : n \in \mathbf{Z}\}.$

Remark. If α be *any* solution (whatsoever) of the equation $\cos x=k$, even then the solution-set of the equation can be easily seen to be $\{2n\pi\pm\alpha : n \in \mathbf{Z}\}$.

16·11 TO SOLVE THE EQUATION $\tan x=k$, k BEING ANY FIXED REAL NUMBER

Let $\alpha=\tan^{-1} k$. In the interval $]-\pi/2, \pi/2[$, $\tan x$ increases strictly and assumes each value exactly once. In particular, it assumes the value k only at α.

By periodicity of tan, all the values of x for which $\tan x=k$ are given by $n\pi+\alpha$, where n is any integer, so that

$$\{x : \tan x=k\}=\{n\pi+\alpha : n \in \mathbf{N}\},$$
$$=\{n\pi+\tan^{-1} k : n \in \mathbf{Z}\}.$$

Hence $\boxed{\{x : \tan x = k\} = \{n\pi + \tan^{-1} k : n \in \mathbf{Z}\}}$.

Alternatively, we may proceed as follows :

Let $\alpha = \tan^{-1} k$. Then the given equation can be written as
$$\tan x = \tan \alpha.$$

Now $\tan x = \tan \alpha \Rightarrow \dfrac{\sin x}{\cos x} - \dfrac{\sin \alpha}{\cos \alpha} = 0, \Rightarrow \dfrac{\sin (x-\alpha)}{\cos x \cos \alpha} = 0,$
$\Rightarrow \sin (x-\alpha) = 0$ (why ?) $\Rightarrow (x-\alpha) \in \{n\pi : n \in \mathbf{Z}\}, \Rightarrow x \in \{n\pi + \alpha : n \in \mathbf{Z}\}$. ...(*i*)

Also, $x \in \{n\pi + \alpha : n \in \mathbf{N}\}, \Rightarrow \tan x = \tan \{n\pi + \alpha\} \Rightarrow \tan x = \tan \alpha$. ...(*ii*)

From (*i*) and (*ii*), we find that
$$\tan x = \tan \alpha \Leftrightarrow x \in \{n\pi + \alpha : n \in \mathbf{Z}\}.$$

Hencc $\{x : \tan x = k\} = \{n\pi + \tan^{-1} k : n \in \mathbf{Z}\}$.

Remark. It can be easily seen that if α be any solution whatsoever of the equation $\tan x = k$, then the solution-set of the equation can be written as $\{n\pi + \alpha : n \in \mathbf{N}\}$.

Corollary. If $k \neq 0$, then the solution-set of the equation $\cot x = k$ is $\{n\pi + \cot^{-1} k : n \in \mathbf{Z}\}$.

Proof. $\cot x = k \Leftrightarrow \tan x = 1/k, (k \neq 0)$,
$\Leftrightarrow x \in \{n\pi + \tan^{-1} (1/k) : n \in \mathbf{Z}\}$,
$\Leftrightarrow x \in \{n\pi + \cot^{-1} k : n \in \mathbf{Z}\}$,
since $\tan^{-1} (1/k) = \cot^{-1} k$.

Example 15. *Find the solution-sets of the following equations :*

(*i*) $\sin x = 1/2$. (*ii*) $\cos x = 1/3$.
(*iii*) $\tan x = -\sqrt{3}$. (*iv*) $\sec x = -\sqrt{2}$.

Solution. (*i*) Since $\sin^{-1} (1/2) = \pi/6$, therefore, the solution-set of the equation $\sin x = \frac{1}{2}$ is
$$\{n\pi + (-1)^n \sin^{-1} \tfrac{1}{2} : n \in \mathbf{Z}\},$$
i.e., $$\{n\pi + (-1)^n \pi/6 : n \in \mathbf{N}\}.$$

(*ii*) The solution-set of the equation $\cos x = \frac{1}{3}$ is
$$\{2n\pi \pm \cos^{-1} \tfrac{1}{3} : n \in \mathbf{Z}\}.$$

(*iii*) Since $\tan^{-1} (-\sqrt{3}) = -\pi/3$, therefore, the solution-set of the equation $\tan x = -\sqrt{3}$ is
$$\{n\pi + \tan^{-1} (-\sqrt{3}) : n \in \mathbf{Z}\},$$
i.e., $$\{n\pi + (-\pi/3) : n \in \mathbf{Z}\},$$
i.e., $$\{n\pi - \pi/3 : n \in \mathbf{Z}\}.$$

(*iv*) $\sec x = -\sqrt{2} \Leftrightarrow \cos x = -1/\sqrt{2}$,

$\Leftrightarrow x \in \{2n\pi \pm \cos^{-1}(-1/\sqrt{2}) : n \in \mathbf{N}\}$,

$\Leftrightarrow x \in \{2n\pi \pm \frac{3}{4}\pi : n \in \mathbf{Z}\}$.

Hence the solution-set of the given equation is

$$\{2n\pi \pm 3\pi/4 : n \in \mathbf{Z}\}.$$

EXERCISE 16 (*e*)

1. Determine the solution-set of each of the following equations :

(*i*) $\sin x = \sqrt{3}/2$. (*ii*) $\cos x = -1/2$.

(*iii*) $\tan x = 1$. (*iv*) $\cot x = -\sqrt{3}$.

(*v*) $\sec x = 2$. (*vi*) $\csc x = -2$.

2. Write down the solution-set of each of the following equations :

(*i*) $\tan 2x = 1/\sqrt{3}$. (*ii*) $\cot 3x = -2$.

3. Solve the following equations :

(*i*) $2\cos x + 1 = 0$. (*ii*) $\sqrt{2}\sin x + 1 = 0$.

16 12. EQUATIONS INVOLVING ONLY ONE TRIGONOMETRIC FUNCTION

Example 16. *Solve the equation* $\sin^2 x = 1/4$.

Solution. $\sin^2 x = 1/4 \Leftrightarrow \frac{1}{2}(1 - \cos 2x) = 1/4$,

$\Leftrightarrow 2(1 - \cos 2x) = 1$,

$\Leftrightarrow \cos 2x = \frac{1}{2}$,

$\Leftrightarrow 2x \in \{2n\pi \pm \pi/3 : n \in \mathbf{Z}\}$,

$\Leftrightarrow x \in \{n\pi \pm \pi/6 : n \in \mathbf{Z}\}$.

Hence the solution-set of the given equation is

$$\{n\pi \pm \pi/6 : n \in \mathbf{Z}\}.$$

Aliter.

$\sin^2 x = \frac{1}{4} \Rightarrow \sin x = \frac{1}{2} \vee \sin x = -\frac{1}{2}$,

$\Rightarrow x \in \{n\pi + (-1)^n \pi/6 : n \in \mathbf{Z}\}$

$\vee\ x \in \{n\pi + (-1)^n(-\pi/6) : n \in \mathbf{Z}\}$,

$\Rightarrow x \in \{n\pi + (-1)^n \pi/6 : n \in \mathbf{Z}\}$

$\vee\ x \in \{n\pi + (-1)^{n+1} \pi/6 : n \in \mathbf{Z}\}$,

$\Rightarrow x \in \{n\pi \pm \pi/6 : n \in \mathbf{Z}\}$. ...(*i*)

Also, $x \in \{n\pi \pm \pi/6 : n \in \mathbf{Z}\} \Rightarrow \sin^2 x = \sin^2 (n\pi \pm \pi/6$,

$\Rightarrow \sin^2 x = \{(-1)^n \sin (\pm\pi/6\}^2)$,

$\Rightarrow \sin^2 x = \frac{1}{4}$. ...(*ii*)

From (*i*) and (*ii*), we find that the solution-set of the given equation is $\{n\pi \pm \pi/6 : n \in \mathbf{Z}\}$.

Example 17. *Solve the equation*

$$2\cos^2 x - 5\cos x - 3 = 0.$$

Solution. Factorizing the left-hand member of the given equation, we have

$$(2\cos x + 1)(\cos x - 3) = 0.$$

By setting each factor to ze, we havero

$$2\cos x + 1 = 0, \text{ or } \cos x - 3 = 0,$$

i.e.,
$$\cos x = -\tfrac{1}{2}, \text{ or } \cos x = 3.$$

The second of these equations has no solution because the cosine of a number cannot exceed unity. Also,

$$\cos x = -\frac{1}{2} \Rightarrow x \in \left\{ 2n\pi \pm \frac{2\pi}{3} : n \in \mathbf{Z} \right\}.$$

Hence the required solution-set is $\left\{ 2n\pi \pm \dfrac{2\pi}{3} : n \in \mathbf{Z} \right\}$.

Check. If $x \in \left\{ 2n\pi \pm \dfrac{2\pi}{3} : n \in \mathbf{Z} \right\}$, then

$$\begin{aligned} 2\cos^2 x - 5\cos x - 3 &= 2\cos^2\left(2n\pi \pm \frac{2\pi}{3}\right) \\ &\quad -5\cos\left(2n\pi \pm \frac{2\pi}{3}\right) - 3, \\ &= 2\cos^2\left(\frac{2\pi}{3}\right) - 5\cos\left(\frac{2\pi}{3}\right) - 3, \\ &= 2\left(-\frac{1}{2}\right)^2 - 5\left(-\frac{1}{2}\right) - 3 = 0. \end{aligned}$$

Example 18. *Solve the equation :*

$$\sec^2 x - 4\tan x + 2 = 0.$$

Solution.
$$\begin{aligned} &\sec^2 x - 4\tan x + 2 = 0, \\ \Leftrightarrow\ &\tan^2 x - 4\tan x + 3 = 0, \\ \Leftrightarrow\ &(\tan x - 1)(\tan x - 3) = 0, \\ \Leftrightarrow\ &\tan x = 1 \vee \tan x = 3; \\ \Leftrightarrow\ &x \in \{n\pi + \pi/4 : n \in \mathbf{Z}\} \vee x \in \{n\pi + \tan^{-1} 3 : n \in \mathbf{Z}\}, \\ \Leftrightarrow\ &x \in \{n\pi + \pi/4 : n \in \mathbf{Z}\} \cup \{n\pi + \tan^{-1} 3 : n \in \mathbf{Z}\}. \end{aligned}$$

Hence the solution-set of the given equation is

$$\{n\pi + \pi/4 : n \in \mathbf{Z}\} \cup \{n\pi + \tan^{-1} 3 : n \in \mathbf{Z}\}.$$

EXERCISE 16 (*f*)

Solve the following equations. Check the solution in each case :

1. $\cos^2 x = \dfrac{1}{4}$.

2. $3\tan^2 x = 1$.

3. $\tan^2 x=3 \csc^2 x-1$.
4. $4 \sin^2 x+4 \sin x+1=0$.
5. $2 \cos^2 x+3 \cos x+1=0$.
6. $\cot^2 x+2 \cot x-3=0$.
7. $\sec^2 x+3 \sec x-4=0$.
8. $\sin^2 x-2 \cos x+\frac{1}{4}=0$.
9. $3 \tan x+\cot x=5 \csc x$.
10. $\csc^3 2x=4 \csc 2x$.

16·13. EQUATIONS INVOLVING TRIGONOMETRIC FUNCTIONS OF TWO OR MORE MULTIPLES OF THE VARIABLE

Example 19. *Solve the equation*

$$\cos px=\cos qx, \ (p\neq q).$$

Solution. $\cos px=\cos qx$

$\Leftrightarrow \cos px-\cos qx=0,$

$\Leftrightarrow 2 \sin \left\{\frac{1}{2}(p+q)\, x\right\} \sin \left\{\frac{1}{2}(q-p)x\right\}=0,$

$\Leftrightarrow \sin \left\{\frac{1}{2}(p+q)x\right\}=0 \vee \sin \left\{\frac{1}{2}(q-p)x\right\}=0,$

$\Leftrightarrow \frac{1}{2}(p+q)x\in\{n\pi : n\in\mathbf{Z}\} \vee \frac{1}{2}(q-p)x\in\{n\pi : n\in\mathbf{Z}\},$

$\Leftrightarrow x\in\left\{\frac{2n\pi}{(p+q)} : n\in\mathbf{Z}\right\}\vee x\in\left\{\frac{2n\pi}{(q-p)} : n\in\mathbf{Z}\right\},$

$\Leftrightarrow x\in\left\{\frac{2n\pi}{(q\pm p)} : n\in\mathbf{Z}\right\}.$

Hence the solution-set of the given equation is :

$$\left\{\frac{2n\pi}{(q\pm p)} : n\in\mathbf{Z}\right\}.$$

Example 20. *Solve the equation*

$$\cos x+\cos 3x+\cos 5x+\cos 7x=0.$$

Solution. $\cos x+\cos 3x+\cos 5x+\cos 7x$

$$=2 \cos 2x \cos x+2 \cos 6x \cos x,$$
$$=2 \cos x (\cos 2x+\cos 6x),$$
$$=4 \cos x \cos 4x \cos 2x.$$

Therefore, the given equation is equivalent to the equation

$$4 \cos x \cos 4x \cos 2x=0$$

which in turn is equivalent to

$$\cos x=0 \vee \cos 4x=0 \vee \cos 2x=0.$$

Now $\{x : \cos x=0\}=\{(2n+1)\, \pi/2 : n\in\mathbf{Z}\},$

$\{x : \cos 4x=0\}=\{x : 4x=(2n+1)\, \pi/2, n\in\mathbf{Z}\},$

$=\{(2n+1)\, \pi/8 : n\in\mathbf{Z}\},$

$$\{x : \cos 2x=0\}=\{x : 2x=(2n+1)\,\pi/2,\ n\in \mathbf{Z}\},$$
$$=\{(2n+1)\,\pi/4 : n\in \mathbf{Z}\}.$$

Therefore, the solution-set is

$$\{(2n+1)\,\pi/2 : n\in \mathbf{N}\} \cup \{(2n+1)\,\pi/4\ ;\ n\in \mathbf{Z}\}$$
$$\cup\ \{(2n+1)\,\pi/8 : n\in \mathbf{Z}\}.$$

EXERCISE 16 (*g*)

Solve the following equations :

1. $\cos 4x=\cos 6x.$
2. $\sin 3x=\sin 5x.$
3. $\tan 4x=\tan 3x.$
4. $\cos 2x+\sin 3x=0.$
5. $\sin px+\sin qx=0.$
6. $\tan mx=\cot nx.$
7. $\cos x+\cos 3x=\cos 2x.$
8. $\sin 7x=\sin x+\sin 3x.$
9. $\cos x+\cos 2x+\cos 3x=0.$
10. $\sin x+\sin 3x+\sin 5x=0.$
11. $\sin x+\sin 2x+\sin 3x+\sin 4x=0.$

16·14. EQUATIONS OF THE FORM a cos x+b sin x=c

Example 21. *Solve the equation*

$$\cos x+\sqrt{3}\sin x=1.$$

Solution. Let us determine a positive real number r and a real number ϕ in $[0, 2\pi]$ such that

$$1=r\cos\phi,\ \sqrt{3}=r\sin\phi. \qquad ...(i)$$

By squaring both sides of each of the equations (*i*) and adding, we have

$$1^2+(\sqrt{3})^2=r^2\cos^2\phi+r^2\sin^2\phi,$$

or

$$4=r^2.$$

Since we wish r to be positive, therefore, we have

$$r=2. \qquad ...(ii)$$

Substituting the above value of r in (*i*), we have

$$\cos\phi=\tfrac{1}{2},\ \sin\phi=\sqrt{3}/2. \qquad ...(iii)$$

The first of these equations has solutions

$$\frac{\pi}{3},\ 2\pi-\frac{\pi}{3} \text{ in } [0, 2\pi],$$

while the second equation has solutions

$$\frac{\pi}{3},\ \pi-\frac{\pi}{3} \text{ in } [0, 2\pi].$$

Therefore, the equations (*iii*) have a unique common solution

$$\phi=\pi/3 \text{ in } [0, 2\pi].$$

We can now write the given equation as

$$r\cos\phi\cos x+r\sin\phi\sin x=1,$$

or $$r\cos(x-\phi)=1,$$

or $$2\cos\left(x-\frac{\pi}{3}\right)=1,$$

or $$\cos\left(x-\frac{\pi}{3}\right)=\frac{1}{2}. \qquad \text{...}(iv)$$

The solution-set of equation (*iv*), and therefore, of the given equation (which is equivalent to (*iv*)) is,

$$\left\{x : x-\frac{\pi}{3}=2n\pi\pm\cos^{-1}\frac{1}{2},\ n\in\mathbf{Z}\right\},$$

i.e., $$\left\{x : x=2n\pi\pm\frac{\pi}{3}+\frac{\pi}{3},\ n\in\mathbf{Z}\right\},$$

i.e., $$\left\{2n\pi : n\in\mathbf{Z}\right\}\cup\left\{2n\pi+\frac{2\pi}{3} : n\in\mathbf{Z}\right\}.$$

Check. $x=2n\pi\Rightarrow\cos x+\sqrt{3}\sin x=\cos 2n\pi+\sqrt{3}\sin 2n\pi,$
$$=1+\sqrt{3}\cdot 0=1.$$

$$x=2n\pi+\frac{2\pi}{3}\Rightarrow\cos x+\sqrt{3}\sin x$$
$$=\cos\left(2n\pi+\frac{2\pi}{3}\right)+\sqrt{3}\sin\left(2n\pi+\frac{2\pi}{3}\right),$$
$$=\cos\frac{2\pi}{3}+\sqrt{3}\sin\frac{2\pi}{3},$$
$$=-\frac{1}{2}+\sqrt{3}\left(\frac{\sqrt{3}}{2}\right)=1.$$

Example 22. *Solve the equation $2\cos x-3\sin x=-2$.*

Solution. Dividing both sides of the given equation by
$$\sqrt{\{2^2+(-3)^2\}},$$
we have

$$\frac{2}{\sqrt{13}}\cos x-\frac{3}{\sqrt{13}}\sin x=-\frac{2}{\sqrt{13}}. \qquad \text{...}(i)$$

Let $\cos^{-1}\left(\frac{2}{\sqrt{13}}\right)=\phi$. Then $0<\phi<\frac{\pi}{2}$, and $\cos\phi=\frac{2}{\sqrt{13}}$,

so that $\sin\phi=\frac{3}{\sqrt{13}}$. We may, therefore, re-write (*i*) as

$$\cos x\cos\phi-\sin x\sin\phi=-\frac{2}{\sqrt{13}},$$

i.e., $$\cos(x+\phi)=-\frac{2}{\sqrt{13}}. \qquad \text{...}(ii)$$

The solution-set of (*ii*) and therefore, of the given equation is

$$\left\{x:\ x+\phi=2n\pi\pm\cos^{-1}\left(-\frac{2}{\sqrt{13}}\right),\ n\in\mathbf{Z}\right\}.$$

i.e.,
$$\left\{x: x=2n\pi\pm\cos^{-1}\left(-\frac{2}{\sqrt{13}}\right)-\phi,\ n\in\mathbf{Z}\right\},$$

i.e.,
$$\left\{2n\pi\pm\cos^{-1}\left(-\frac{2}{\sqrt{13}}\right)-\cos^{-1}\left(\frac{2}{\sqrt{13}}\right): n\in\mathbf{Z}\right\}.$$

EXERCISE 16 (*h*)

Solve the following equations and check the solution in each case :

1. $\sqrt{3}\cos x+\sin x=\sqrt{2}$.
2. $\cos x-\sqrt{3}\sin x=1$.
3. $\cos x+\sin x=1$.
4. $\sin x-\cos x=\sqrt{2}$.
5. $\sqrt{3}\sin x-\cos x=\sqrt{2}$.
6. $\cos x+\sqrt{3}\sin x=-1$.

16·15. SOLUTION OF PAIRS OF EQUATIONS

Example 23. *Find the solution-set of the pair of equations*

$$\cos x=\frac{1}{2} \text{ and } \sin x=-\frac{\sqrt{3}}{2}.$$

Solution. We shall first find all those x in $[0, 2\pi]$ that satisfy both the given equations and then determine the solution-set by using the fact that the sine and cosine functions are periodic and have a common period 2π.

Now $\left\{x: \cos x=\frac{1}{2}\wedge 0\leqslant x\leqslant 2\pi\right\}=\left\{\frac{\pi}{3}, \frac{5\pi}{3}\right\}$.

$\left\{x: \sin=-\frac{\sqrt{3}}{2}\wedge 0\leqslant x\leqslant 2\pi\right\}=\left\{\frac{4\pi}{3}, \frac{5\pi}{3}\right\}$.

Therefore,

$$\left\{x: \cos x=\frac{1}{2}\wedge \sin x=-\frac{\sqrt{3}}{2}\wedge 0\leqslant x\leqslant 2\pi\right\}$$

$$=\left\{x: \cos x=\frac{1}{2}\wedge 0\leqslant x<2\pi\right\}$$

$$\cap\left\{x: \sin x=-\frac{\sqrt{3}}{2}\wedge 0\leqslant x<2\pi\right\},$$

$$=\left\{\frac{\pi}{3}, \frac{5\pi}{3}\right\}\cap\left\{\frac{4\pi}{3}, \frac{5\pi}{3}\right\},$$

$$=\left\{\frac{5\pi}{3}\right\}.$$

By periodicity of the sine and cosine functions,

$$\left\{x: \cos x=\frac{1}{2}\wedge \sin x=-\frac{\sqrt{3}}{2}\right\}$$

$$=\left\{ 2n\pi+\frac{5\pi}{3} : n\in\mathbf{Z} \right\}.$$

Hence the desired solution-set is $\left\{ 2n\pi+\frac{5\pi}{3} : n\in\mathbf{Z} \right\}.$

Remark. The solution-set can also be written as

$$\left\{2n\pi-\frac{\pi}{3} : n\in\mathbf{Z}\right\}.$$

EXERCISE 16 (*i*)

Find the solution-set of each of the following pairs of equations :

1. $\cos x=-\frac{1}{2}$, $\sin x=\frac{\sqrt{3}}{2}$.
2. $\cos x=-\frac{\sqrt{3}}{2}$, $\sin x=\frac{1}{2}$.
3. $\cos x=\frac{1}{\sqrt{2}}$, $\sin x=-\frac{1}{\sqrt{2}}$.
4. $\cos x=\frac{1}{2}$, $\sin x=-\frac{1}{2}$.
5. $\sin x=\frac{\sqrt{3}}{2}$, $\tan x=-\sqrt{3}$.
6. $\sin x=-\frac{1}{2}$, $\tan x=-\frac{1}{\sqrt{3}}$.
7. $\cos x=\frac{1}{2}$, $\tan x=-\sqrt{3}$.

TEST YOUR UNDERSTANDING XVI

In each of the following problems four alternatives are given. Put a tick mark (✓) against the correct alternative :

1. The value of $\sin^{-1}\left(-\frac{1}{2}\right)$ is

   (*a*) $\frac{\pi}{3}$ (*b*) $\frac{\pi}{6}$ (*c*) $\frac{-\pi}{3}$ (*d*) $\frac{-\pi}{6}$.

2. The value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is

   (*a*) $\frac{\pi}{6}$ (*b*) $\frac{5\pi}{6}$ (*c*) $\frac{\pi}{3}$ (*d*) $\frac{2\pi}{3}$.

3. The value of $\sin^{-1}(1/4)+\cos^{-1}(1/4)$ is

   (*a*) π (*b*) $\frac{\pi}{3}$ (*c*) $\frac{\pi}{2}$ (*d*) 2π.

4. The value of $\tan\left(\sin^{-1}\frac{1}{2}+\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)\right)$ is

(a) 0 (b) 1 (c) $\frac{1}{\sqrt{3}}$ (d) $\sqrt{3}$.

5. The value of $\tan^{-1}\left[2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right]$ is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{2\pi}{3}$.

6. If $\sin\alpha=\sin\beta$, then α and β are related by

(a) $\alpha=2n\pi\pm\beta$ (b) $\alpha=n\pi+\beta$

(c) $\alpha=n\pi+(-1)^n\beta$ (d) $\alpha=n\pi-\beta$.

7. If $\sin x=1/2$, then x equals

(a) $2n\pi\pm\frac{\pi}{6}$ (b) $n\pi+\frac{\pi}{6}$

(c) $n\pi+(-1)^n\frac{\pi}{6}$ (d) $(2n+1)\frac{\pi}{2}$.

8. If $\cos x=1/2$, then x equals

(a) $n\pi+\frac{\pi}{3}$ (b) $2n\pi\pm\frac{\pi}{3}$

(c) $n\pi+(-1)^n\frac{\pi}{3}$ (d) $(2n+1)\frac{\pi}{2}$.

9. If $\tan x=\sqrt{3}$, then x equals

(a) $n\pi+\frac{\pi}{6}$ (b) $2n\pi+\frac{\pi}{3}$

(c) $n\pi+\frac{\pi}{3}$ (d) $n\pi-\frac{\pi}{6}$.

10. If $\cos x+\sin x=1$, then x equals

(a) $2n\pi\pm\frac{\pi}{4}-\frac{\pi}{4}$ (b) $2n\pi+\frac{\pi}{4}$

(c) $n\pi-\frac{\pi}{4}$ (d) $\frac{n\pi}{2}$.

REVIEW EXERCISE XVI

1. Evaluate $\tan\left[\sin^{-1}\left(\frac{-1}{\sqrt{5}}\right)\right]$.

2. Solve : $\tan^{-1} x+\tan^{-1} 3x=\cot^{-1}\frac{1}{2}$.

3. Evaluate $\tan\left\{\sin^{-1}\left(\frac{-5}{13}\right)+\cos^{-1}\frac{8}{17}\right\}$.

4. Express $\sin^2(\tan^{-1} x)$ in terms of x.

5. Prove that $\tan^{-1}\frac{1}{7}+2\tan^{-1}\frac{1}{3}=\frac{\pi}{4}$.

Solve the following equations :

6. $2(\cos^4 x+\sin^4 x)=1$.

7. $\tan x+\tan\left(\frac{\pi}{4}+x\right)=2$.

8. $\cot x+\tan x=2\csc x$.

9. $\cot x-\tan x=2$.

10. $2\sin^4 x-\cos^2 x=0$.

SUMMARY

1. (*a*) If $0\leqslant x\leqslant 1$, then $0\leqslant \sin^{-1} x\leqslant \pi/2$, $0\leqslant \cos^{-1} x\leqslant \pi/2$.

   (*b*) $\sin^{-1}(-x)=-\sin^{-1} x$, provided $-1\leqslant x\leqslant 1$.

   (*c*) $\cos^{-1}(-x)=\pi-\cos^{-1} x$, provided $-1\leqslant x\leqslant 1$.

   (*d*) $\cos^{-1} x+\sin^{-1} x=\pi/2$, provided $-1\leqslant x\leqslant 1$.

2. (*a*) If $x\geqslant 0$, then $0\leqslant \tan^{-1} x<\pi/2$, $0<\cot^{-1} x\leqslant \pi/2$.

   (*b*) $\tan^{-1}(-x)=-\tan^{-1} x$, for all $x\in \mathbf{R}$.

   (*c*) $\cot^{-1}(-x)=\pi-\cot^{-1} x$, for all $x\in \mathbf{R}$.

   (*d*) If $-1\leqslant x\leqslant 1$, then

$$\tan^{-1} x+\cot^{-1} x=\pi/2.$$

3. (*a*) If $0\leqslant xy<1$, then

$$\tan^{-1} x+\tan^{-1} y=\tan^{-1}\frac{x+y}{1-xy}.$$

   (*b*) If $xy=1$, then

$$\tan^{-1} x+\tan^{-1} y=\begin{cases}\pi/2, \text{ if } x>0, y>0\ ;\\ -\pi/2, \text{ if } x<0, y<0.\end{cases}$$

   (*c*) If $xy>1$, then

$$\tan^{-1} x+\tan^{-1} y=\begin{cases}\pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right), \text{ if } x>0, y>0\ ;\\ -\pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right), \text{ if } x<0, y<0.\end{cases}$$

4. If $|k| \leqslant 1$, then $\{x : \sin x = k\} = \{n\pi + (-1)^n \sin^{-1} k : n \in \mathbf{Z}\}$.

5. If $|k| \leqslant 1$, then

$\{x : \cos x = k\} = \{2n\pi \pm \cos^{-1} k : n \in \mathbf{Z}\}$.

6. If $k \in \mathbf{R}$, then

$\{x : \tan x = k\} = \{n\pi + \tan^{-1} k : n \in \mathbf{Z}\}$.

□□

PART IV : STATISTICS AND LINEAR PROGRAMMING

★ ★

World Wars can be good too ! For the first time in history, a war, the Second World War to be precise, made it possible for mathematics to be applied to social sciences. Linear programming was created during the Second World War for defence purposes.

Statesmen (politicians) can be good too ! We at least owe the origin (and growth) of a branch of mathematics, viz., statistics, to them.

RONALD A. FISHER (1890-1962)

Sir Ronald A. Fisher was one of the most eminent scientists of the century who made important contributions to genetics and statistics. His work had a great impact on improvement of agricultural production all over the world. It demonstrates how statistics could be useful in real life.

CHAPTER 17

Statistics

17.1. DATA AND VARIABLES

Times change ; with time, our mode of living changes ; our habits change ; in fact our way of thinking changes. Since time immemorial, change has been the way of the world. The rate of change, however, has been generally very slow with the exception of last one century or so. During recent times there has been a rapid change in every gamut of our life. The reason is not difficult to find. Ways and means have been discovered which have greatly facilitated the dissemination of information. Our media has become very strong. Information about almost everything is almost always available in a suitable form. Newspapers, magazines, television etc. give useful information in an easily comprehensible manner. Information about wages, prices, sports, weather, medicine, agricultural produce, industrial goods, taxes of various kinds, population, per capita income, annual export and import of goods, even things like fashions, crime and trend of voters is scattered all round us. Recall that this information is known as **data** and that a large amount of data that we come across are in the form of numbers.

You do appreciate that there are many agencies at work that help us to receive information in a particular form. For example, consider the daily minimum and maximum temperature of a metropolitan city and the weather forecast for various regions of our country. For us, the minimum and maximum temperature may be something like 8·5 degrees celsius and 40·3 degrees celsius respectively. The weather for the next twenty-four hours might involve something like '*generally dry*', '*heavy rains*', '*cloudy*', '*strong winds*', '*light snow fall*' etc. How do you think the statements like this are issued ? No doubt, you know that there is a whole big agency known as the Meteorological Department which is responsible for information and prediction about weather. This Department constantly records the temperature, relative humidity, amount of rainfall, atmospheric pressure etc. It also receives a large number of photographs of earth (and its ionosphere) through satellites. The amount of data so collected is unbelievably huge. These data in their initial form are known as *raw* data. We cannot make head or tail out of them. The experts in the Meteorological Department (MD) *process* these data, that is to say, they organize these data and present them

systematically in a form suitable for handling the problem in hand. It is only after the data have been so organised that any conclusions can be drawn from them or information based on them can be given **out.**

Let us now see how the experts in MD obtain and process raw data to obtain a particular information, say determining the minimum temperature during a certain period of twenty-four hours. To begin with, they must record the temperature at various instants. Clearly, the temperature will *vary* from instant to instant. At 5 AM, the temperature might be 6·7 degrees celsius whereas at 2.00 PM it might be 18·9 degrees celsius. Since the temperature keeps on *varying*, we can call it a **variable** The recorded temperatures at various instants become then the values of this variable *viz.*, temperature. The collection of the various values of this variable temperature constitutes the raw data. This collection can be organized in ascending or descending order and the minimum temperature can then be easily determined. Notice that each of these values is obtained by *observing* the temperature at a particular instant. The various instants are thus the source of our information or data. These sources are, therefore, known as the **units of observation**. Various values of the variable pertain to the units of observation. Units of observation can be persons, places or any objects.

In the example considered above, the variable was *temperature*. The units of observation were various *instants*. The values of the variable were certain numbers. The variable represented a *quantity* which could be *measured and expressed* in numbers. Such variables are known as **quantitative variables**. Arithmetic operations can be performed meaningfully on the values of a quantitative variable.

Let us have another example. Suppose we want to find what percentage of people in a certain locality of a town is male. This time our source of information or units of observation are the *people* living in this locality. For each person, we must examine the sex. The variable is, therefore, the *sex* of people. What would our data consist of ? Now a person can either be a male or a female. Thus the variable in question, *viz*., sex, can have only two possible vaues, *male* and *female*. Our data will, therefore, be a collection of 'Male' and 'Female' records. Notice that this time our variable cannot be *measured* and cannot, therefore, yield numerical values. We may choose to designate 'male' and 'female' with certain numbers if we like. However, it would not be meaningful to perform any arithmetic operation on these numbers. The values of the variable in this case express the various states (being male/female) of a unit of observation (person). Such variables are known as **qualitative variables** (as against quantitative variables). Other examples of a qualitative variable are colours of flowers, kinds of crops, religions of people etc.

EXERCISE 17 (*a*)

1. Which of the following are qualitative variables ?

(*a*) Place of job (*b*) Complexion
(*c*) Height (*d*) Weight
(*e*) Age (*f*) Area
(*g*) Income (*h*) Taste
(*i*) Caste (*j*) Arithmetic operation
(*k*) Subject of study (*l*) Rainfall.

2. Which of the following could be regarded as a variable and which as a value of this variable ?

(*a*) 100 *cm* (*b*) Weight
(*c*) 2·5 *kg* (*d*) Length
(*e*) Nationality (*f*) Christian.

3. What could be possible units of observation for the following variables ?

(*a*) Marks (*b*) Number of pages
(*c*) Colour (*d*) Number of voters.

17·2. FREQUENCY DISTRIBUTIONS

Raw data are useless unless they are processed and organised in a suitable form. You already know that to make data useful and comprehensible, we can represent them in a tabular form, so as to bring out the essential features and chief characteristics. Tables help us in making answers available to our questions regarding given data. Let us illuatrate this through an example.

Example 1. *Arrange the figures contained in the following passage in a tabular form inserting appropriate headings :*

USA has been the most successful nation in the history of modern Olympic Games. At the thirteenth Olymic Games held in 1948 at London, immediately after World War II, it bagged as many as 38 gold medals, 27 silver medals and 19 bronze medals. At the 1952 Olympics, the corresponding figures were 40 gold medals, 19 silver medals and 17 bronze medals. Four years later, at the Melbourne Olympics in 1956, it won 32 golds, 25 silvers and 17 bronzes. Continuing its medal hunt at the Rome Olympics in 1960, it ended up with 34 golds, 21 silvers, and 16 bronzes. Four years later in 1964, its medal tally rose to 90, out of which there were 36 gold medals and 26 silver medals, the rest being bronze medals. At the next three Olympic games held in 1968 (Mexico), 1972 (Munich) and 1976 (Montreal), it continued its medal winning spree with 45, 33 and 34 gold medals, 28, 31, and 35 silver medals, and 34, 30 and 25 bronze medals respectively. Unfortunately it boycotted the 1980 Moscow Olympic Games. At the XXII Olympic Games held in

1984 on its homeground at Los Angeles, its achievements rose to new heights (partly due to non-participation by the countries belonging to the communist block), amassing 83 gold medals, 61 silver medals, and 30 bronze medals.

Solution. We can organize the above data in the form of a table given below :

United States Medal Tally for Olympic Games 1948-1984

Year	*Gold Medals*	*Silver Medals*	*Bronze Medals*	*Total*
1948	38	27	19	84
1952	40	19	17	76
1956	32	25	17	74
1960	34	21	16	71
1964	36	26	28	90
1968	45	28	34	107
1972	33	31	30	94
1976	34	35	25	94
1980*	—	—	—	
1984	83	61	30	174

*USA did not participate in Olympic Games.

Source : *Sportsworld*, 3-9 August, 1988.

By looking at the above table we can immediately tell the total medal tally of United States or the number of medals of any type obtained at an Olympic Game held between 1948 and 1984.

In a table for a qualitative variables, first of all, the various alternatives or values of the variable are written separately. For example, if figures for food production in various states of India are given and they are to be shown in tabular form, then the variable according to which figures will be divided is 'States of India', and its various values will be Punjab, Bihar, U.P. etc. The required table will then be formed by placing against various values (Punjab, Bihar, U.P., etc.) the production in these states. So if there is only one variable, a one-way table is formed ; if there are two or more than two variables, a two way or many-way table will be formed. Hence to prepare a table with qualitative characteristics, the following steps are taken :

1. Pick out from the data, various variables and the alternatives corresponding to each one of them.

2. Draw a blank table showing all the variables with their alternatives along rows and columns respectively.

3. Find, from the given data, and enter the values in the proper cells formed by the intersection of rows and columns.

Since tabulation is a process of reducing the data to an easily understandable form, a table should not be too complicated. All details must be briefly given. Before drawing a table one must have a clear idea of the facts to be presented, the contrast to be stressed and the points upon which emphasis is to be laid. While drawing a blank table special attention should be given to the following points :

1. A title must be given to the table. It should include three things. Firstly, the things which are represented; secondly, the place to which the data belong and lastly the time to which it refers.

2. Brief headings should be given to all columns and rows.

3. Things to be compared should be placed side by side or one above the other.

4. Characteristics should be so adjusted that figures to be added should come one under the other *i.e.*, should fall in the same column.

5. Quantitative characteristics must be so written that, if possible, value of the variable increases from left to right (in a row) or from top to bottom (in a column).

6. Proper lines (single lines, double lines, bold lines, etc.) should be drawn in the table. This will not only place various characteristics into contrast, or emphasise some of them, but will also make the table attractive.

7. Units in which the quantities tabulated are measured must be given. If there are only one or two units used throughout the table, they should be written at the right-hand top corner. But if there are more units used in the table, they should be given along with the heading of the rows or the columns in which the figures corresponding to them occur.

8. The source from which the data are taken, if known, should be given at the bottom of the table.

9. Along both the vertical and horizontal sides, main divisions of the table should be done according to the characteristic which is of major importance or should be done according to a characteristic which is to be shown quite prominently. For others, each main division should be sub-divided, *e.g.*, in a table showing literacy among men and women, if emphasis is to be given to the number of literates and illiterates. two main columns should be taken, one for literates and the other for illiterates. Each column should then be sub-divided into two parts, one for males and the other for females. If on the other hand emphasis is to be laid on the number of men

and women, main division should be done according to men and women and sub-divisions according to literates and illiterates.

10. If certain things in the table are not self evident and require explanations, foot notes corresponding to them should be attached to the table.

Let us recall that a table with quantitative variables is usually called a **frequency table** *or* a **frequency distribution.** Frequency means the number of times an item with the same characteristic occurs. To prepare a frequency table, the following steps are taken :

(*a*) Arrange the raw data in an array, *i.e.*, in ascending or descending order of magnitude.

(*b*) Draw three columns, the second being sufficiently bigger than the third.

(*c*) Write in the first column all the items one under the other.

(*d*) Take the given observations one by one. Corresponding to each of them place a bar in the second column against the item in which the observation lies. These are called **tally bars**.

(*e*) When the observations are all over, count the bars in each row. In the third column, place against each item the number of tally bars counted. These numbers are the *frequencies* because they tell us how *frequently* the values belonging to a particular item lie in the data.

(*f*) Figures in the first and third column taken together represent the frequency table.

Note. To facilitate the counting of tally bars, it is customary to combine the tally bars in groups of five each by placing four bars in vertical position and the fifth in slanting position, cutting the first four.

When the number of items themselves becomes unwieldy, we divide the items into groups or classes The frequency, in this case, of a particular class is the number of times the items occur in that class. The method of preparing the table is the same as explained above in the case of frequency table with the only exception that now the items of a column are to be replaced by the classes. When the various classes are placed against their frequencies, then this representation is called a **grouped frequency distribution** or simply a **grouped distribution.**

An example will illustrate the above mentioned concepts.

Example 2. *Following are the marks obtained by 70 students out of a maximum of 60 marks. Prepare a frequency table as also a grouped frequency table with equal class intervals of five marks :*

43, 28, 19, 25, 48, 12, 31, 47, 10, 34, 0, 47, 17, 5, 46, 0, 47, 15, 20, 37, 0, 31, 48, 15, 37, 26, 33, 50, 5, 16, 31, 37, 26, 22, 50, 19, 10, 50, 31, 41, 35, 30, 59, 25, 9, 13, 50, 19, 22, 32, 6, 52, 12, 20, 52, 26, 25, 34, 8, 53, 12, 17, 22, 54, 26, 30, 57, 45, 41, 40.

Frequency Table of Marks of 70 Students

Marks	*Tally Bars*	*Frequency (No. of Studets)*
0	III	3
5	II	2
6	I	1
8	I	1
9	I	1
10	II	2
12	III	3
13	I	1
15	II	2
16	I	1
17	II	2
19	III	3
20	II	2
22	III	3
25	III	3
26	IIII	4
28	I	1
30	II	2
31	~~IIII~~	5
32	I	1
33	I	1
34	II	2
35	I	1
37	III	3
40	I	1
41	I	1
43	I	1
45	I	1
46	I	1
47	III	3
48	II	2

Marks	*Tally Bars*	*Frequency (No. of Students)*
50	IIII	4
52	II	2
53	I	1
54	I	1
57	I	1
59	I	1
		70

Solution. The data in the array form (ascending order) are :

0, 0, 0, 5, 5, 6, 8, 9, 10, 10, 12, 12, 12, 13, 15, 15, 16, 17, 17, 19, 19, 19, 20, 20, 22, 22, 22, 25, 25, 25, 26, 26, 26, 26, 28, 30, 30, 31, 31, 31, 31, 31, 32, 33, 34, 34, 35, 37, 37, 37, 40, 41, 43, 45, 46, 47, 47, 47, 48, 48, 50, 50, 50, 50, 52, 52, 53, 54, 57, 59.

Grouped Frequency Table of Marks of 70 students

Marks (Class)	*Tally Bars*	*Frequency*
0— 5	III	3
5—10	卌	5
10—15	卌 I	6
15—20	卌 III	8
20—25	卌	5
25—30	卌 III	8
30—35	卌 卌 I	11
35—40	IIII	4
40—45	III	3
45—50	卌 II	7
50—55	卌 III	8
55 —60	II	2
		70

Remarks. **1.** In each class, the lower limit is included and the upper limit is excluded.

2. The maximum marks are 60.

In connection with the above representation of grouped data, the following terms should be noted :

(*i*) Each of the groups of marks is called a **class**, *e.g.*, 0—5. In the above representation, we have included marks between 0 and under 5. Of course we can include 5 also, if we interpret it that way.

(*ii*) The left hand member is called the **lower limit** of the class and the right-hand member, the **upper limit** of the class. Thus 10 and 15 are the lower and upper limits respectively of the class 10—15 (irrespective of their being included or excluded in the class).

(*iii*) The difference between the upper limit and the lower limit is called the **width** of the class or the **class interval**. In the above representation, the width of the classes is same, *viz.*, 5, but it may not be same in all cases.

(*iv*) One-half of the sum of the upper limt and lower limit is called the **mid-value of the class.** Thus $\frac{0+5}{2}$, *i.e.*, 2·5 is the mid-value of the class 0—5.

In case the class intervals are given, we can prepare the table as explained above. But the class intervals are usually not given. The following points should always be kept in view while choosing class intervals :

(*a*) As far as possible, the width of all classes should be equal and the upper and lower limits should be integers.

(*b*) The number of classes should not ordinarily exceed 20, because otherwise the classes themselves become unmanageable.

(*c*) The classes should be defined precisely, *i.e.*, whether you are including or excluding the upper or the lower limit.

Remark. Throughout the book, unless mentioned otherwise, we shall be including the lower limit of class and excluding the upper, whenever a grouped frequency distribution is tabulated in the above form.

There are two other ways of presenting a table for a grouped frequency distribution, which are quite often used. We give below two examples to illustrate these.

Marks obtained by 43 students at an annual examination

Marks	*Frequency*
310 and under 320	9
320 and under 330	9
330 and under 340	6
340 and under 350	5
350 and under 360	10
360 and under 370	2
450 and under 460	1
500 and under 510	1

Remark. The maximum marks for the examination are 800.

Source. Result sheets of the Examination.

Marks of 21 students

Marks	*Frequency*
500—509	3
510—519	2
530—539	1
540—549	3
550—559	3
560—569	1
580—589	2
610—619	2
620—629	2
630—639	2

Note. The maximum marks for the above examination are 800.

Source. Result of the Examination.

You should note the relative merits and demerits of the above-said three ways of preparing a grouped frequency table. The first one of these is the one most commonly used because the middle values turn out to be similar figures generally, *e.g.*, if the successive classes are 0—10, 10—20, 20—30, ... then the middle values of the classes will be 5, 15, 25,......, but had we taken the classes to be 0—9 10—19, 20—29,...the middle values would have been 4·5, 9·5, 14·5,...

etc. If one prepares a table in the first way, then one must mention which of the two limits, lower or the upper, is included, whereas there is no such difficulty in the second and the third method.

17·2·1. Cumulative Frequency. Before we proceed further, we give the idea of cumulative frequency.

Cumulative frequency corresponding to a particular item means the total of all the frequencies preceding the item in the table and the frequency of the item itself. Cumulative frequency of a class means the total of all the frequencies upto the class considered. A frequency table containing an extra column listing the various cumulative frequencies is known as a **cumulative frequency distribution**. An example of a Cumulative Frequency Distribution is given below :

Marks	*Frequency*	*Cumulative Frequency*
0— 5	3	3
5—10	5	8
10—15	6	14
15—20	8	22
20—25	5	27
25—30	8	35
30—35	11	46
35—40	4	50
40—45	3	53
45—50	7	60
50—55	8	68
55—60	2	70

The cumulative frequency (C.F.) of the class 0—5 is the same as its frequency because there is no preceding class. The C.F. of the class 5—10 is the sum of the frequencies of the classes 0—5 and 5—10, and is hence $3+5=8$. To get the C.F. of the class 10—15, we need only add the C.F. of the class 5—10 and the frequency of the class 10—15. Why ? Would the C.F. of the last class be the total frequency ?

EXERCISE 17 (*b*)

1. The following marks were given to a batch of candidates :

   45, 62, 66, 43, 63, 44, 55, 58, 79, 32, 51, 56, 60, 51, 49, 25, 42, 54, 54, 58, 70, 58, 50, 52, 38, 67, 50, 59, 48, 65, 71, 30, 46, 55, 82, 51, 45, 53, 40, 35, 56, 70, 52, 67, 55, 57, 30, 63, 42, 74.

Draw a frequency table and show the cumulative frequencies also in the table.

2. Arrange the following data of heights in centimetres of a group of 31 students in the form of a cumulative frequency table :

130, 135, 148, 130, 140, 148, 149, 135, 140, 135, 145, 140, 135, 145, 145, 130, 140, 140, 146, 150, 146, 140, 145, 149, 157, 148, 145, 146, 148, 145, 148.

3. Give an example of
   (*a*) frequency table,
   (*b*) grouped frequency table,
   (*c*) cumulative frequency table,
   (*d*) cumulative grouped frequency table.

4. Can we obtain a frequency table from a grouped frequency table ? Is the converse possible ? Illustrate your answer by examples.

5. The following table gives the life times of 400 tubes tested in a laboratory :

Life times of tubes tested (in days)	*No. of tubes*
300— 400	14
400— 500	46
500— 600	58
600— 700	76
700— 800	68
800— 900	62
900—1000	48
1000—1100	22
1100—1200	6
	400

(*a*) Find the upper limit of the fifth class.

(*b*) Find the lower limit of the eighth class.

(*c*) Find the width of each class.

(*d*) Find the frequency of the fourth class.

(*e*) Find the percentage of tubes whose life times are less than 600 days.

(*f*) Find the percentage of tubes whose life times are greater than or equal to 900.

6. Make a frequency table having grades of wages with class-intervals of Rs. 2 each from the following data of daily wages (in rupees) received by 30 employees of a certain factory :

14, 16, 16, 14, 22, 13, 15, 24, 12, 23, 14, 20, 17, 21, 22, 18, 18, 19, 20, 17, 16, 15, 11, 12, 21, 20, 17, 18, 19, 23.

7. The rents of 25 shops in rupees are given below. Put them in the form of a frequency table with class intervals of five rupees :

140, 160, 170, 180, 160, 170, 190, 180, 170, 180, 180, 200, 190, 250, 190, 190, 200, 210, 220, 280, 210, 260, 230, 240, 220. Find out the cumulative frequencies also.

8. The weights in grammes of 50 apples picked out at random from a consignment are given as follows :

82, 118, 80, 110, 104, 84, 106, 107, 76, 82, 109, 107, 115, 93, 187, 95, 123, 125. 111, 92, 86, 70, 126, 68, 130, 129, 139, 119, 115, 128, 100, 186, 84, 99, 113, 204, 111, 141, 136, 123, 90, 115, 98, 110, 78, 90, 107, 81, 131, 75.

Form the grouped frequency table of the above data taking all the classes to be of equal width, in such away that the mid-value of the first class corresponds to 70 grammes.

9 The following is a cumulative frequency table :

Class	*Cumulative Frequency*
Below 5	4
5—10	6
10—15	12
15—20	20
20—25	27
25—30	29
30—35	35
35—40	55
40—45	60
45—50	67
50 and above	80

Study the table carefully and tell which of the two limits (lower or upper) is included in any class.

Prepare the grouped frequency table of the above data.

10. The following is a record of weights of 70 students in Kilograms. Tabulate the data in the form of a frequency distribution, taking the lowest class as 30—35 and the size of each class as 5 :

31, 37, 47, 54, 56, 38, 39, 35, 48, 36, 44, 48, 55, 42, 43, 42, 46, 38, 46, 46, 51, 45, 51, 45, 39, 53, 45, 43, 57, 51, 57, 36, 39, 59, 48, 32, 50, 41, 51, 53, 44, 45, 46, 53, 55, 38, 43, 44, 53, 54, 31, 47, 37, 56, 58, 43, 42, 53, 49, 38, 33, 41, 52, 43, 46, 43, 39, 36, 34, 44.

11. Form an ordinary grouped frequency distribution table from the following data :

Height (in centimetres)	*Number of Plants*
Above 70	2
,, 60	5
,, 50	10
,, 40	18
,, 30	27
,, 20	40
,, 10	42

[**Hint** : How many plants have height between 10 and 20 cm ? etc.]

12. Form an ordinary grouped frequency distribution table from the following data :

Weight (in Kilograms)	*Number of Candidates*
Below 15	10
,, 25	15
,, 35	22
,, 45	28
,, 55	37
,, 65	42
,, 70	50

13. Throw a six-faced die 40 times. After each throw, record the dots uppermost. Tabulate your data.

14. Throw two dice together 20 times. After each throw, record the total of the dots uppermost on the two dice. Tabulate your data.

15. Remove all the face cards from an ordinary deck of cards. Shuffle the cards that remain, draw two cards from the deck, record the total of the spots on the two cards and replace the cards in the deck. Repeat this operation 30 times. Tabulate your data.

16. Take a 10-card deck consisting of the ace, 2, 3, 4, 5, 6, 7, 8, 9, and 10 of hearts. Shuffle the deck and then turn them face up

one at a time. Before each card is turned, guess what its denomination will be. Record the number of correct guesses. Repeat this 20 times. Summarize.

17. As a class project, collect the following data on each student in the class and summarize the data in an appropriate way or ways :
   (*i*) number of letters in full name ;
   (*ii*) age in months ;
   (*iii*) number of magazines and/or newspapers subscribed to ;
   (*iv*) number of names of States in India that the student can write down ; and
   (*v*) number of formulae that the student can write down in five minutes.

18. Tabulate the following information :

In a trip organized by a college, there were 80 persons, each of whom paid Rs 15·50 on an average. There were 60 students each of whom paid Rs 16. Members of the teaching staff were charged at a higher rate. The number of servants was 6 (all males) and they were not charged anything. The number of ladies was 20% of the total, of which 1 was a lady staff member.

19. Present the information contained in the following text in a suitable tabular form, giving an appropriate title :

"The total rural population of India according to 1951 census is 2,948 lakhs, of which 2,404 lakhs belong to agricultural classes. Of the total urban population of 618 lakhs, 531 lakhs belong to non-agricultural classes.

Of the rural agricultural classes, 687 lakhs are self-supporting persons, 1,414 lakhs are non-earning dependents and 303 lakhs are earning dependents. The rural non-agricultural population comprises 544 lakhs, of which 170 lakhs are self-supporting persons, 326 lakhs are non-earning dependents and 48 lakhs are earning dependents. In the urban agricultural classes, 23 lakhs are self-supporting, 56 lakhs are non-earning dependents and 8 lakhs are earning dependents. The urban non-agricultural population comprises 531 lakhs of persons, of whom 153 lakhs are self-supporting persons, 347 lakhs are non-earning dependents and 31 lakhs are earning dependents."

20. Arrange the information given in the following passage in tabular form giving appropriate headings :

"After having been away from modern Olympic Games for nearly forty years, USSR returned to the Olympic Games in 1952. Represented by a large contingent, they won 22 gold

medals, 35 silver medals and 19 bronze medals. In 1956, at the Melbourne Olympic Games, Russia turned out to be the top sporting nation bagging 98 medals, 37 of which were gold and 29 were silver, relegating USA to the second place. Four years later, at Rome Olympics in 1960 it again occupied the first place with a haul of 43 gold, 29 silver and 31 bronze medals. At the the next three Olympic Games held in 1964 (Tokyo), 1968 (Mexico) and 1972 (Munich) it bagged 30, 29 and 50 gold medals, 31, 32 and 27 silver medals, and 35, 30, and 22 bronze medals respectively. At the Montreal Olympic games in 1976, it further consolidated its winning spree by capturing 49 gold, 41 silver and 35 bronze medals. The most glorious moment was, however, in 1980 when on the home-ground (Moscow), the Russian participants won as many as 80 gold, 69 silver and 46 bronze medals. It bycotted (alongwith other countries of the communist block), the XXII Olympic Games held at Los Angeles in 1984."

17·3. RELATIVE FREQUENCY DISTRIBUTIONS

Quite often we wish to compare two similar situations for a certain attribute. For example, we may want to know whether girls perform better than boys at a certain examination or whether a certain brand of tea is more popular in a city than another brand and so on. Clearly, we shall make such a comparison on the basis of information or data available to us. The relevant data are generally organized as a frequency table. Thus the job at hand may require the comparison of two similar frequency tables. For example, in case of the performance of girl *vis-a-vis* boys at a certain examination say A, we may have the following frequency distributions :

Performance of Girls at Exam. A

Status	*Number of Girls*
Pass	700
Fail	300
Total=	1000

Performance of Boys at Exam. A

Status	*Number of Boys*
Pass	936
Fail	1064
Total=	2000

It appears from the tables above that more boys (936) have been declared pass than girls (700). Does it mean that boys have performed better ? A little reflection will show that this is not true.

It is true that more boys have been declared pass, but then the number of boys appearing at this examination was double the number of girls appearing at this examination. We notice that more than two-thirds of the girls appearing at the examination passed whereas not even half the boys passed the examination. This means girls certainly performed better than boys. Thus wee see that a comparison of frequencies may be quite misleading. What we should have compared was not the frequencies but the ratio of the frequencies to the total frequency. This ratio is known as the **relative frequency.** Since relative frequencies are ratios, they are generallv expressed as a percentage. This amounts to changing the data proportionately assuming the tolal frequency to be 100. We reproduce the above tables again with a column representing the relative frequencies as well. Such tables are known as **relative frequency tables.**

Performance of Girls at Exam. A

Status	*Number of girls*	*Relative frequency*	*Relative frequency as a %*
Pass	700	·7(=700/1000)	70(=·7×100)
Fail	300	·3(=300/1000)	30(=·3×100)
Total	=1000		

Performance of Boys at Exam. A

Status	*Number of boys*	*Relative frequency*	*Relative frequency as a %*
Pass	936	·468	46·8
Fail	1064	·532	53·2

The comparison is easy now. Since the pass percentage 70 in case of girls is higher than 46·8 in case of boys, the performance of girls is better.

Example 3. *Calculate the relative frequencies for the frequency distribution given below. Also express these relat ve frequencies as a percentage.*

Classification of inhabitants of India below the age of 25 according to 1981 Census

Age	*Frequency*
0— 9	177,389,757
10—14	85,674,734
15—19	64,037,433
20—24	57,307,790
Total	=384,409,714

Solution The relative frequency of class 0—9

$$=\frac{177,389,757}{384,409,714}=\cdot 46.$$

The above value is correct upto two decimal places and is only an approximate value. To get it as a percentage, we multiply it by 100 and get 46. In a similar fashion, the relative frequencies of the other classes can be obtained. The completed table showing the approximate values of relative frequencies (correct upto two decimal places) is given below :

Classification of inhabitants of India below the age of 25 according to 1981 Census

Age	*Frequency*	*Relative frequency*	*Relative frequency* (%)
0— 9	177,389,757	·46	46
10—14	85,674,734	·22	22
15—19	64,037,433	·17	17
20—24	57,307,790	·14	14
Total	=384,409,714		

Remarks. 1. When relative frequencies are approximate values, they need not add upto 1 as in the above example.

2. When frequencies are large, several facts about the distribution may remain latent till we look at the relative frequencies. For example, it is not clear from the first table in Example 3 above that the number of children in the age group 0—9 is more than thrice the number of young men in age group 20—24. But a cursory glance at the second table makes this fact clear. Similarly, we can see immediately from the second table that the ratio of persons in the age-groups 10—14 and 15—19 is 22 : 17. This is not evident from the first table. Relative frequencies are very useful when frequencies happen to be large.

3. Comparison between two frequency tables cannot be made unless the variables are same. In case of grouped frequency distributions, the class-intervals should tally too.

EXERCISE 17 (*c*)

Calculate all answers correct upto two decimal places.

1. Calculate the relative frequencies of the various classes in the frequency table given below and express them as a percentage :

Marks	: 0—9	10—19	20—29	30—39	40 and above
Frequency :	15	27	38	40	10

Remark. Notice that the data have been arranged a little differently here. Instead of listing the classes and the frequencies vertically in the form of columns, the same have been written horizontally as rows. This is done frequently to save space.

2. The following table gives information about smokers in two villages A and B. Compare the two villages as regards smokers in various age-groups.

Smokers in Villages A and B

Age-group	*Village A*	*Village B*
0—20	125	50
21—40	365	400
41—60	410	500
61 and above	420	300

3. The following table gives information about the milk-yield of she-buffaloes in two neighbouring villages. Compare the villages as regards milk-yield class by class :

Milk-yield of buffaloes in villages A and B

Yield per day (*in l*) Above—upto	*Number of she-buffaloes* *Village A*	*Village B*
0— 3	25	10
3— 6	50	47
6— 9	80	93
9—12	125	218
12—20	30	202
Total	310	570

17·4. GRAPHICAL REPRESENTATION OF FREQUENCY DISTRIBUTIONS

As they say, a picture is worth a thousand words. So it is with data. A tabular presentation of data is more useful than its word description. But we can do better and represent a frequency distribution by means of pictures. Representation of a frequency distribution by means of any kind of pictures is known as a **graphical**

representation of the distribution. We list below some advantages of graphical representation of data :

1. It is attractive.
2. It is effective in as much as several salient features of the distribution can be discerned at a glance.
3. It makes data simple and intelligible.
4. It facilitates comparison.
5. It saves time and labour.

Graphical representation of data is universal. People in business, industry, commerce, science and technology, all use various types of graphical representation for data. We shall consider here only two of these *viz.*, bar diagrams and pie charts.

17·4·1. Bar Diagrams

To draw simple bar diagrams, we begin by choosing a pair of perpendicular lines called axes. On one of the axes, equal lengths are marked, one for each class in case of quantitative variables and one for each alternative/value of the variable in case of qualitative variables. This length has nothing to do with the lengths of the various class-intervals which may or may not be equal. On the other axis, we choose a suitable scale to represent frequencies. On each of the lengths marked on the first axis, we erect a rectangle (or bar) whose height/width represents, on the chosen scale, the frequency of the class (or value) represented by this particular length. Thus the heights/widths of the various rectangles are proportional to the frequencies of the various classes (or values). Let us represent the following frequency distribution of students in various classes of a school by means of a simple bar diagram.

Distribution of Students in a School

Class	*No. of Students*
I	120
II	100
III	60
IV	30
V	20
	Total=330

Let us choose a horizontal axis OX and a vertical axis OY. On the horizontal axis OX, let us mark five equal lenghts of 1 *cm* each corresponding to each of the five classes I to V. Let us choose a scale of 1 *cm* to represent 20 students. Thus the frequency 120 of class I is represented by a length of 6 *cm*. Hence on the first length on the horizontal axis which represents class I, we erect a

Distribution of students in a school

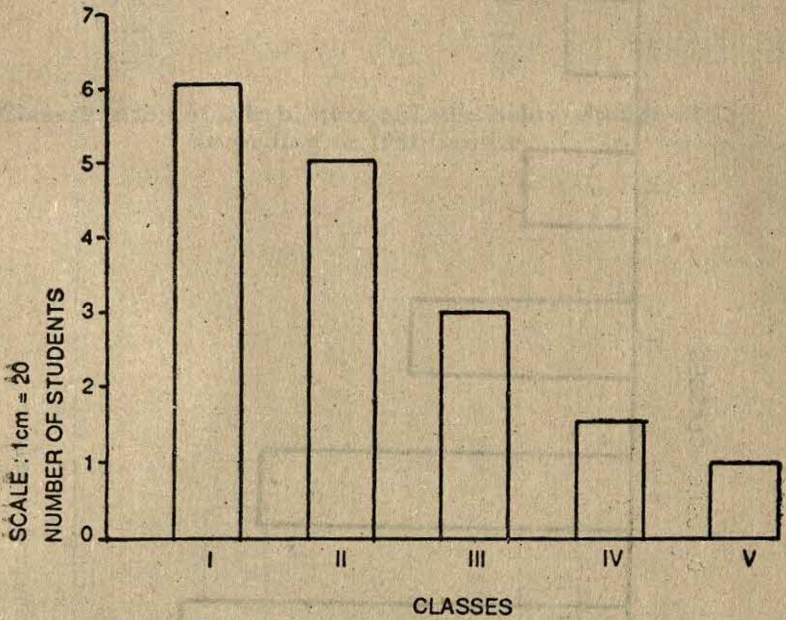


Fig. 17·1.

rectangle of length 6 *cm*. Similarly for the other classes II to V, rectangles of length 5, 3, 1·5 and 1 *cm* respectively are raised.

Remarks. 1. The above pictorial representation is called a **bar diagram** because of the bar-like appearance of the rectangles.

2. Since we chose the horizontal axis for representing classes and the vertical axis for representing frequencies, our rectangles are vertical. Had we chosen the vertical axis for representing classes and the horizontal axis for frequencies, our bar diagram would have looked as shown in Fig. 17·2.

However, by and large we choose the vertical bars instead of horizontal ones.

3. On a different scale, the diagram given on next page can be taken to represent the corresponding relative frequency distribution. The relative frequency (%) of class I is (approximately) 36. Thus had we chosen the scale 1 $cm = 6$ on the vertical axis, the diagram could be taken to be that of the corresponding relative frequency distribution.

Distribution of students in a school

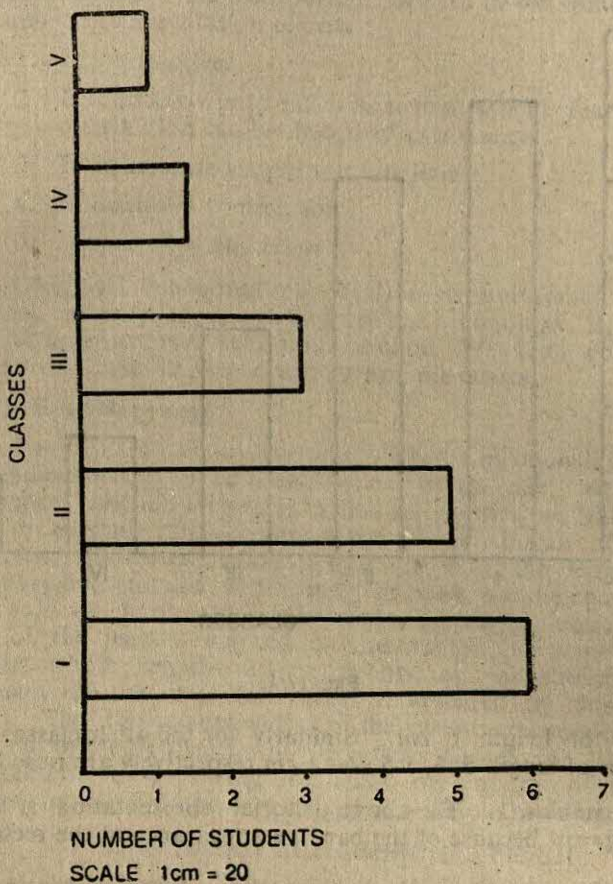


Fig 17·2.

4. While making a bar diagram, the following points should be kept in mind :

(*a*) Scale should be so chosen that the diagram is neither too big nor too small. The scale should be mentioned near the axis representing the frequencies.

(*b*) Every diagram, like a table, should have a title.

(*c*) One should use graph paper and ruler as far as possible so as to make the diagram as accurate as possible.

Example 4. *The relative frequency distribution of Example 3, with the last two classes merged in one, and its bar diagram are shown on next page :*

Age	*Relative frequency* (%)
0— 9	46
10—14	22
15—24	31

Classification of inhabitants of India below the age of 25 according to 1981 Census

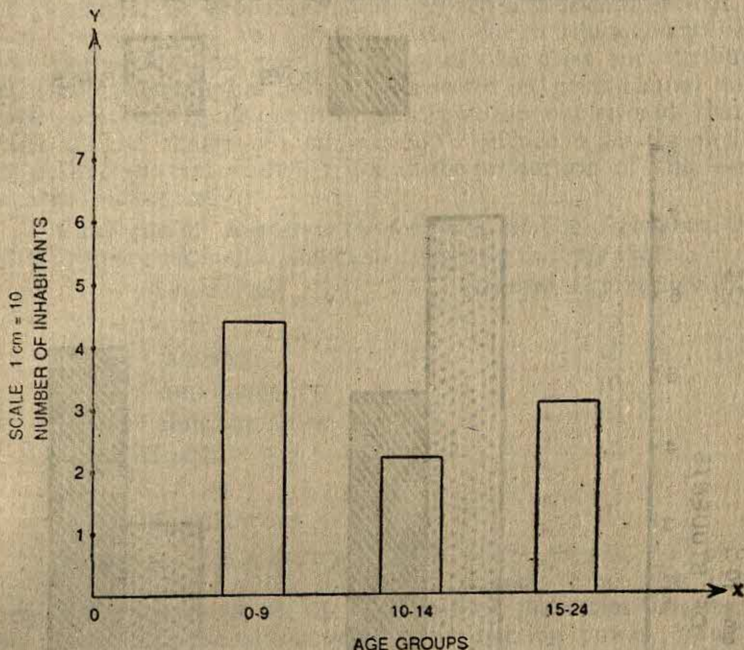


Fig. 17·3.

Remarks. 1. Notice that even though the class-intervals are different, yet the lengths representing the same on the horizontal axis are the same.

2. Many-way tables can be represented by a single bar diagram. Also, where variable values or class-intervals are the same, a single diagram can be used to represent more than one frequency distribution as shown in Example 5 below. This facilitates comparison.

Example 5. *The relative frequency distributions of the performance of boys and girls considered earlier are reproduced below (in the form of a two-way table) and their bar diagrams are drawn in a single picture below :*

Performance of boys and girls at Exam. A

Status	*Relative frequency* (%)	
	Girls	*Boys*
Pass	70	46·8
Fail	30	53·2

Solution.

Performance of boys and girls at Exam. A

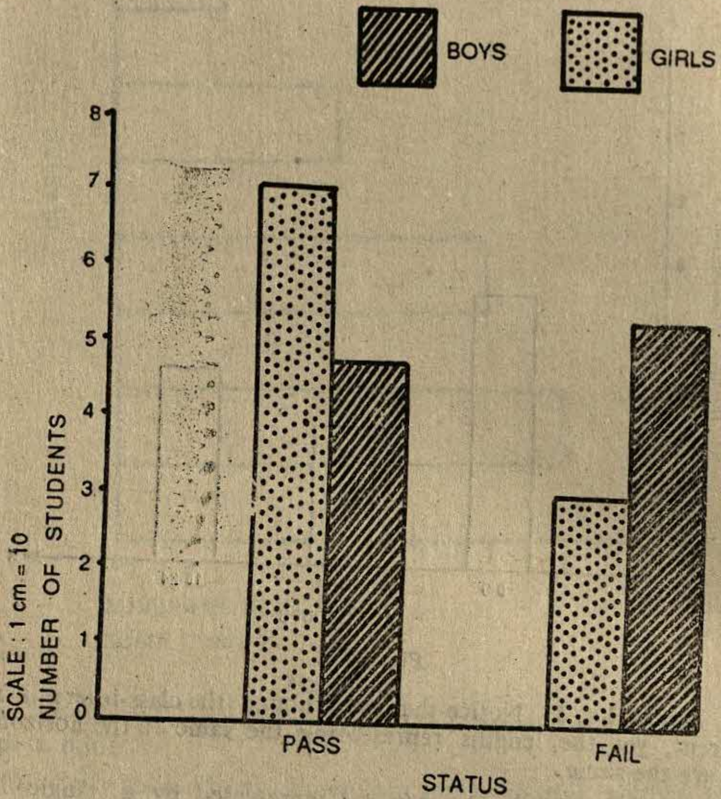


Fig. 17·4.

To represent a many-way table or two frequency distributions on the same bar diagram, we erect the rectangles corresponding to same value/class of various variables side by side. The rectangles corresponding to different variables are shaded differently. For example, we have shown the rectangles corresponding to girls by dots and those corresponding to boys by slant lines. This scheme should be shown near the bar diagram. Such bar diagrams are

known as **double bar diagrams** as against the simple bar diagrams drawn earlier.

17·4·2. Pie Charts

Another popular pictorial device for representing relative frequency distributions is a pie chart. A **pie chart** is nothing but a circular region (of arbitrary radius) divided into several sectors, one each corresponding to various classes (or attributes) and having area proportional to the frequency of the class (or attribute) it represents Recall that the area of a sector is proportional to the angle subtended by its arc at the centre of the circle, and that the angle at the centre of a circle is 360°. Hence to get the angles of the desired sectors, we divide 360° in the proportion of the various relative frequencies

Example 6. *Represent the following relative frequency distribution of the budget of a municipal committee in a pie chart.*

Budget Head	*Relative Expenditure* (%)
Power	15
Transport	15
Communication	10
Public amenities	18
Health	12
Salaries	11
Miscellaneous	19

Solution. Since there are seven budget heads, our circular region would be divided into seven sectors. Since the relative frequencies are 15, 15, 10, 18, 12, 11 and 19, the angles A_1, A_2,......, A_7 subtended by the various sectors representing power, transport, ..., miscellaneous are in the proportion 15 : 15 : 10 : 18 : 12 : 11 : 19. Note that $15+15+10+\cdots\cdots+19=100$, so that

$$\angle A_1=\frac{15}{100}\times 360°=54°,$$

$$\angle A_2=54°,$$

$$\angle A_3=\frac{10}{100}\times 360°=36°,$$

$$\angle A_4=\frac{18}{100}\times 360°=64{\cdot}8°=65° \text{ approx.},$$

$$\angle A_5=\frac{12}{100}\times 360°=43{\cdot}2°=43° \text{ approx.},$$

$$\angle A_6=\frac{11}{100}\times 360°=39{\cdot}6°=40° \text{ approx.},$$

$$\angle A_7=360°-(54°+54°+36°+65°+43°+40°),$$

$$=68°\left(=\frac{19}{100}\times 360°=68{\cdot}4°=68° \text{ approx.}\right).$$

The resulting pie chart is shown below.

Budget of a Municipal Committee

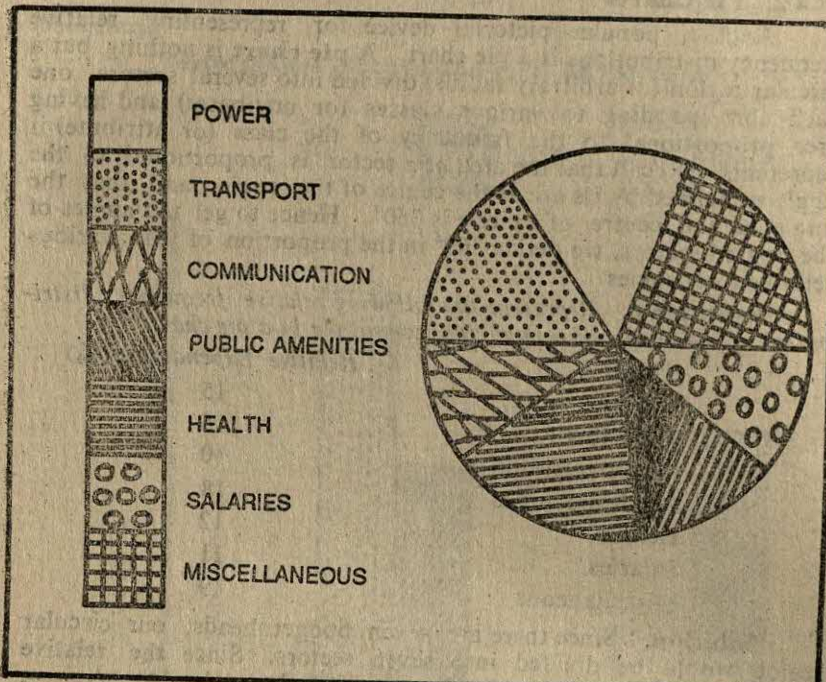


Fig. 17·5.

EXERCISE 17 (*d*)

1. Draw a bar diagram to represent the following data :

Year	*1982*	*1983*	*1984*	*1985*
Exports (crores of Rs.)	7,805	8,803	9,770	11,855

2. Draw a bar diagram to represent the following data :

Population of India

Year	*Population* (*in crores*)
1901	23·6
1911	25·2
1921	25·1
1931	27·9
1941	31·9
1951	36·1

Year	Population (in crores)
1961	43·9
1971	54·8
1981	68·5

3. Prepare the relative frequency distribution from the frequency distribution of Exercise 1 and represent the same by a
   (a) bar diagram,
   (b) pie chart.
4. Prepare the relative frequency distribution from the frequency distribution of Exercise 2 and represent the same by a
   (a) bar diagram,
   (b) pie chart.
5. Represent the following data by means of a double bar diagram :

Year	*1983*	*1984*	*1985*
Exports (Crores of Rs.)	8,803	9,770	11,855
Imports (Crores of Rs.)	14,306	15,831	17,173

6. Represent the following data by means of a suitable bar diagram :

Age and Marital Status of Inhabitants of India

Age group	*Unmarried*	*Married*
0—14	134,628,189	1,190,313
15—24	45,887,676	16,678,303
25—34	7,048,269	39,561,552
35—44	1,162,271	35,626,701
45—54	665,817	26,731,640
55—64	368,498	15,361,824
65 and above	363,187	9,415,162

Source : Census of India 1981.

[**Hint :** Take suitable approximations.]

17·5. SUMMARISING DATA

We have already studied how raw data can be condensed or summarized in the form of a frequency table. Though better than raw data, frequency tables may still be a big jumble of figures from which it may not be possible to extract the desired information and discover the salient features or the characteristic properties of the given data. It is usual to further summarise the information con-

tained in the data or to represent the same by a few numerical values (in case of quantitative data) which generally answer questions of the following type :

(A) Which is (are), the item(s) around which data items have a tendency to cluster.

(B) How far scattered are the various data items.

Numerical values which summarise or represent data in this manner are known as **averages**. An average may or may not be a data item. Averages are used when we have to compare two or more similar types of data. Instead of comparing all the items of the data separately, we just compare their representative items and draw relevant conclusions.

Many types of averages have been studied, and unfortunately, any type of average cannot be used for all types of data. Different types of averages are best suited for different types of data. Thus, one may get absurd results by using an average which is not suitable for a particular type of data. We shall point out the types of data for which a particular average is best suited. Some desirable features of an ideal average are listed below :

(*a*) An average should be rigidly defined.

(*b*) An average should not be of too abstract a mathematical character.

(*c*) An average should be based on all the observations.

(*d*) An average should be calculable with reasonable ease and rapidity.

(*e*) An average should lend itself to algebraic treatment.

17·6. MEASURES OF LOCATION

We shall first study averages which answer questions of the type (A) above. These are known as **averages of the first kind** or more meaningfully, **measures of central tendency** or **measures of location**.

17·7. ARITHMETIC AVERAGE

Arithmetic average is the most commonly used average and is defined as *the quotient of the sum of all the observations divided by the number of observations.*

Thus, if $x_1, x_2, \ldots\ldots\ldots, x_n$ be n observations (for example, $x_1, x_2, \ldots\ldots, x_n$ may be the points obtained when we throw a six-faced die n times), then

$$\text{Arithmetic average} = \frac{x_1 + x_2 + \ldots + x_n}{n} \text{ per unit of observation,} \quad \ldots(1)$$

$$= \frac{1}{n} \sum_{i=1}^{n} x_i \text{ per unit of observation.}$$

Arithmetic average is also called "*Arithmetic mean*" and is written as 'A.M.' in abbreviated form. Note that the arithmetic average is not completely specified unless we express the unit in which the various data items are measured as also the units of observation. This is illustrated in the following example.

Example 7. *The rainfall in a city A in five successive years is 20, 30·4, 59·2, 18·1 and 42·3 centimetres. Calculate the average rainfall per year during this five-year period.*

Solution. $$\text{A.M.}=\frac{1}{n}\sum_{i=1}^{n} x_i \ cm \text{ per year},$$

$$=\frac{20+30{\cdot}4+59{\cdot}2+18{\cdot}1+42{\cdot}3}{5}\ cm.\text{ per year},$$

$$=\frac{170}{5}\ cm.\text{ per year},$$

$$=34\ cm.\text{ per year.}$$ **Ans.**

Remark. Notice that the A.M. is not a data item.

17·7·1. A.M. of Frequency Distributions

In the case of frequency distributions, when the ith observation x_i occurs with frequency f_i, *i.e.*,

x_1 occurs f_1 times,

x_2 ,, f_2 ,, ,

....

....

x_n ,, f_n ,, ,

then since $x_i+x_i+\ldots+x_i(f_i$ times) adds upto f_ix_i, therefore,

$$\text{A.M.}=\frac{f_1x_1+f_2x_2+\ldots+f_nx_n}{f_1+f_2+\ldots+f_n} \text{ per unit of observation},$$

$$=\frac{\sum_{i=1}^{n} f_ix_i}{\sum_{i=1}^{n} f_i} \text{ per unit of observation.} \qquad \ldots(2)$$

Example 8. *The following table gives the weekly salaries of persons employed in a factory. Calculate the mean weekly salary per person.*

Salary (*in Rs*) :	110	130	150	170	190	210	230
No. of persons :	5	7	10	15	13	16	14

Solution. We form the following table :

x_i	f_i	$f_i x_i$
110	5	550
130	7	910
150	10	1500
170	15	2550
190	13	2470
210	16	3360
230	14	3220
	$\Sigma f_i = 80$	$\Sigma f_i x_i = 14560$

$\therefore$ $\quad$ A.M. $= \dfrac{\Sigma f_i x_i}{\Sigma f_i}$ per person,

$=$ Rs $\dfrac{14560}{80}$ per person $=$ Rs 182 per person. **Ans.**

17·7·2. A.M. of Grouped Frequency Distributions

When the data are given in the form of a grouped frequency distribution, then we first find the mid-values of the classes, and then apply formula (2) with x_i replaced by the ith mid-value. In other words, the mid-values of the classes are themselves treated as the observations. See example 9.

Remark. Please note that in case of a grouped frequency distribution, the formula for A.M. given above gives us only an approximate value of the A.M. because while computing A.M. by this formula, instead of taking the actual value of any observation, we take for it the mid-value of the class to which it belongs.

17·7·3. Short-cut Method for Computing A.M.

Sometimes when the data are complicated, we assume any convenient number, say a, as the *provisional* or *working mean*. Then the actual arithmetic mean is given by the following formula :

$$\text{A.M.} = \left(a + \frac{\sum\limits_{i=1}^{n} f_i (x_i - a)}{\sum\limits_{i=1}^{n} f_i} \right) \text{ per unit of observation } \cdots(3)$$

Proof. We easily obtain formula (3) from formula (2) by shifting the origin to the point with abscissa α as shown below :

$$\frac{\sum\limits_{i=1}^{n} f_i(x_i-\alpha)}{\sum\limits_{i=1}^{n} f_i}=\frac{\sum\limits_{i=1}^{n} f_i x_i}{\sum\limits_{i=1}^{n} f_i x_i}-a\frac{\sum\limits_{i=1}^{n} f_i}{\sum\limits_{i=1}^{n} f_i}$$

$$=\text{A.M.}-a, \text{ by using (2).}$$

$$\therefore\quad \text{A.M.}=\left(a+\frac{\sum\limits_{i=1}^{n} f_i(x-a)}{\sum\limits_{i=1}^{n} f_i}\right) \text{ per unit of observation.}$$

17·7·4. Step Deviation Method for Calculating A.M.

Formula (3) is used while computing the A.M. from a frequency distribution. In case we have to compute the A.M. from a grouped frequency distribution with equal class intervals, the following formula may be used :

$$\text{A.M.}=\left(a+\frac{h\sum\limits_{i=1}^{n} y_i f_i}{\sum\limits_{i=1}^{n} f_i}\right) \text{ per unit of observation,} \qquad \ldots(4)$$

where a is any provisional mean, h is the width of each class and

$$y_i=\frac{x_i-a}{h},$$

x_i's being the mid-values of the classes.

Example 9. *Compute the mean of the following frequency table by (i) direct method, (ii) by short-cut method, and (iii) by step deviation method :*

Class	*Frequency*
5—10	10
10—15	6
15—20	4
20—25	12
25—30	8
30—35	4
35—40	2
40—45	1
45—50	3
	50

Solution. Direct Method :

Class	*Mid-value* x_i	*Frequency* f_i	$f_i x_i$
5—10	7·5	10	75·0
10—15	12·5	6	75·0
15—20	17·5	4	70·0
20—25	22·5	12	270·0
25—30	27·5	8	220·0
30—35	32·5	4	130 0
35—40	37·5	2	75·0
40—45	42·5	1	42·5
45—50	47·5	3	142·5
		50	1100·0

$\therefore$ A.M. $=\frac{\Sigma f_i x_i}{\Sigma f_i}=\frac{1100}{50}=22$ **Ans.**

Short-cut Method :

Since 12 is the maximum frequency, therefore, let us suppose

$a=$provisional mean$=22·5$.

x_i	x_i-a $(=x_i-22·5)$	f_i	$f_i(x_i-a)$
7·5	−15	10	−150
12·5	−10	6	− 60
17·5	− 5	4	− 20
22·5	0	12	0
27·5	5	8	40
32·5	10	4	40
37·5	15	2	30
42·5	20	1	20
47·5	25	3	75
		50	−25

$$\therefore \quad \text{A.M.}=a+\frac{\Sigma f_i(x_i-a)}{\Sigma f_i},$$

$$=22·5+\frac{-25}{50},$$

$$=22·5-·5=22.$$ **Ans.**

Step-deviation Method :

Let $a=22{\cdot}5$. Also. let $h=5$.

x_i	$y_i=\dfrac{x_i-a}{h}$	f_i	f_iy_i
7·5	−3	10	−30
12·5	−2	6	−12
17·5	−1	4	−4
22·5	0	12	0
27·5	1	8	8
32·5	2	4	8
37·5	3	2	6
42·5	4	1	4
47·5	5	3	15
		50	−5

$$\therefore \quad \text{A.M.}=a+h\,\frac{\sum\limits_{i=1}^{n} f_iy_i}{\sum\limits_{i=1}^{n} f_i},$$

$$=22{\cdot}5+5\times\frac{-5}{50},$$

$$=22{\cdot}5-{\cdot}5=22. \quad \textbf{Ans.}$$

Note. Observe the rapidity of calculations of the step-deviation method.

Remark. Arithmetic average kind of tells you that if every data item had the same value, the sum of all the values remaining fixed, then every item will have the value equal to the A.M. Hence A.M. should be used to summarize the data when all items have equal importance.

Merits of A.M.

(*a*) It is very easily understandable.

(*b*) It can be calculated easily.

(*c*) It lends itself to algebraic treatment, *e.g.*, the A.M. of two or more data can be calculated from their separate means.

(*d*) It considers all the observations and gives them importance proportional to their weights (frequencies).

(*e*) It can be calculated even when we only know the sum of the observations and the number of observations and not necessarily the observations themselves.

Demerits of A.M.

(*a*) A.M. may not be an item in the actual observations.

(*b*) The fact that it gives importance proportional to the frequency of each observation is sometimes harmful. For example, in a factory, a manager is getting Rs 2065 and 100 labourers are getting Rs. 45 each as weekly salary. Then to say that the average weekly salary in the factory is Rs 65 per person is a very fallacious statement and does not represent the actual facts.

(*c*) In case the data are given only qualitatively, it cannot be used with advantage.

(*d*) It cannot be calculated even if one of the observations is missing or in cases where the extreme ends are open, *e.g.*, "above 100" or "below 50" etc.

EXERCISE 17 (*e*)

1. Following are the marks obtained by ten students :
   4, 5, 4, 6, 5, 5, 6, 2, 1, 2.
   Calculate the A.M.

2. The monthly salaries of 10 domestic servants are given below in rupees :

   175, 225, 160, 130, 230, 370, 105, 185, 230 and 260.

   Find the average monthly salary per servant.

3. Calculate the mean of the data given below :

x:	1	2	3	4	5	6	7	8	9	10
f:	1	2	3	4	5	6	7	8	9	10

4. The following tables give the heights in centimetres of persons living in two different families A and B. In which family do the persons seem to be taller ?

Family A

Height	150	142	130	60	80	100
No. of persons :	3	1	2	1	2	1

Family B

Height	140	160	70	85	95	120
No. of persons :	2	1	2	1	1	3

5. Calculate to the nearest paisa, the mean of the following grouped frequency distribution by two methods :

Weekly wages (in Rs.)	*No. of workers*
12·5—17·5	2
17·5—22·5	22
22·5—27·5	19
27·5—32·5	14
32·5—37·5	3
37·5—42·5	4
42·5—47·5	6
47·5—52·5	1
52·5—57·5	1

6. Eight coins were tossed together and the number X of heads resulting was observed. The operation was repeated 256 times and the frequencies that were obtained for the values of X are shown in the table below. Calculate the mean number of heads turning up per toss.

X:	0	1	2	3	4	5	6	7	8
F:	2	6	26	59	72	52	32	7	0

7. The frequency distribution below gives the marks obtained by 188 students in a test. Obtain the arithmetic mean.

Class	*Frequency*
2— 6	1
6—10	9
10—14	21
14—18	47
18—22	52
22—26	36
26—30	19
30—34	3

8. Show that the mean of the first n natural numbers is $\frac{1}{2}(n+1)$.

9. Show that the mean of the squares of the first n natural numbers is $(n+1)(2n+1)/6$.

10. The following frequency table was formed by a *patwaree* while finalizing his records about the land owned by farmers in his locality :

Land (in hectares) ...and below	*No. of farmers*
0—1	5
1—2	10
2—3	14
3—4	18
4—5	25
5—6	26
6—7	12
7—8	4
8—9	1
9 and above	3

Find the mean land owned per farmer.

[**Hint**. Clearly, you cannot fix up the mid-value of the last class. Leave it and find the mean for the rest. The last class contains the exceptional cases and had better be ignored for a realistic representation of the data by means of the A.M.]

11. Calculate the mean for the following frequency distribution :

Class	0-10	10-20	20-30	30-40	40-50	50-60
Frequency	3	6	8	15	10	8

(*AISSE 1987 C*)

12. For the following distribution, find the mean :

Class	100-120	120-140	140-160	160-180
Frequency	10	8	4	4
Class	180-200	200-220	220-240	
Frequency	3	1	2	

(*AISSE 1987*)

13. Given the following frequency distribution, calculate the arithmetic mean :

Monthly wages	*Workers*	*Monthly wages*	*Workers*
12·5—17·5	2	37·5—42·5	4
17·5—22·5	22	42·5—47·5	6
22·5—27·5	19	47·5—52·5	1
27·5—32·5	14	52·5—57·5	1
32·5—37·5	3		

(*AISSE 1983*)

14. The mean height of 15 students is 154 cm. It is discovered later on that while calculating the mean the reading 175 cm. was wrongly read as 145 cm. Find the correct mean height.

(*DBSSE 1987*

15. The average monthly wage of a group of 10 persons is Rs 1500. One member of the group, whose monthly wage is Rs 1300, left the group and is replaced by a new member whose monthly wage is Rs 1200. Find the new average monthly wage. *(AISSE 1987 C)*

16. The mean height of 20 students is 155 cm. It is observed later on that while calculating the mean, the reading 149 cm was wrongly read as 189 cm. Find the correct mean. *(AISSE 1987)*

17·8. MEDIAN

Like A.M., median also pertains to quantitative variables alone. Suppose that the values of all the observations are arranged in ascending or descending order of magnitude. **Median** *is the value of an observation which lies in the middle of the observations when arranged in ascending or descending order of magnitude.* This means that as many data items have value smaller than or equal to the median value as have value bigger than or equal to the median value. If the variable has an odd number of values say $m=2n+1$, certainly, when arranged in the ascending or descending order, the $(m+1)/2$th or the $(n+1)$th observation will meet this requirement. n data items will have value less than or equal to the value of the $(n+1)$th observation and n observations will have value greater than or equal to the value of the nth observation. If, however, the number of observations is even say $2n$, then such a requirement cannot be met with. In this case, the nth and the $(n+1)$th data items can be thought of as lying in the middle of the data. If we take any value M lying between the values of these two observations, then n data items will be less than or equal to M and n data items will be greater than or equal to M. Hence M may be regarded as a median of the given data.

Remark. Notice that the A.M. of the $(n+1)$th and the nth data itmes is also a value lying between these two data items. It is conventional to take this A.M. as the median.

Example 10. *Calculate the median of the following observations :*

18, 15, 13, 11, 3, 4, 5, 9, 6, 1, 2, 10, 8.

Solution. Arranging the items in ascending order of magnitude, we get the sequence

1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 13, 15, 18.

No. of observations $=13$.

$$\therefore \quad \text{Median} = \frac{13+1}{2}\text{ th observation,}$$

$$= 7\text{th observation,}$$

$$= 8. \ \textbf{Ans.}$$

Remarks. 1. You must carefully note the distinction between '7' and the '7th data item'. There are 13 data items. So the *7th item* is the median. Hence 8, and not 7, is the median. However, if the data item 8 were replaced by 7, then the 7th observation would be 7 itself and hence the median value (by chance) would also be 7.

2. Had our list contained one more item, 100 say, then there would have been 14 items in all. In this case there would have been 2 middle items, *viz.*, the 7th and the 8th. These would have been 8 and 9. So 8·5 would have been taken as the median value.

Let us see now how to find the median value when the data are arranged in the form of a frequency distribution.

Example 11. *Calculate the median height from the following frequency table :*

Height (in centimetres)	*No. of boys*
139	6
121	4
111	8
123	6
125	7
104	5
107	4

Solution. Let us arrange the boys in ascending order of heights and also form a cumulative frequency table as shown below :

Height (in cm)	*No. of boys*	*Cumulative frequency*
104	5	5
107	4	9
111	8	17
121	**4**	**21**
123	6	27
125	7	34
139	6	40

Recall that the 'frequency of 104 is 5' means that '5 data items are (have value) 104 each'. Similarly, from the above table we notice that 4 observations have value 107 each. The C.F. column shows that 5 data items are less than or equal to 104, 9 data items are less than or equal to 107, 17 observations have value less than or equal to 111, and so on. The last entry 40 in this column shows that there are 40 observations in all. This is an even number, the middle two observations being the 20th and the 21st. The table given above shows that 4 items, *viz*, 18th, 19th, 20th and

21st, all have value equal to 121. Thus we find that both of the 20th and the 21st observation have value 121. Hence the median height is 121 cm ; 20 observations have value less than or equal to 121 cm and 20 observations have value greater than or equal to 121 cm.

Remark. Suppose we modify the above table so that the frequencies of 121 and 123 become 3 and 7 instead of 4 and 6 respectively. The C F. column would then contain the entries 5, 9, 17, 20, 27, 34 and 40. This would mean that the 20th observation would be 121 and the 21st 123. The median value would be 122 cm now. Now 20 items are smaller than the median and 20 items are bigger than the median.

When the data are given in the form of a grouped frequency distribution, then the first thing that we notice is that this time our median value, like the A.M. for grouped frequency distributions, can only be an approximate value. Hence when the total frequency is an even integer, N say, we try to find the value (approximate of course) of the N/2th item and do not bother about the middle two items. Thus the first step consists of *finding the class which contains the N/2th item*. This is know as the **median class.** We now make the assumption that the values in any class are spread uniformly over that class This means that if there are 10 values in a class of length 1, then the difference between any two successive values is 0·1. This is only an assumption and the actual situation may be quite different. But since we have no way of knowing the actual data, this assumption would do. Let us take an example to see how median is calculated for a grouped frequency distribution on the basis of this assumption.

Example 12. *Calculate the median from the following table* :

Percentage recovery of Sugar on Cane	*No. of factories*
8·0— 8·2	2
8·2— 8·4	5
8·4— 8·6	4
8·6— 8·8	11
8·8— 9·0	11
9·0— 9·2	11
9·2— 9·4	13
9·4— 9·6	10
9·6— 9·8	7
9·8—10·0	6
10·0—10·2	3
10·2—10·4	1
10·4—10·6	1

Solution. We first frame the cumulative frequency table :

Percentage recovery of Sugar on Cane	*No. of factories*	*Cumulative frequency*
8·0— 8·2	2	2
8·2— 8·4	5	7
8·4— 8·6	4	11
8·6— 8·8	11	22
8·8— 9·0	11	33
9·0— 9·2	**11**	44
9·2— 9·4	13	57
9·4— 9·6	10	67
9·6— 9·8	7	74
9·8—10·0	6	80
10·0—10·2	3	83
10·2—10·4	1	84
10·4—10·6	1	85

No. of observations$=85$

$\therefore$ Median observation$=\dfrac{85+1}{2}$th observation,

$=43$rd observation.

To determine the median class, we have to find out the class to which 43rd observation belongs. Looking at the column which contains the cumulative frequencies, we observe that the first two observations belong to the first class; next five or 3rd to 7th belong to the second class: next four or 8th to 11th observations belong to the third class;...; 34th to 44th observations belong to the 6th class or the class 9·0—9·2. Thus we see that the 43rd observation belongs to the class 9·0—9·2 and this, therefore, is the median class.

Now the width of the median class is 9·2—9·0, *i.e.*, 0·2 cm. There are 11 items in the median class. Of these 11, we are interested in the 10th item, because the 43rd item from the beginning is the (43—33)rd item in this class, as 33 items from the beginning belong to the earlier classes. If we divide the length 0·2 cm of the median class into 11 equal parts, each part would be 0·2/11 cm long. Hence the 10th item would be at a distance $10\times\dfrac{0{\cdot}2}{11}$cm from the beginning of this class, *i.e.*, from the lower limit 9·0 cm of this class. Hence the median value is

$$\left(9{\cdot}0+10\times\frac{0{\cdot}2}{11}\right)\text{ cm or } 9{\cdot}18 \text{ cm.}$$ **Ans.**

Remark. Notice that here

$9 \cdot 0$=lower limit of the median class,

$10 = \frac{1}{2}$ [the total frequency—C.F. of the class preceding the median class],

$0 \cdot 2$=width of the median class, and

11=frequency of the median class.

This suggests the following formula for calculating the median value of a grouped frequency distribution :

$$\text{Median} = L_1 + \frac{N/2 - C}{f_{median}} \times h,$$

where L_1=Lower limit of the median class,

N=total frequency,

C=cumulative frequency of the class preceding the median class,

f_{median}=frequency of the median class,

h=width of the median class.

The median can now be very easily calculated by substituting the various symbols in the formula given above.

To determine the median class, we have simply to divide the total frequencies by 2 and look out in the cumulative frequency column (which we have to first form if it is not already available) for the first number which is greater than or equal it. The class corresponding to this number in the C.F. column is the median class.

Uses of Median

The median is useful in the following fields :

(*a*) for comparison of data, especially when individual values are not capable of measurement.

(*b*) for determining the typical or middle type in social problems like wages, distribution of wealth etc., because half the items are below and half above it.

Merits of median

(*a*) It is very easy to calculate the median. In fact, sometimes it can be found by inspection.

(*b*) If the class intervals are open at the ends, median can be calculated, provided we know the corresponding frequencies only. Recall that A.M. cannot be computed in such cases.

(*c*) Unlike arithmetic mean, it is not affected by the inclusion of very large or very small items.

Demerits of median

(*a*) It is not amenable to algebraic treatment.

(*b*) Like arithmetic mean, the median that we calculate may not be an item in the data *e.g.*, the median of the observations 1, 3, 5, 7, 9, 10 can be taken as 6, but it is none of the actual observations.

(*c*) Sometimes when the items near the central item vary very slowly, the median is usually indeterminate.

(*d*) It does not give equal importance to all the items. It actually recognises the place where the items are put.

EXERCISE 17 (*f*)

Calculate all answers correct to two decimal places.

1. Following are the marks obtained by a class. Determine the median marks :

   180, 140, 130, 123, 215, 211, 220, 80, 105, 189, 220, 165, 149, 127, 78, 55, 46, 20, 120, 160, 97, 163, 183, 203, 133, 152, 62, 77, 111, 99, 110.

2. Calculate the median of the following figures :

   100, 109, 117, 106, 76, 101, 109, 118, 110, 101, 109, 118, 140, 81, 86, 101, 110, 119, 86, 8, 89, 90, 91, 94, 95, 96, 99, 97, 100, 102, 111, 120, 104, 112, 121, 104, 104, 105, 113, 113, 122, 123, 114, 124, 106, 114, 125, 107, 108, 115, 128, 116, 117, 108, 134, 129.

3. Calculate the median of the following data :

Variable :	1	2	3	4	5	6	7	8	9	10
Frequency :	1	2	3	4	5	6	7	8	9	10

4. *Find the median of the following distribution :

x :	8	5	6	10	9	4	7
y :	6	4	5	8	9	6	4

(*DBSSE 1987*)

5. The weights (in kilogram) of 15 students are as follows :

   31, 35, 27, 29, 32, 43, 37, 41, 34, 28, 36, 44, 45, 42, 30.

   Find the median. If the weight 44 kg is replaced by 46 kg and 27 kg by 25 kg, find the new median.

   (*DBSSE 1987 C*)

6. The daily earnings of 10 workers in a factory are :

   16, 8, 19, 7, 12, 6, 13, 14, 16, 17.

   Find the median earning. (*AISSE 1979*)

7. The marks obtained by 12 students out of 50 are :

   25, 24, 23, 32, 40, 27, 30, 25, 20, 10, 15, 45.

   Find the median score. (*AISSE 1979 C*

8. The following table gives the number of fami'ies residing in different rent class houses. Find the average rent paid by a family by computing the median.

Monthly Rent (in Rs.)	*No. of families*
Less than Rs 45	192
45—50	147
50—55	70
55—60	27
60—70	29
70—80	9
80—90	3
90—110	4
110—150	3
150 and above	1

9. The yields of grain (x kg) from 500 small plots are grouped into classes with a common class interval (0·2 kg) in the table below, the values of x given being the mid-values of the classes. Show that the mean of the distribution is 3·95 kg and the median is also 3·95 kg

x	f	x	f
2·8	4	4·2	69
3·0	15	4·4	59
3·2	20	4·6	35
3·4	47	4·8	10
3·6	63	5·0	8
3·8	78	5·2	4
4·0	88		

10. Find the median for the following :

Marks	*No. of Students*
Below 10	15
,, 20	35
,, 30	60
,, 40	84
,, 50	96
,, 60	127
,, 70	198
,, 80	250

11. Find the median for the following :

Marks	*No. of students*
0—4	10
4—8	12
8—12	18
12—14	7
14—18	5
18—20	8
20—25	4
25 and above	6
	70

12. The scores on a reading comprehension test of 1000 students are given below :

Scores (out of 75)	*Frequency*	*Scores (out of 75)*	*Frequency*
0— 5	6	25—30	250
5—10	12	30—35	185
10—15	50	35—40	110
15—20	120	40—45	32
20—25	225	45—50	10

Find the median score. (*AISSE 1979*)

17 9. MEASURES OF DISPERSION

The measures of location studied already can be regarded as representatives of the data but they do not tell us anything about the *spread, scatteredness* or *variability* of the data. We can easily construct examples of data whose variations are different, but for which the means are the same. Consider for example the following data representing the marks obtained in a paper out of a maximum of 20 marks by two sections of the same class :

1, 2, 3, 4, 6, 7, 8, 10, 19, 20

and 4, 5, 5, 7, 8, 8, 9, 10, 11, 13.

The arithmetic means of both the data are same, *viz.* 8, but there is a wide difference in the formation of the above data. The first data start from 1 and end at 20 whereas the second data start from 3 and end at 13 only. This, in other words, means that the first section has students who are dull and some bright students also, whereas the students in the second section are almost average students. Such a variation in data is called **dispersion**. Measures of dispersion are also known as the **averages of the second kind.**

We generally study variation or dispersion relative to a measure of location. Thus we may be interested in knowing how the data items are dispersed about the mean or about the median. In this section, we shall study one measure *viz.*, *standard deviation*, which measures dispersion about the A.M. and one measure known as the *mean deviation about the median* which measures dispersion about the median.

17·10. MEAN DEVIATION ABOUT THE MEDIAN

Mean deviation about the median is defined as the arithmetic mean of the absolute deviations of the values of the various observations from their median. Notice that the sum of the deviations of the values of the observations from their A.M. is always zero. Absolute deviation means taking the positive difference or the absolute value of the difference.

Thus, if the observation with value x_i occurs with frequency f_i and M is the median of this frequency distribution, then

Mean deviation from the median

$$=\frac{\Sigma_i f_i \mid x_i - \text{M} \mid}{\Sigma f_i}.$$

The notation $\mid x_i - \text{M} \mid$ means that $x_i - \text{M}$ is always to be considered as positive, *i e.*, the sign of $x_i - \text{M}$ is not to be considered.

Example 13. *Calculate the mean deviation from the median of the following distribution :*

Class	: *0—10*	*10—20*	*20—30*	*30—40*	*40—50*
Frequency	: *10*	*12*	*25*	*19*	*8*

Solution. To calculate the mean deviation from the median, we first have to calculate the median. We thus have

Class	*Frequency=f*	*Cumulative frequency*
0—10	10	10
10—20	12	22
20—30	**25**	**47**
30—40	19	66
40—50	8	74

Here, N=74, N/2=37. The number in the C.F. column which is just greater than 37 is 47. Hence the median class is 20—30.

$$\therefore \quad \text{Median}=20+\frac{37-22}{25}\times 10=26.$$

To calculate the mean deviation, we have to calculate $x_i - 26$ for various values of x_i. Hence we frame the following table :

x_i	f_i	x_i-26	$f_i \mid x_i-26 \mid$
5	10	-21	210
15	12	-11	132
25	25	-1	25
35	19	9	171
45	8	19	152
	74		690

$$\therefore \text{ Mean deviation}=\frac{\Sigma f_i \mid x_i-26 \mid}{\Sigma f_i}=\frac{690}{74},$$

$$=9{\cdot}32. \quad \textbf{Ans.}$$

EXERCISE 17 (g)

1. Calculate the mean deviation from the median of :
2354, 2780, 3011, 3020, 3541, 4150, 5000.

2. Calculate the mean deviation from the median :

x :	0	1	2	3	4	5	6	7	8	9	10	11	12
y :	15	16	21	10	17	8	4	2	1	2	2	0	2.

3. Calculate the mean deviation from the median of the following distribution :

Class :	0—10	10—20	20—30	30—40	40—50
Frequency :	5	8	15	16	6

4. Calculate the mean deviation from the median of the following data :

Class	*Frequency*
0—5	449
5—10	705
10—15	507
15—20	281
20—25	109
25—30	52
30—35	16
35—40	4

5. Calculate the mean deviation from the median :

Class	*Frequency*
15—16	0
16—17	1
17—18	3
18—19	8
19—20	12
20—21	14
21—22	14
22—23	5
23—24	2
24—25	3
25—26	1
26—27	0
27—28	1

17·11. STANDARD DEVIATION

The mean deviation has the drawback that the signs of the deviations are discarded artificially. No rigid justification can be given for the dropping of the signs. Standard deviation, the most powerful measure of dispersion, avoids this artificiality by first squaring the deviations (so that negative deviations also become positive) and afterward taking the square root.

Thus, if x_i occurs with frequency f_i, then

$$\text{Standard deviation} = \text{S.D.} = \sqrt{\frac{\sum\limits_i f_i(x_i-\bar{x})^2}{\sum\limits_i f_i}},$$

where $\bar{x}$ is the mean of the data.

S D. is usually denoted by the greek letter σ.

Remarks. 1. Unlike mean deviation. S.D. is calculated from the mean.

2. The square of S.D. is called **variance.**

17·11·1. Short-cut Method to Calculate S.D.

In almost all cases, since the mean is usually in decimals, to avoid cumbersome calculations we assume any number a, arbitrarily chosen, as the mean and apply the following formula to calculate the S.D.

$$\text{S.D.} = \sqrt{\left[\frac{1}{\sum\limits_i f_i}\sum f_i(x_i-a)^2 - \left\{\frac{1}{\sum\limits_i f}\sum\limits_i f_i(x_i-a)\right\}^2\right]}.$$

This number a is usually chosen as the value corresponding to maximum frequency.

Remarks. 1. The above formula can be proved as follows :

$$\sum_i f_i(x_i-\bar{x})^2=\sum_i f_i\{(x_i-a)-(\bar{x}-a)\}^2,$$

$$=\sum_i f_i(x_i-a)^2-2(\bar{x}-a)\sum_i f_i(x_i-a)+(\bar{x}-a)^2\Sigma f_i \quad ...(i)$$

Also,

$$\sum_i f_i(x_i-a)=\sum_i f_i x_i-a\sum_i f_i,$$

$$=\bar{x}\sum_i f_i-a\sum_i f_i,$$

$$=(\bar{x}-a)\sum_i f_i,$$

so that

$$\bar{x}-a=\frac{\sum_i f_i(x_i-a))}{\sum_i f_i}. \quad ...(ii)$$

Substituting the above value of $\bar{x}-a$ in the right-hand side of (i), we have

$$\sum_i f_i(x_i-\bar{x})^2=\sum_i f_i(x_i-a)^2-\frac{2\{\sum_i f_i(x_i-a)\}^2}{\sum_i f_i}+\frac{\{\sum_i f_i(x_i-a)\}^2}{\sum_i f_i},$$

$$=\sum_i f_i(x_i-a)^2-\frac{\{\sum_i f_i(x_i-a)\}^2}{\sum_i f_i}.$$

Therefore,

$$\text{S.D.}=\sqrt{\left[\frac{\{\sum_i f_i(x_i-\bar{x})^2\}}{\sum_i f_i}\right]},$$

$$=\sqrt{\left\{\frac{1}{\sum_i f_i}\sum_i f_i(x_i-a)^2-\left(\frac{1}{\sum_i f_i}\Sigma f_i(x_i-a)\right)^2\right\}}.$$

2. Taking $a=0$ in the above formula, we also have

$$\text{S.D.}=\sqrt{\left\{\frac{1}{\sum_i f_i}\sum_i f x_i^2-\bar{x}^2\right\}},$$

since $\sum_i f_i x_i/\sum_i f_i=\text{A.M.}=\bar{x}$.

17·11·2. Coefficient of Variation

Sometimes we are required to compare the consistency of two data. For this, we define a dimensionless number called **coefficient of variation** by the following formula :

$$C.V. = \text{Coefficient of variation} = 100 \times \frac{\text{S.D.}}{\text{Mean}},$$

and say that lesser the coefficient of variation, the more consistent are the data.

Thus to compare the consistency of two data, we calculate their coefficients of variation and say that the data with lesser coefficient of variation are more consistent.

Example 14. *Calculate the S.D. and the coefficient of variation from the following table :*

Class	*Frequency*
1— 5	7
6—10	10
11—15	16
16—20	32
21—25	24
26—30	18
31—35	10
36—40	5
41—45	1

Solution. We frame the table as follows to calculate the mean :

Class	x=*mid-value*	f=*frequency*	fx
1—5	3	7	21
6—10	8	10	80
11—15	13	16	208
16—20	18	32	576
21—25	23	24	552
26—30	28	18	504
31—35	33	10	330
36—40	38	5	190
41—45	43	1	43
		123	2504

$$\therefore \qquad \text{A.M.} = \frac{2504}{123} = 20{\cdot}36 \text{ (approx.)}$$

Now we suppose that a=arbitrary mean=18. Thus we have the following table :

x	f	$x-18$	$f(x-18)$	$f(x-18)^2$
3	7	-15	-105	1575
8	10	-10	-100	1000
13	16	-5	-80	400
18	32	0	0	0
23	24	5	120	600
28	18	10	180	1800
33	10	15	150	2250
38	5	20	100	2000
43	1	25	25	625
	123		290	10250

Now, $$\text{S.D.}=\sqrt{\left[\frac{1}{\Sigma f}\,\Sigma f(x-18)^2-\left\{\frac{1}{\Sigma f}\,\Sigma f(x-18)\right\}^2\right]},$$

$$=\sqrt{\frac{1}{123}\times 10250-\left(\frac{1}{123}\times 290\right)^2},$$

$$=\frac{1}{123}\sqrt{10250\times 123-(290)^2},$$

$$=\frac{1}{123}\sqrt{1196650},$$

$=8{\cdot}82$ approx. **Ans.**

$\therefore$ Coefficient of variation $=100\times\dfrac{\text{S.D.}}{\text{Mean}}$,

$$=100\times\frac{8{\cdot}82}{20{\cdot}36}=43{\cdot}32. \quad \textbf{Ans.}$$

EXERCISE 17 (*h*)

1. Calculate the standard deviation of 60, 60, 61, 62, 63, 63, 63, 64, 64, 70.

2. Calculate the S.D. of :

x :	8	10	12	14	16	18	20	22	24	26
f :	64	100	144	196	256	324	400	484	576	676

3. Calculate the standard deviation of the following values of the world's annual gold output (in millions of pounds) of 19 different years :

94, 95, 96, 93, 87, 79, 73, 69, 68, 78, 82, 89, 95, 103, 108, 117, 130, 97.

4. Four students get the following marks :

(*i*) 1st student gets 5 marks ;

(ii) 2nd student gets 7 marks ;
(iii) 3rd student gets 9 marks ;
(iv) 4th student gets 11 marks.

Find the standard deviation.

5. Calculate the standard deviation for the following frequency distribution :

Class	0—4	4—8	8—12	12—16
Frequency	4	8	2	1

6. Find the S.D. and the coefficient of variation from the following table, giving the wages of 130 persons :

Wages (in Rs)	*No. of persons*
70—80	12
80—90	18
90—100	35
100—110	42
110—120	50
120—130	45
130—140	20
140—150	8

7. Calculate the standard deviation for the following table :

Wages (in years)	*No. of members*
20—30	3
30—40	61
40—50	132
50—60	153
60—70	140
70—80	51
80—90	2

8. The scores of two golfers, A and B, for 10 rounds each are :

A :	58	59	60	54	65	66	52	75	69	52
B :	83	56	92	65	86	78	44	54	78	68

Which player may be regarded as more consistent ? Why ?

9. Find the mean and the standard deviation for the following data :

Age (years)	*No. of teachers*
25—30	30
30—35	23
35—40	20
40—45	14
45—50	10
50—55	3

(*DBSSE, 1985*)

10. Find the mean and standard deviation for the following set of observations :

Observation :	9·7	9·8	9·9	10·0	10·1	10·2
Frequency :	2	3	4	6	4	1

(*Roorkee Entrance 1980*)

11. Calculate the mean and standard deviation for the following distribution :

Class Interval	*Frequency*
0—4	2
4—8	5
8—12	8
12—16	16
16—20	14
20—24	10
24—28	8
28—32	3

(*AISSE 1984 C*)

12. Find the mean and S.D. for the following distribution :

Marks	*No. of students*
5—10	5
10—15	6
15—20	15
20—25	10
25—30	5
30—35	4
35—40	2
40—45	2

(*AISSE 1985*)

13. Find the mean and the S.D. of the following distribution :

Class interval :	30—40	40—50	50—60	60—70	70—80
Frequency :	12	19	28	25	16

(*DBSSE 1988 C*)

14. Find the mean and the standard deviation of marks of 20 students as given below :

Class Interval	*Frequency*
0—10	2
11—20	3
21—30	10
31—40	19

Class Interval	Frequency
41—50	30
51—60	47
61—70	54
71—80	28
81—90	5
91—100	2

(*Roorkee Entrance 1981*)

15. The following frequency table gives the ages of a group of 50 children invited to a birthday party. Find the mean and standard deviation of the distribution.

Age (Classes) :	5—7	7—9	9—11	11—13	13—15
Frequency :	16	13	10	6	5

(*IIT, JEE 1978*)

16. Calculate the mean, median and standard deviation of the following distribution :

Age (in years)	Number of Marks
20—30	3
30—40	61
40—50	132
50—60	153
60—70	140
70—80	51
80—90	2

(*DBSSE 1984*)

17. The following data give the distribution of wages of 230 persons. Calculate the mean and the standard deviation for the distribution :

Wages of rupees	No. of persons
70—80	12
80—90	18
90—100	35
100—110	42
110—120	50
120—130	45
130—140	20
140—150	8

(*AISSE 1988*)

18. Find the mean and the standard deviation of the following distribution :

Class interval	*Frequency*
5—15	15
15—25	15
25—35	23
35—45	22
45—55	25
55—65	10
65—75	5
75—85	10

(*DBSSE 1986*)

19. Calculate the mean and standard deviation of the following data :

Interval	*Frequency*
10—15	7
15—20	8
20—25	6
25—30	14
30—35	20
35—40	15
40—45	11
45—50	10

(*AISSE 1987 C*)

20. In a study on patients, the following data were obtained. Find the standard deviation of the data.

Age (*in years*)	*Number of cases*
10—19	1
20—29	0
30—39	1
40—49	10
50—59	17
60—69	38
70—79	9
80—89	3

(*AISSE 1979*)

TEST YOUR UNDERSTANDING XVII

1. The sum of all the relative frequencies in a relative frequency distribution must be

(*a*) greater than 1

(*b*) equal to 1
(*c*) equal to zero
(*d*) greater than zero.

2. The *middle* aspect of a data are best described by
   (*a*) A.M. and S.D.
   (*b*) A.M. and mean deviation
   (*c*) A.M. and median
   (*d*) S.D. and mean deviation.

3. The *variability* or *spread* of a data is best described by
   (*a*) A.M. and median
   (*b*) A.M. and S.D.
   (*c*) Averages of the first kind
   (*d*) Averages of the second kind.

4. Averages cannot be calculated for
   (*a*) Quantitative data
   (*b*) Qualitative data
   (*c*) Any grouped frequency distribution
   (*d*) Any frequency distribution.

5. The sum of all the values of a variable is 350. The A.M. is 7. The total number of data items is
   (*a*) 350×7 (*b*) $350 \div 7$. (*c*) $350 - 7$ (*d*) $350 + 7$.

6. The A.M. of ten observations is 5. One observation is replaced by 16. The new data have an A.M. equal to 6. The observation replaced by 16 was :
   (*a*) 1 (*b*) 6 (*c*) 10 (*d*) 16.

7. Each observation in a data has its value increased by 10 :
   (*a*) The A.M. changes but not the S D.
   (*b*) The S.D. changes but not the A.M.
   (*c*) Neither of A.M and S.D. changes
   (*d*) Both of A.M. and S.D. change.

8. Which of the following cannot be negative ?
   (*a*) A.M. (*b*) Median (*c*) Mean deviation (*d*) C.V.

9. For the data items 2, 3, 5, 10 and 20, the value 5 is the
   (*a*) A.M. (*b*) S.D. (*c*) Median (*d*) Mean deviation.

10. For the data 10^7, 2^{102}, 31, 4^{16}, which of the following must be a data item ?
    (*a*) A.M. (*b*) Median (*c*) S.D. (*d*) Mean deviation.

REVIEW EXERCISE XVII

1. Comment :

(a) **Annual time-schedule of a student**

No. of days	*Spent as*
122	Sleeping (8 h per day)
52	Sunday holidays
60	Summer vacations
10	Dussehra break
7	X-mass holidays
20	Other holidays
40	Eating/bathing/dressing up etc.
20	Examinations
331	Total

Hence the student goes to school on $365-331=34$ days only in a year.

(b) More people were killed in rail accidents in the year 1979 than in 1929. Hence railway travel was safer in 1929 than in 1979.

(c) Only 1% of people using brand A of tooth-paste have gum-troubles ; hence brand A is good for gums.

(d) Over a one year period, it was observed that 98% of the cars which got involved in road accidents, were being driven by men and only 2% by women. Hence women are better car drivers than men.

(e) A statistician was to cross a stream with his family including two young children. The statistician tells his wife that since the average depth of the stream is less than the average height of their family, therefore, it is safe for each member of the family to cross the stream on foot.

2. A die is thrown 120 times and the results tabulated below. Find the mean and median scores :

Score :	1	2	3	4	5	6
Frequency :	22	19	16	21	18	24

3. In a quality control test, the time to the nearest minute for which 100 candles of a certain type burned is given below :

Time (in minutes) :	460—464	465—469	470—474	475—479
No. of candles :	1	6	14	25
Time (in minutes) :	480—484	485—489	490—494	495—499
No. of candles :	28	18	6	2

Find the mean and median to the nearest minute.

4. The masses in grammes of 10 packets of sugar are found to be 1002, 1001, 1003, 999, 1004, 1002, 1001, 1000, 1001, 1003.

   Find the mean mass per packet and the standard deviation.

5. A book is checked for printing mistakes and the mistakes found per page are recorded below :

No. of mistakes :	0	1	2	3	4	5	6
No. of pages :	23	36	17	10	3	2	1

   Find the mean per page and the standard deviation of the number of mistakes.

6. Calculate the relative frequencies of the frequency distribution in Exercise 5 above correct upto two decimal places and express them as percentages. Is the sum of these 100 ? If not, why not ?

7. Draw a bar diagram for the frequency distribution of Exercise 3 above.

8. Make a pie chart for the data of Exercise 5 above.

9. Express the data of Exercise 2 above in the form of a grouped frequency distribution of class-width 2 each. Calculate the mean and median of this new distribution. Do these values represent the data better or the values computed earlier ?

10. Write a short note on the utility and limitations of graphical representation of data.

11. Find the mean, median and mode from the following table :

Class interval	*Frequency*
0— 7	19
7—14	25
14—21	31
21—28	72
28—35	51
35—42	43
42—49	28

(*IIT, JEE 1978*)

12. Calculate the mean, median and variance of the following data :

Height (in cm)	*Number of Children*
95—105	19
105—115	23
115—125	36
125—135	70
135—145	52

(*AISSE 1981 C*)

13. Calculate the mean, median and variance from the following data :

Height (in cm)	No. of students	Height (in cm)	No. of students
136—140	3	156—160	27
140—144	8	160—164	20
144—148	15	164—168	9
148—152	19	168—172	4
152—156	35		

(*AISSE 1982*)

14. Calculate the mean, median and standard deviation for the following data :

Height (in cm)	No. of boys	Height (in cm)	No. of boys
135—140	4	155—160	24
140—145	9	160—165	10
145—150	18	165—170	5
150—155	28	170—175	2

(*AISSE 1982*)

15. Find the mean, median and standard deviation for the following data ;

Class Interval	Frequency
10—20	3
20—30	1
30—40	1
40—50	8
50—60	17
60—70	38
70—80	9
80—90	3

(*AISSE 1985 C*)

16. Calculate the mean, median and standard deviation for the following data :

Weekly wages (in Rs)	Number of workers
20— 40	8
40— 60	12
60— 80	20
80—100	30
100—120	40
120—140	35
140—160	18
160—180	7
180—200	5

(*AISSE 1983 C*)

17. Calculate the mean, median and standard deviation of the following distribution :

Age in years	*Number of Marks*
20—30	3
30—40	61
40—50	132
50—60	153
60—70	140
70—80	51
80—90	2

(*DBSSE 1984*)

18. Calculate the mean, median and standard deviation of the following distribution :

Class Interval	*Frequency*	*Class Interval*	*Frequency*
20—40	6	120—140	15
40—60	9	140—160	16
60—80	11	160—180	8
80—100	14	180—200	7
100—120			

(*DBSSE 1982*)

19. On testing similar rods in the laboratory, breaking loads are found to be as follows. Find the mean and S.D. of the loads. Class interval of 2·5 kg may be taken.

66·4	67·8	78·0	74·8	65·2	56·4	60·2	65·6
69·0	61·2	66·6	67·4	66·2	72·8	70·2	69·0
63·2	72·4	67·8	70·0	72·2	69·0	66·4	70·6
64·0	70·8	73·0	72·2	68·6	69·2	69·0	79·2
62·6	59·8	70·2	60·4	71·0	74·0	74·4	69·8

(*Roorkee Entrance 1982*)

20. In a study to test the effectiveness of a new variety of seeds, an expriment was performed with 50 experimental fields and the following results of yield per hectare (in quintals) were obtained :

Yield	*No. of Fields*	*Yields*	*No. of Fields*
31—35	2	51—55	16
36—40	3	56—60	5
41—45	8	61—65	2
46—50	12	66—70	2

Find the Mean Deviation from the Mean and Standard Deviation. (*DBSSE 1981*)

SUMMARY

1. **Quantitative Variable :** A variable which can be measured and takes numerical values.
2. **Qualitative Variable :** A variable which cannot be measured.
3. **Frequency distribution :** Collection of various discrete values of a variable together with the corresponding frequencies.
4. **Frequency table :** Tabular from of a frequency distribution.
5. **Grouped frequency distribution :** Frequency distribution of a quantitative variable when frequencies are assigned to various classes of possible values of the variable.
6. **Relative frequency :** Frequency divided by the total frequency.
7. **Measures of location/Measures of central tendency/Averages of the first kind :** Item of the data around which other items have a tendency to cluster.
8. **Arithmetic Mean :** $\text{A.M.}=\bar{x}=\sum_i x_i/n,$

$$=\sum_i f_i x_i/\sum_i f_i,$$

$$=\alpha+\sum_i f_i(x_1-\alpha)/\sum_i f_i,\ \alpha \text{ arbitrary constant},$$

$$=\alpha+h\sum_i f_i\left(\frac{x_i-\alpha}{h}\right)\Big/\sum_i f_i,\ h \text{ class-length}.$$

9. **Median :** Middle observation, when observations have been arranged in descending or ascending order. For a grouped frequency distribution,

$$\text{Median}=L_1=\frac{(N/2)-C}{f_{median}}\times h.$$

10. **Measures of dispersion/Averages of the second kind :** Measures which specify the variability or spread or scatteredness of data.
11. **Mean deviation or average deviation about the median M :**

$$\sum_i f_i\,|\,x_i-M\,|\,/\sum_i f_i.$$

12. **Standard deviation :** $\sigma=\text{S.D.}=\left(\sum_i(x_i-\bar{x})^2/n\right)^{\frac{1}{2}}$

$$=\{\sum_i f_i(x_i-\bar{x})^2/\sum f_i\}^{\frac{1}{2}},$$

$$=\left[\frac{1}{\sum_i f_i}\sum f_i(x_i-a)^2-\left(\frac{1}{\sum f_i}\sum f_i(x_i-a)\right)^2\right]^{\frac{1}{2}},$$

a arbitrar :

$$=\left[\frac{1}{\sum_i f_i}\sum_i f_i x_i^2-\bar{x}^2\right]^{\frac{1}{2}}.$$

13. **Variance :** Square of the standard deviation.

14. **Coefficient of variation :** 100 × S.D./Mean.

HISTORICAL NOTE AND CURRENT STATUS

Once a king asked his wise minister—"What is fear and what is surprise ?" —The minister answered that fear is nothing but a form of ignorance and surprise is nothing but a form of non-understanding. Whether or not there is truth in what the minister said, you will soon verify the validity of at least the latter half of his statement in case of Statistics. Do you know that Statistics was called Political Science in the nineteenth century ? Today the terms Statistics and Political Science characterize altogether different concepts but the 1840 edition of Penny Cyclopedia defines Statistics as that department of Political Science which is concerned in collecting and arranging facts illustrative of the condition and resources of a State. This definition of Statistics would not seem to be surprizing if we appreciate the fact that around that time the role of the subject was limited to passive calculation and compilation of tables of a government's empirical data, and that it is a rather recent matter that Statistics has permeated through other spheres of human activity. In fact, Statistics has become a way of life today.

Weather reports are all based on Statistics. Information about wages, prices, stocks etc., are all based on Statistics. Businessmen and industrialists study such information carefully and decide what to buy and sell, and what to manufacture and store etc. Educationists use statistics to judge the effectiveness of various teaching devices. Doctors studying medicine use statistics to determine the effectiveness of new drugs. Agriculturists use statistics in studying the effects of various fertilizers on crops. Governments use statistics to know the amount of taxes to be expected etc. Economists use statistics to determine the optimal costs and productions. Surveys are used to determine the trend of voters or fashions even. There is hardly a circle of life to which the line of statistics is neither a secant nor a tangent. In other words, statistics pervades every sphere of our life.

□ □

JOHN VON NEUMANN (1903-1957)

John Von Neumann, one of the greatest mathematicians of the twentieth century, was born in Budapest in 1903. From his early youth he exhibited remarkable talent in mathematics, physics, chemistry and engineering. After obtaining a degree in Chemical Engineering in 1923, he spent the early part of his career in Germany. In 1933 he was appointed professor of mathematics at the Institute for Advanced Study in Princeton, U.S.A.

Von Neumann was one of the founders of the computer age. He played a central role in the design of some of the first U.S. electronic computers and in the development of programming techniques.

Linear Programming

18·1. LINEAR INEQUATIONS

Recall that an equality between two algebraic expressions (including at least one variable), if satisfied by some values of the variable but not all, is said to be an equation. We get an inequation, if we have an *inequality* ($<$, $\leqslant$, $>$ or $\geqslant$) instead of the equality. Thus $2x < 3$, $5x-7 \geqslant 0$, $6x+y-5 \leqslant 9$ are all inequations. An inequation is said to be **linear**, if the various terms containing the variables are of degree one.

A linear inequation in the single variable x can always be written as

$$x < a, x \leqslant a, x > a \text{ or } x \geqslant a,$$

a being a real number. The **solution-set of an inequation** in x is the set of those values of x which make it true. Recall that unlike a linear equation, the solution-set of a linear inequation is always an infinite set. We can graph the solution-set on the number line easily.

Illustrations. The bold portions show the solution-sets of the inequations on the left. An empty circle at the end shows that the point is not included in the solution-set.

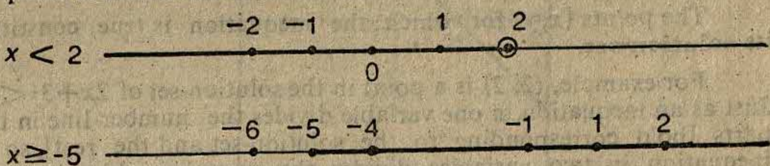


Fig. 18·1.

The following operations on inequations produce equivalent inequations :

1. Addition (or subtraction) of a real number to both sides of an inequation.
2. Multiplication (or division) by a positive real number.
3. Multiplication (or division) by a negative real number with the inequality sign reversed.

4. Collecting and combining like terms on either side.
5. Removing/inserting brackets on either side according to rules obeyed by these operations.

Illustrations

(*a*) $x<2$ and $x+3<5$ are equivalent inequations, *i.e.*, $x<2 \Leftrightarrow x+3<5$.

(*b*) $x \geqslant -5 \Leftrightarrow x-2 \geqslant -7$.

(*c*) $3x>9 \Leftrightarrow x>3$.

(*d*) $x<6 \Leftrightarrow 5x<30$.

(*e*) $x>-2 \Leftrightarrow -2x<4$.

(*f*) $x+(2-3x) \leqslant 5 \Leftrightarrow -2x+2 \leqslant 5$.

Example 1. *Solve the inequation* $3x+5<x-9$.

Solution. $3x+5<x-9$,

$\Leftrightarrow 3x+5-5<x-9-5$,

$\Leftrightarrow 3x<x-14$,

$\Leftrightarrow 3x-x<x-14-x$,

$\Leftrightarrow 2x<-14$,

$\Leftrightarrow x<-7$.

Thus the solution-set is the set of all real numbers less than -7.

18·2. LINEAR INEQUATIONS IN TWO VARIABLES

Typical linear inequations in two variables x and y are of the form

$$ax+by \leqslant c,\ ax+by<c,\ ax+by>c \text{ or } ax+by \geqslant c \text{ etc.}$$

The points (x, y) for which the inequation is true, constitute its **solution-set**.

For example, $(2, 2)$ is a point in the solution-set of $2x+3y<42$. Just as an inequation in one variable divides the number-line in two parts [bold corresponding to the solution-set and the rest], so an inequation in two variables divides the cartesian plane into two portions ; points corresponding to the solution-set, and the rest.

18·2·1. Graphing the solution-set of a Linear Inequation in Two Variables

Observe that if we replace the inequality sign in an inequation by the equality sign, we shall get the equation of a straight line. Just as in inequation $x<a$, a is the dividing point which separates its solution-set from the rest of the line, so the line mentioned above in case of an inequation in two variables separates the part of the plane which corresponds to the solution-set from the rest of the plane. Therefore, to graph an inequation $ax+by \leqslant c$ etc., we adopt the following procedure :

Step 1. Replace the inequality sign by equality and plot the resulting line. In case of the strict inequalities $<$ and $>$, draw the line dotted. In case of $\leqslant$ and $\geqslant$, draw the line thick. (This divides the plane in two portions.)

Step 2. Choose a point [if possible (0, 0)] not lying on the dividing line. Substitute its co-ordinates in the inequation. If the resulting inequality is correct, shade the portion of the plane which contains the chosen point. In the other event, shade the portion which does not contain the chosen point.

The shaded portion represents the solution-set. The dotted line is not a part of the shaded region ; the thick line is !

Example 2. *Graph the (solution-set of) inequation*

$$x+y-1\geqslant 0.$$

Solution. *Step 1.* We first plot the dividing line $x+y-1=0$, which has been obtained from the inequation by replacing '$\geqslant$' by '$=$'. Now

$$x+y-1=0 \Leftrightarrow \frac{x}{1}+\frac{y}{1}=1,$$

showing that this line cuts off intercepts 1 and 1 on the axes, or that it meets the axes in the points, (1, 0) and (0, 1). We plot the line through these points. Since the inequality is weak, we draw the line thick.

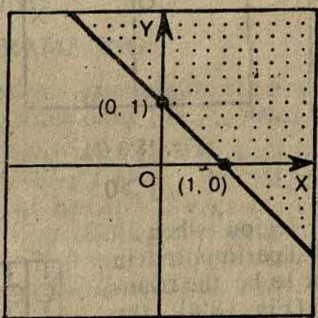


Fig. 18·2.

Step 2. Choose (0, 0) as the test-point. Putting $x=0=y$ in the given inequation, we get

$$0+0-1\geqslant 0,$$

which is *false*. Hence we shade the portion which *does not* contain the origin.

Every point in the shaded part and every point on the border line is in the solution-set and makes the inequality a true statement.

18·3. SYSTEM OF LINEAR INEQUATIONS IN TWO VARIABLES (SIMULTANEOUS INEQUATIONS)

A system of linear inequations or simultaneous inequations can be treated in a manner analogous to that in which you treat a system of linear equations or simultaneous equations. The **solution set of a system of linear inequations** in two variables is the set of all points (x, y) which satisfy all the inequations in the system simultaneously. This amounts to identifying that region of the plane which is included in all the portions corresponding to the solution-sets of the various inequations. Oh yes ! there might not be any point common to all the regions. For example, consider $x+y>5$ and $x+y<4$. There cannot be any point (x, y) for which $x+y$ is simultaneously greater than 5 and less than 4. In such cases, the solution-set is nothing but the empty set.

Example 3. *Find the solution of the simultaneous inequations* $x \geqslant 0$, $y \geqslant 0$, *and* $x+y \leqslant 2$.

Solution. Fig. 18·3 (*a*), 18·3 (*b*) and 18·3 (*c*) show the solution-sets (shaded regions) of the given inequations.

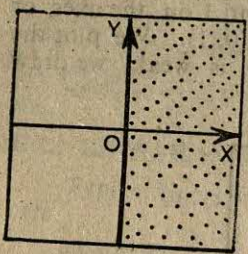


Fig. 18·3 (*a*).

$x \geqslant 0$

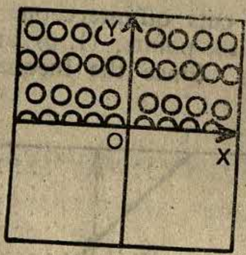


Fig. 18·3 (*b*).

$y \geqslant 0$

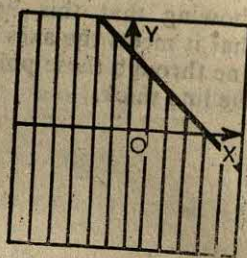


Fig. 18·3 (*c*).

$x+y \leqslant 2$

The common region when all three graphs are superimposed in one graph turns out to be the triangular region OBC (Fig. 18·4), the boundary lines being part of the region.

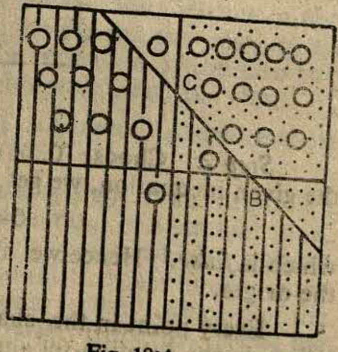


Fig. 18·4.

EXERCISE 18 (*a*)

1. Graph the solution-sets of the following inequations :

   (*a*) $x+y \geqslant 3$. (*b*) $x-y \leqslant 2$.

   (*c*) $2x+y>1$. (*d*) $2x-3y<4$.

   (*e*) $x \geqslant y-5$. (*f*) $y-3 \leqslant 2x$.

2. Solve the following pairs of simultaneous inequations graphically :

   (*a*) $x \geqslant 5$ and $x \leqslant -10$.

   (*b*) $x+y \geqslant 3$ and $2x+y-5 \leqslant 4$.

   (*c*) $2x+y>1$ and $y-3 \leqslant 2x$.

   (*d*) $x \geqslant y-5$ and $2x-3y<4$.

   (*e*) $y \geqslant -1$ and $y \leqslant -1$.

3. Solve the following systems of linear inequations :

   (*a*) $x+y \geqslant 1$, $x \geqslant 0$ and $y \leqslant 0$.

   (*b*) $x \geqslant 0$, $y \geqslant 2$ and $2x-y+3 \leqslant 5$.

   (*c*) $x \geqslant 0$, $y \geqslant 1$ and $x+y+3 \leqslant 4$.

4. Construct a system of linear inequations (of your own !) in x and y whose solution-set is

   (*a*) the empty set.

   (*b*) An open region.

   (*c*) A bounded region.

18·4. LINEAR PROGRAMMING

Let us now consider an example to see how inequations help us in finding solutions to some daily life problems.

Example 4. *A farmer has Rs 300 available for purchase of hens and he has a place to keep 8 hens. An old hen costs Rs 30 and a young one Rs 60. An old hen lays 3 eggs a week and a young one 5 per week. Assuming it costs Re 1 per week to feed a hen (ald or young) and an egg sells for 75 paise, find, how many hens of each type should he buy so as to maximize his profit per week.*

Solution. Let us suppose that for maximum profit, the farmer should purchase x old hens and y young hens. These hens would cost him (in rupees) $30x+60y$. Since he only has Rs 300, x and y must be such that

$$30x+60y \leqslant 300,$$

$$\Leftrightarrow \qquad x+2y \leqslant 10. \qquad \ldots(1)$$

Since, he can keep at the most 8 hens, therefore, we must also have

$$x+y \leqslant 8 \qquad \ldots(2)$$

Now for profit. The total number of eggs he would get per week is

$$3x+5y.$$

Each egg sells for Re 0·75. Therefore, by selling $3x+5y$ eggs he would get rupees

$$0{\cdot}75\,(3x+5y).$$

But he spends Rs $(x+y)$ in feeding the hens. So his profit in rupees really is

$$0{\cdot}75\,(3x+5y)-(x+y),$$
$$=1{\cdot}25x+2{\cdot}75y.$$

Also, he cannot buy a negative number of hens, so that

$$x\geqslant 0,\ y\geqslant 0. \qquad \ldots(3)$$

Keeping in view the *limitations* or *constraints* (1), (2) and (3), let us see what are the various possibilities for the values of x and y. For this we graph the solution-set of the system of inequations (1), (2) and (3) because *all* the three constraints have to be satisfied. This is done in Fig. 18.5 below :

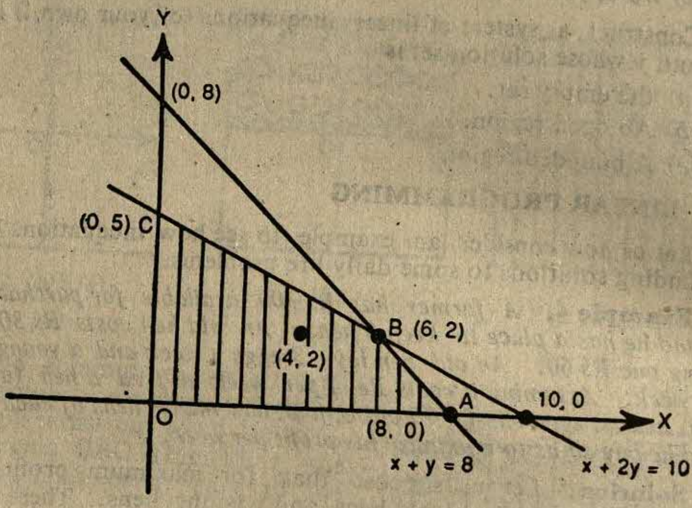


Fig. 18·5.

The shaded region represents those pairs (x, y) which satisfy all the three constraints. Since x, y must be integers (you cannot, for instance, buy half a hen and get eggs !) all possible pairs (x, y) in this region with x and y integers give a *possible* or a *feasible* number of hens to be purchased. For example, since (4, 2) lies in this region, he can buy 4 old hens and 2 young hens (x represents old hens etc.) and satisfy the constraints (1), (2) and (3). The question

is, which of these *feasible* integer pairs would maximize his profit. Note that x and y are non-negative integers and $x+y \leqslant 8$. Hence $x \in \{0, 1, 2, \ldots, 8\}$. Further, since a young hen (y) costs Rs 60 and he has Rs 300 only, $y \in \{0, 1, 2, 3, 4, 5\}$. This is clear from Fig. 18 5 also. With these values of x and y we can form 54 distinct pairs (x, y). If we use the constraint $x+y \leqslant 8$, the number of these pairs comes down to 39. A further consideration $30x+60y \leqslant 300$ reduces the number of these pairs to 25. Each of these 25 gives us a combination obeying all the constraints. For each of these pairs we can compute the profit Rs $(1{\cdot}25\,x+2{\cdot}75\,y)$ and pick up the pair or pairs which give the highest value. Verify that the pair (0, 5) gives a maximum profit, Rs 13·75 per week.

Are you by any chance thinking that since problems such as above involve a lot of computational work, it would not be worthwhile studying them? If yes, we must first disappoint you. Such problems occur very often in real life situations and involve lot more computations than in the case above. Even computers fail to cope. This makes two things clear. Firstly, such problems will have to be solved. Secondly, methods will have to be found which reduce the labour involved in solving such problems by direct verification as we have done above. We shall first learn where such problems arise and then see how mathematical reasoning can be used to find the best or near-best solutions to such problems without tears. By the way, this latter is what has made Indians like Narendra Karmarkar famous in the world.

18·4·1. Linear Programming Problems

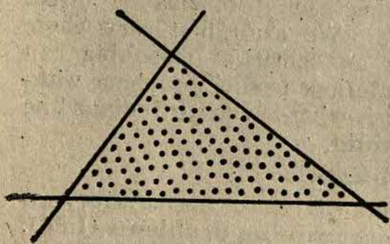
Problems like the one just solved, fall into the category of problems known as the **linear programming problems** (LPP). *Linear*, because the involved inequations and the function to be maximized (or minimized) are linear ; and *programming* because you have to *plan* or *programme* your scarce resources in order to maximize profit (or minimize cost etc.)

Such problems occur frequently in business, industry, commerce, government ; as a matter of fact, in any situation where we wish to make the *best* or *optimal* use of our limited resources, be these in the form of labour, materials or money. The limitations on the resources can often be expressed in the form of linear inequations and are known as linear **constraints**. Subject to these constraints, we wish to maximize something (*e.g.*, profit as in the above example) or minimize something (*e.g.*, cost). This something is known as the **objective function.** The objective function of an LPP is a linear function of the involved variables. The maximum or the minimum value of the objective function is known as the **optimal** value. A set of values of the variables which satisfies all the constraints is known as a **feasible solution.** A feasible solution which leads to an optimal value of the objective function is known as an **optimal solution.** The processes or techniques of obtaining the optimal values are called **optimization techniques.**

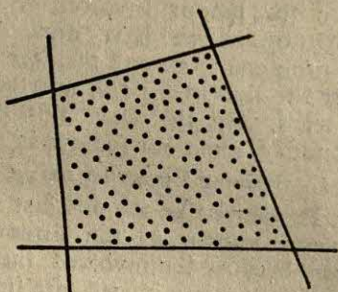
18·4·2. Graphical Solution of an LPP

Having convinced ourselves about the importance of linear programming problems, let us see how to solve them without too many computations. We shall restrict ourselves to the case of LPP in two variables. Generally, these variables would be non-negative because the quantities represented by the same may be something like material used, money required or time spent etc. This is going to give us two linear constraints of the type $x \geqslant 0$ and $y \geqslant 0$. Naturally, there would be other constraints for reason of limited resources. Thus, we may assume that there are at least there linear constraints. The term *constraint* henceforth would denote a linear constraint unless stated otherwise.

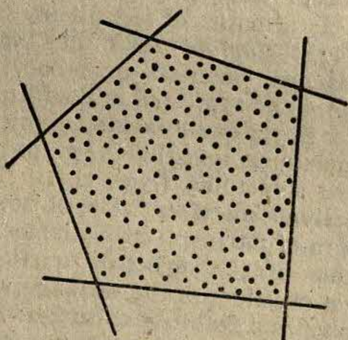
Each constraint (being an inequation/equation) gives rise to a line in the plane. As we have seen before, for simultaneous satisfaction of all the constraints, we consider the region common to the (graphs of the) solution-sets of all the constraints. Now there are several possibilities for this region. We shall concentrate on the case when the solution-set of the system of inequations involved is of the type shown below :



(3 CONSTRAINTS)



(4 CONSTRAINTS)



(5 CONSTRAINTS)

Fig. 18·6.

Now what is the characteristic of the above regions ? First of all they are closed regions bounded by straight lines. Secondly, the points of intersection of any pair of these lines are distinct. In other words, no point in the plane lies on more than two of these lines. Such a closed region as you know, is called a *polygon.* Thirdly, these regions have another property : if you take any two points in any one of these regions, then all the points lying on the line segment joining the points also lie in the same region. Because of this property, the above regions are known as *convex.* (Is a circular region convex ?) A non-convex polygon is shown in the following diagram. Clearly, *not all* the points of the line segment joining P and Q lie within the polygon.

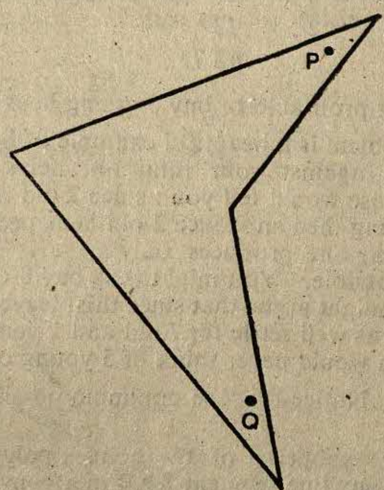


Fig. 18·7.

Convex regions have very nice properties. One of their beautiful properties is the following :

The maximum and minimum values of a linear function over a convex polygon occur at some vertex (corner) or the other.*

Let us try to understand this property by means of Example 4. Notice that it is an LPP. There are four constraints *viz.*,

$$x \geqslant 0,\ y \geqslant 0,\ x+y \leqslant 8 \text{ and } x+2y \leqslant 10.$$

The objective function, call it P, is

$$P = 1{\cdot}25\,x + 2{\cdot}75\,y.$$

The set of points (x, y) in the region OABC with both x and y integers, constitutes the set of feasible solutions. Now, for each

*It is known as the Fundamental Theorem of Linear Programming.

point (x, y) of the region OABC, P has a certain value. The above property tells us that the minimum and maximum values of P occur at some or other of the points O, A, B and C. What a boon ! Using this property, we can immediately say that the maximum value of P can be found by calculating its value at the points O, A, B and C. There is no need to worry about all those 54 feasible solutions and and so on. Now A is the point of intersection of the line $x+y=8$ with the x-axis. It is, therefore, (8, 0). Similarly, B and C are seen to be (6, 2) and (0, 5) respectively. The values of P for the various points O, A, B and C are shown in the following table :

Point	x	y	$P\ (=1{\cdot}25x+2{\cdot}75y)$
O	0	0	0
A	8	0	10
B	6	2	13
C	0	5	13·75

Thus it is most profitable to buy 5 young hens alone.

The above problem is a beautiful example of how mathematical reasoning, even against your intuition, helps you get better results. Common sense would tell you : since 2 old hens cost only as much as a young hen and since 2 old hens produce 6 eggs per week whereas a young one produces only 5, it is best to buy as many old ones as possible. You might thus buy 8 old hens. If you think better, you might argue that since this leaves you Rs 60 uninvested, you might as well settle for 7 old and 1 young or 6 old and 2 young ones. You would never think of 5 young ones.

Remarks. 1. Notice that the minimum possible profit occurs at the vertex O.

2. The above property of the convex polygons is based on the fact "if we take any line segment AXB in a convex region, then denoting the value of a linear function P at a point T by P(T), we must have $P(A) \geqslant P(X) \geqslant P(B)$ or else $P(A) \leqslant P(X) \leqslant P(B)$."

This means that if A and B are adjacent vertices of a convex polygon and $P(A)=P(B)$, then for every point X of the edge (side) AB of the polygon, the value of P(X) is the same as that of P(A). If P is the profit function for an LPP, then all points of the line segment AB produce the *same* profit. Such a line is then called an **iso-profit** line. Also note that in the jargon of LPP, the term *profit* is used for any objective function which is supposed to be maximized.

3. To solve an LPP, the following steps need be taken :

Step I : Determine the convex polygonal region by making use of the various constraints.

Step II : Find the vertices of this polygon.

Step III : Find the value of the objective function at all the vertices.

Step IV : Examine which vertex gives the optimal value.

4. There is another convenient graphical method to find the optimal solution of an LPP. Consider Example 4 again. We wish to find that value of P for which $P=1{\cdot}25x+2{\cdot}75y$ is maximum. Now various feasible solutions (x, y) produce various values of P. But for different values of P, $1{\cdot}25x+2{\cdot}75y=P$ produces a set of (iso-profit) parallel lines. Since the feasible solutions (x, y) all lie in the polygon OABC, we graph the line $1{\cdot}25x+2{\cdot}75y=P$ for various values of P so that at least one point of the polygon OABC is on the line. The dotted lines in Fig. 18·8 show the line $1{\cdot}25x+2{\cdot}75y=P$ for different values of P. Notice that by writing this equation in the form

$$\frac{1{\cdot}25}{\alpha}x+\frac{2{\cdot}75}{\alpha}y=\frac{P}{\alpha},$$

where $\quad \alpha=[(1{\cdot}25)^2+(2{\cdot}75)^2]^{1/2},$

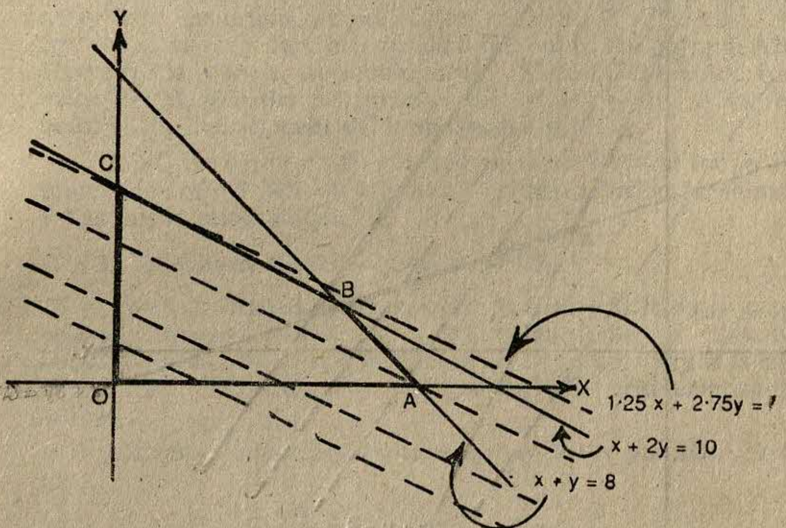


Fig. 18·.8

we see that α is a constant and P/α is the distance of the origin from the line $1{\cdot}25x+2{\cdot}75y=P$. Hence for maximum value of P, we want the line to be as far away from the origin as possible, yet having at least one point of the polygon OABC on it. This happens when the line passes through C. But if C (0, 5) lies on it, then

$$(1{\cdot}25)0+(2{\cdot}75)5=P,$$

$$\Leftrightarrow \qquad 13{\cdot}75=P.$$

Thus we obtain the desired value Rs 13·75 again.

Example 5. *Maximize $14x+10y$ subject to the constraints*

$$x \geqslant 0,\ y \geqslant 0,$$
$$x+3y \geqslant 6,$$
$$2x+y \geqslant 4,$$
and
$$7x+5y \leqslant 35.$$

Solution.

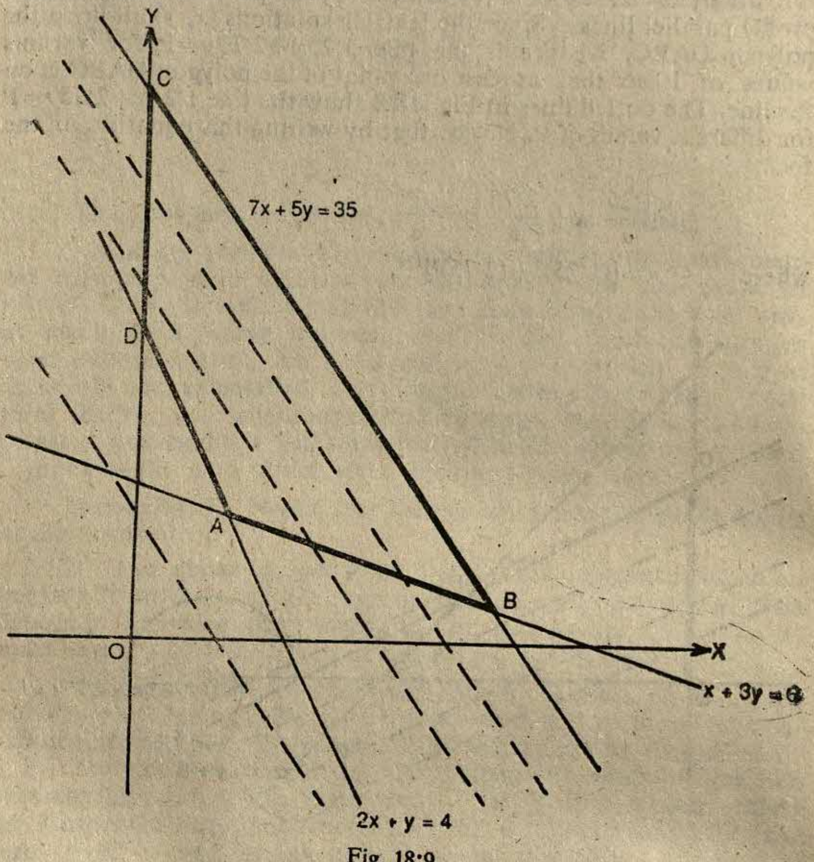


Fig. 18·9.

Solution.

First Method. We graph the lines $x=0$, $y=0$, $x+3y=6$, $2x+y=4$ and $7x+5y=35$. The region bounded by these lines is the quadrilateral region ABCD and is the solution-set of the given inequations. Every point on this quadrilateral and in its interior, is a feasible solution of the given problems. The optimal solution

is to be found among the vertices A, B, C and D which happen to be (6/5, 8/5), (75/16, 7/16), (0, 7) and (0, 4) respectively. We tabulate the value of the profit function (objective function) for these points below :

Point	x	y	$P=14x+10y$
A	6/5	8/5	164/5 (=32·8)
B	75/16	7/16	70
C	0	7	70
D	0	4	40

Thus both of B(75/16, 7/16) and C(0, 7) provide optimal solutions giving a maximum value 70. Since B and C are adjacent vertices, we know that every point on the line segment BC also gives a maximum value of $14x+10y$. Verify this by taking some points on BC. Thus you have as many optimal solutions as you want.

Second Method. We plot the line $14x+10y=P$ for various values of P, shown dotted in Fig. 18·9. Note that BC is also parallel to $14x+10y=P$. Hence as we move $14x+10y=P$ farther and farther away from the origin, we get BC (for the value 70 of P) also. If we try to move any farther, we shall fall out of the polygon ABCD. Hence BC is as far as we should move. Since BC is realized for the value P=70, 70 is the maximum value of $14x+10y$. Also, every point of the line segment BC is an optimal solution.

Is BC the iso-profit line for this problem ? What is the minimum value of P ? What would be an optimal solution for minimum P ? Is this solution unique ?

18·4·3. Mathematical Formation of LPP

You know that a problem where you have to find non-negative values of two variables x and y which maximize or minimize a linear function of x and y under some linear constraints is known as the LPP. We can state the general LPP mathematically as follows :

Maximize (minimize) $P=a_0x+b_0y$ *(objective function), subject to*

$$\left.\begin{array}{l} a_1x+b_1y\leqslant c_1, \\ a_2x+b_2y\leqslant c_2, \\ \vdots \quad \vdots \quad \vdots \\ a_nx+b_ny\leqslant c_n, \end{array}\right| \quad \textit{(Linear Constraints)}$$

$x\geqslant 0,\ y\geqslant 0$ *(non-negativity conditions).*

A feasible solution of the above LPP is a pair (x, y) of non-negative real numbers such that all the constraints are satisfied. A feasible solution which gives the maximum (minimum) value of P is called an optimal solution.

Remarks. 1. Since $ax+by \geqslant c \Leftrightarrow -ax-by \leqslant -c$, we can express our constraints in terms of '$\leqslant$' alone if we choose. It is not necessary to do so however.

2. Sometimes a constraint may be in the form of an equality also.

3. An optimal solution need not be unique. In fact, as we have seen, there might be infinitely many optimal solutions.

Example 6. *A regional computer firm assembles two types of computers C_1 and C_2. The market survey shows that no more than 50 and 255 computers of these types respectively can be sold in the region. As per the contract with the parent firm, it must produce at least 5 computers of each type. The assembly of these computers requires three processes P_1, P_2, P_3 to be performed. The time required for processing, the available time for each process and profit per computer is shown in the table below. We want to determine how many computers of each type should be produced.*

Computer	*Production time per Computer (in h)*			*Profit per computer (in thousand Rs)*
	P_1	P_2	P_3	
C_1	15	35	2	30
C_2	12	30	3	15
Available time (in h)	1154	9100	900	

Formulate this information as an LPP.

Solution. Appears tough ! It is not so tough ! We shall take each bit of information and translate it into the form of an inequation. Our *first problem is to determine the variables* involved. Since the firm has the option to determine the number of computers of each type, these numbers are the variables. Since there are two types, there have to be two variables. Let us call them x_1 and x_2. Thus let us suppose that the firm produces x_i computers of type C_i, $i=1, 2$. Since a negative number of computers cannot be assembled,

$$x_1 \geqslant 0, \; x_2 \geqslant 0. \qquad \text{...(1)}$$

Our second problem is to determine the objective function P.

Now what would the firm wish to maximize or minimize ? Clearly, profits are to be maximized and cost to be minimized. Since nothing is said about the expenditure in producing the computers of various types, we shall maximize the profit. Now we find from the table that x_1 computers of type C_1 earn a total profit $30x_1$ (in thousand rupees). Similarly for the other type. The total profit P is given by

$$P=30x_1+15x_2. \qquad \text{...(2)}$$

Now start reading the problem. The first information is given about the sale potential Since the firm would not like to assemble the computers it cannot sell, therefore, we must have

$$x_1 \leqslant 50,\ x_2 \leqslant 225. \qquad \text{...(3)}$$

Next comes the contract with the parent firm. This requires

$$x_1 \geqslant 5,\ x_2 \geqslant 5. \qquad \text{...(4)}$$

Now we come to the processes P_1, P_2, P_3. We note that one computer of type C_1 requires 15 h in process P_1. So x_1 computers of this type would require 15 x_1 h in this process. The total time (in h) required for this process by all the computers of both types is $15x_1+12x_2$. From the table we find that the total time available for this process is $1154h$. Hence we must have

$$15x_1+12x_2 \leqslant 1154. \qquad \text{...(5)}$$

Similarly, time limitation for the other processes P_2 and P_3 give the inequations

$$35x_1+30x_2 \leqslant 9100,$$

$$\Leftrightarrow \quad 7x_1+6x_2 \leqslant 1820, \qquad \text{...(6)}$$

and

$$2x_1+3x_2 \leqslant 900. \qquad \text{...(7)}$$

The whole information has now been put into mathematical relations (1) to (7). We can now organize our data as follows :

Maximize $P=30x_1+15x_2$	(Objective function)
subject to the constraints	
$x_1 \geqslant 0,\ x_2 \geqslant 0,$	(Non-Negativity conditions)
$\left.\begin{array}{r} x_1 \leqslant 50, \\ x_2 \leqslant 225, \\ x_1 \geqslant 5, \\ x_2 \geqslant 5, \\ 15x_1+12x_2 \leqslant 1154, \\ 7x_1+6x_2 \leqslant 1820, \\ 2x_1+3x_2 \leqslant 900. \end{array}\right\}$	(Constraints)

Remark. We have used x_1, x_2 here for the variables but you can use x, y if you like. However, there is no harm in getting used to x_1, x_2. Later on when you would come across several variables, $x_1, x_2, x_3, x_4, \ldots\ldots$etc., would be quite convenient.

EXERCISE 18 (*b*)

Solve each of the following LPP's :

1. Maximize $P=x+y$, subject to the constraints

$$2x+y \leqslant 40,\ 2x+5y \leqslant 180,\ x_1,\ x_2 \geqslant 0.$$

2. Minimize $C=100x+600y$ subject to

$$x+y \leqslant 10,\ x \geqslant 3,\ y \geqslant 2,\ x+3y \leqslant 16 \text{ and } x \geqslant 0,\ y \geqslant 0.$$

3. A patient has been advised that his daily diet must contain at least 1250 calories, 3500 units of vitamins and 60 units of

minerals. Of the two foods A and B available to him, A costs Rs 5 and B costs Rs 6 per unit. One unit of A contains 300 units of vitamins, 2 units of minerals and 60 calories. These three for the other food B are respectively 353, 1 and 67. Express these data as an LPP.

[**Hint.** What would the patient try to *minimise* ? The cost. Suppose he buys x units of A and y units of B etc.]

Express as an LPP and solve graphically :

4. Shyam has a workshop to produce nuts and bolts. To prepare a packet of nuts, one hour on machine A and three hours on machine B are required. The respective time periods for a packet of bolts are the other way round. He earns a profit of Rs 7·5 on a packet of nuts and a profit of Rs 3 on a packet of bolts. Each of his machines can be operated for at the most twelve hours per day. How many packets of nuts and bolts each should he produce per day to maximize his daily profit ? What is the maximum profit ?

5. A company produces two types of belts, A and B say. Profits on these types are Rs 2 and Rs 1·5 each belt respectively. A belt of type A requires twice as much time as a belt of type B. The company can produce at the most 1000 belts of type B per day. Material for 800 belts only per day is available. At the most 400 buckles for belts of type A and 700 for those of type B are available per day. How many belts of each type should the company produces so as to maximize profit.

[**Hint.** If x belts of type A and y of type B are produced, time constraint is $2x+y \leqslant 1000$. Be careful about the polygon which gives the feasible solutions.]

TEST YOUR UNDERSTANDING XVIII

In each of the following problems, four alternatives are given. Put a tick-mark (✓) against the correct alternative.

(*All inequations are linear and in two variables unless otherwise stated.*)

1. The inequation $x+2y \geqslant 3$ has
   (*a*) 1 solution.
   (*b*) 2 solutions.
   (*c*) finitely many but more than 3 solutions
   (*d*) infinitely many solutions.
2. A system of two inequations
   (*a*) may have a unique solution.
   (*b*) can never have a unique solution.
   (*c*) always has infinitely many solutions.
   (*d*) always has finitely many solutions.

3. The solution-set of a system of two inequations
   (*a*) may be the empty set.
   (*b*) may be a finite set.
   (*c*) may be represented by a bounded region in the plane.
   (*d*) is always represented by an open region in the plane.
4. The graph of the solution-set of a system of three inequations CANNOT be a
   (*a*) (single) point. (*b*) line.
   (*c*) triangular region. (*d*) rectangular region.
5. If A is a square and B is a hexagon, then
   (*a*) A is convex but B is not.
   (*b*) B is convex but A is not.
   (*c*) both of A and B are convex.
   (*d*) none of A and B is convex.
6. An LPP
   (*a*) always has an optimal solution.
   (*b*) always has a feasible solution.
   (*c*) may have an optimal solution but not a feasible solution.
   (*d*) may have a feasible solution which is not optimal.
7. Which of the following statements is not necessarily true in reference to an LPP ?
   (*a*) The objective function is linear.
   (*b*) The constraints are linear.
   (*c*) The variables are positive.
   (*d*) The optimal solution (if there is one) is obtained at a vertex of the convex polygon which constitutes the solution-set of the constraints.
8. If (x, y) is an optimal solution of an LPP, then
   (*a*) both of x and y may be zero.
   (*b*) none of x and y can be zero.
   (*c*) one of x and y must be zero.
   (*d*) none of the above.
9. A "maximize..." type LPP can be re-stated as a "minimize..." type LPP, but then
   (*a*) the objective function P will change, constraints remaining same.
   (*b*) P will not change, constraints will change.
   (*c*) both will change.
   (*d*) neither will change.

REVIEW EXERCISE XVIII

1. Graph the inequation $2x+3y \leqslant 7$.
2. Graph the solution-set of the inequation $x-2y > 5$.
3. Give an example of a system of inequations whose solution-set is empty.
4. Write the inequation which represents the dotted region in the adjoining diagram.

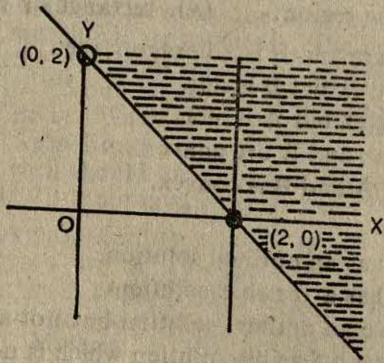


Fig. 18·9.

5. Write the system of inequations which represents the third quadrant, axes excluded.
6. Shade a triangular region in the plane. Write three inequations whose solution-set is this region.
7. Graph the solution-set of the system

$$3x+4y > 1,\ x \geqslant 0,\ y \leqslant 0.$$

8. Give an example of an LPP which has no feasible solution.
9. Draw a convex polygon bounded by five lines.
10. A chemist requires 10, 12 and 12 units of chemicals A, B and C respectively for an experiment. One product X contains 5, 2 and 1 units of A, B and C respectively per jar and costs Rs 3 per jar. Another product Y contains 1, 2 and 4 units respectively of A, B and C per jar and costs Rs 2 per jar. How many jar of each of X and Y should the chemist purchase in order to minimize cost subject to the above constraints ?

SUMMARY

1. **Linear Inequation in two Variables X and Y :** An expression of any of the forms $ax+by \geqslant C$, $ax+by > C$, $ax+by \leqslant C$, $ax+by < C$.
2. **Graph of an Inequation :** Collection of points satisfying the inequation.

3. **Linear Programming Problem (LPP) :** A problem where we maximize or minimize a linear function *w.r.* to linear constraints (inequations) and restrictions on the involved variables to be non-negative.

4. **Objective Function :** The function which is to be optimized (*i.e.*, maximized or minimized) in an LPP.

5. **Iso-profit Line :** A line at each point of which the objective function has the same value.

6. **Convex Set :** A region in the plane having the property that every point on the line joining any two of its points lies in the plane.

HISTORY AND CURRENT STATUS

A popular belief about mathematics is that it is an *exact* science. However, as you learn more and more of mathematics, you believe less and less in the above theory. Consider for example the fact that theoretically the sum of the relative frequencies must be one. However, we generally have to take approximations, and the resulting sum is hardly ever exactly one.

During the Second World War, something quite spectacular happened. So far, mathematics could be used in real life applications for drawing conclusions when more or less complete information was available. Thus mathematics was being applied in natural sciences, but it could not pervade the social sciences because here the basic atom is *human behaviour* which is so very uncertain and unpredictable. During the Second World War, the British Government was trying to find out ways and means to defend their country against the German bombing by making use of their radars. During the process, simple mathematical techniques were devised by a team of British and American mathematicians to deal with situations where the information was *inadequate* or *uncertain* or *both*. This broke new ground. Now it became possible to solve the linear programming problems. The main persons behind the theory of linear programming were John von Neumann, G. B. Dantzig, T. C. Koopmans and some other mathematicians, statisticians and economists. The first application of LP was made by Marshall Wood and his staff in the Air Forces Project *Scientific Computation of Optimum Programme*. Later, the technique became very popular among the *managers* anywhere and everywhere.

Today, as you must have already appreciated, the LP technique is being used in all fields of life. The *real* life applications of LP may involve a large number of variables and constraints. That means our graphical technique for solving an LPP would not work. The two variable LPP gives us a convex polygon in the 2-dimensional plane. The case of three variables would give us a convex polyhedron in the 3-dimensional space. What about the case of four or more variables ? The first and the most general technique (non-graphical of course !) to solve an LPP is the *simplex method*. This was developed by Dantzig. Several other techniques have since been developed but these apply to special types of LPP. The real life LPP is generally so complex that people are happy to have a solution which is *not the best but nearly best*, if it means a moderate cut in the time and expenditure involved. Karmarkar became famous because he gave a computer algorithm to obtain a near-best solution which drastically cuts down the expenditure and time involved in solving an LPP.

□□

Assorted Problems

1. A group of 123 workers went to a canteen for cold drinks, ice-cream and tea. 42 workers took ice-cream, 36 tea and 30 cold drinks, 15 works purchased ice-cream and tea, 10 ice-cream and cold drinks, and 4 cold drinks and tea but not ice-cream, 11 took ice-cream and tea but not cold drinks. Determine how many workers did not purchase any thing.

   (*Roorkee Entrance 1989*)

2. A class has 175 students. The following table shows the number of students studying one or more of the following subjects in this class :

Subject	*Number of students*
Mathematics	100
Physics	70
Chemistry	46
Mathematics and Physics	30
Mathematics and Chemistry	28
Physics and Chemistry	23
Mathematics, Physics and Chemistry	18

   How many students are enrolled in Mathematics alone, Physics alone and Chemistry alone ? Are there students who have not offered any of these three subjects ?

   (*Roorkee Entrance 1984*)

3. The following three relations are defined on the set of natural numbers **N**

   $R=\{(x, y) : x<y, x \in \mathbf{N}, y \in \mathbf{N}\}$
   $S=\{(x, y) : x+y=10, x \in \mathbf{N}, y \in \mathbf{N}\}$
   $T=\{(x, y) : x=y \text{ or } x-y=1, x \in \mathbf{N}, y \in \mathbf{N}\}$

   Explain clearly which of the above relations are (*i*) Reflexive (*ii*) Symmetric (*iii*) Transitive. (*Roorkee Entrance 1986*)

4. Let R be the relation defined on the set of natural numbers **N** as

   $R=\{(x, y) : x \in \mathbf{N}, y \in \mathbf{N}, 2x+y=41\}.$

   Find the domain and range of this relation R. Also, verify whether R is (*i*) reflexive (*ii*) symmetric (*iii*) transitive.

   (*Roorkee Entrance 1988*)

5. For each set of ordered pairs below, state whether it is a function or not giving reasons.

   (*i*) $\{(3, 2), (4, 2), (5, 2)\}$
   (*ii*) $\{(2, 3), (2, 4), (2, 5)\}$

   (*Roorkee Entrance 1987*)

6. Give the domain and range of the function $(x^2-1)^{-\frac{1}{2}}$.

(*Roorkee Entrance 1987*)

7. A mapping is defined as $f : \mathbf{R} \to \mathbf{R}$, $f(x) = \cos x$. Show that it is neither one-one nor surjective. (*Roorkee Entrance 1989*)

8. Find real values of x and y if

$$(x^4+2xi)-(3x^2+yi)=(3-5i)+(1+2yi).$$

(*Roorkee Entrance 1984*)

9. What is wrong with the following calculation ?

$$1=\sqrt{1}=\sqrt{(-1)(-1)}=\sqrt{-1}\times\sqrt{-1}=i\times i=-1.$$

(*Roorkee Entrance 1987*)

10. For every real value of $a > 0$, determine the complex numbers which will satisfy the equation :

$$|z|^2-2iz+2a(1+i)=0.$$

(*Roorkee Entrance 1990*)

11. Find out the range in which the value of the function $\dfrac{x^2+34x-71}{x^2+2x-7}$ lies for all real values of x. Justify your answer.

(*Roorkee Entrance 1983*)

12. Solve for x the equation

$$\frac{2\sqrt{x}+1}{3-\sqrt{x}}=\frac{11-3\sqrt{x}}{5\sqrt{x}-9}$$

(*Roorkee Entrance 1985*)

13. Solve the following equation :

$$(x^2+2)^2+8x^2=6x\,(x^2+2).$$

(*Roorkee Entrance 1986*)

14. Solve the following equation for x :

$$3x^3=[x^2+\sqrt{18}x+\sqrt{32}]\,[x^2-\sqrt{18}x-\sqrt{32}]-4x^2$$

(*Roorkee Entrance 1988*)

15. $\log_{(2x+3)}(6x^2+23x+21)=4-\log_{(3x+7)}(4x^2+12x+9)$.

(*I.I.T. J.E.E., 1987*)

16. Balls are arranged in rows to form an equilateral triangle. The first row consists of one ball, the second of 2 balls, and so on. If 669 more balls are added, then all the balls can be arranged in the shape of a square, and each of its sides then contains 8 balls less than each side of the triangle did. Determine the initial number of balls. (*Roorkee Entrance 1985*)

17. Solve the following equations for x and y

$$\log_{10}x+\log_{10}x^{1/2}+\log_{10}x^{1/4}+\dots=y,$$

$$\frac{1+3+5+\dots\dots+(2y-1)}{4+7+10+\dots+(3y+1)}=\frac{20}{7\log_{10}x}$$

(*Roorkee Entrance 1987*)

18. The sum of the first ten terms of an arithmetic progression is equal to 155, and the sum of the first two terms of a geometric progression is 9. Find these progressions if the first term of the arithmetic progression equals the common ratio of the geometric progression and the first term of the geometric progression equals the common difference of the arithmetic progression.

(*Roorkee Entrance 1987*)

19. Determine the sum of the following series to infinity

$$1/(1.3.5)+1/(3.5.7)+1/(5.7.9)+\dots\dots$$

(*Roorkee Entrance 1989*)

20. Obtain the sum of :

$$\frac{1}{(x+1)}+\frac{2}{(x^2+1)}+\frac{4}{(x^4+1)}+\dots+\frac{2^n}{(x^{2^n}+1)}$$

(*Roorkee Entrance 1990*)

21. Sum the series

$$\sum_{n=0}^{\infty}\frac{1}{n!}[n+(n-1)x^2]^2\frac{x^{2n}}{(1+x^2)^{n+2}}\exp\left[\frac{-x^2}{1+x^2}\right]$$

(*Roorkee Entrance 1988*)

22. Determine b satisfying

$$\log_e 2\log_b 625=\log_{10}16\log_e 10.$$

(*Roorkee Entrance 1986*)

23. Construct an equation whose roots are the n^{th} power of the roots of the equation :

$$x^2-2x\cos\theta+1=0.$$ (*Roorkee Entrance 1989*)

24. Find the values of x satisfying the equation :

$$|x-1|^{\log_3 x^2-2\log_x 9}=(x-1)^7.$$

(*Roorkee Entrance 1990*)

25. There are ten points in a plane. Of these ten points, four points are in a straight line and with the exception of these four points, no other three points are in the same straight line. Find (*i*) the number of straight lines (*ii*) the number of triangles, that can be formed by joining these ten points.

(*Roorkee Entrance 1984*)

26. From 4 officers and 8 jawans, in how many ways can 6 be chosen
    (*i*) to include exactly one officer ;
    (*ii*) to include at least one officer.
    (*Roorkee Entrance 1985*)

27. Find the sum of
    $$3\ {}^nC_0 - 8\ {}^nC_1 + 13\ {}^nC_2 + \cdots \text{up to } (n+1) \text{ terms.}$$
    (*Roorkee Entrance 1988*)

28. In the Binomial expansion of $(1+y)^n$, where n is a natural number, the coefficients of the fifth, sixth and seventh terms are in arithmetic progression. Find n.
    (*Roorkee Entrance 1984*)

29. Find out that term of the expansion of
    $$(\tfrac{1}{2}x^{1/3} + x^{-1/5})^8$$ (*Roorkee Entrance 1985*)
    which does not contain x.

30. Determine b satisfying
    $$\log_{\sqrt{8}} b = 3\tfrac{1}{3}.$$ (*Roorkee Entrance 1985*)

31. If A be the sum of the odd terms and B the sum of even terms in the expansion of $(x+a)^n$, prove that
    $$A^2 - B^2 = (x^2 - a^2)^n.$$ (*Roorkee Entrance 1986*)

32. Find out the sum of the coefficients in the expansion of the binomial $(5p-4q)^n$, where n is a positive integer.
    (*Roorkee Entrance 1987*)

33. Find the coefficient of x^{50} in the expression :
    $$(1+x)^{1000} + 2x(1+x)^{999} + 3x^2(1+x)^{998} + \ldots + 1001x^{1000}.$$
    (*Roorkee Entrance 1990*)

34. A line is such that its segment between the straight lines $5x-y-4=0$ and $3x+4y-4=0$ is bisected at the point $(1, 5)$. Obtain its equation. (*Roorkee Entrance 1988*)

35. Derive the conditions to be imposed on B so that (0, B) should lie on or inside the triangle having sides
    $$y+3x+2=0,\ 3y-2x-5=0 \text{ and } 4y+x-14=0.$$
    (*Roorkee Entrance 1990*)

36. A ray of light is sent along the line $x-2y-3=0$. Upon reaching the line $3x-2y-5=0$, the ray is reflected from it. Find the equation of the line containing the reflected ray.
    (*Roorkee Entrance 1990*)

37. Find the internal angles of the triangle formed by the pair of the straight lines :
    $$x^2 - 4xy + y^2 = 0$$
    and the straight line $x+y+4\sqrt{6}=0$.

Give the co-ordinates of the vertices of the triangle so formed and also the area of the triangle.

(*Roorkee Entrance 1983*)

38. From a point A(1, 1), straight lines AL and AM are drawn at right angles to the pair of straight lines $3x^2+7xy+2y^2=0$. Find the equation of the pair of straight lines AL and AM. Also find the area of the quadrilateral ALOM, where O is the origin of co-ordinates. (*Roorkee Entrance 1984*)

39. The opposite angular points of a square are (3, 4) and (1, —1). Find the co-ordinates of the other two vertices.

(*Roorkee Entrance 1985*)

40. The base of a triangle passes through a fixed point (f, g) and its sides are respectively bisected at right angles by the lines

$$y^2-8xy-9x^2=0.$$

Determine the locus of its vertex. (*Roorkee Entrance 1985*)

41. Show that the four straight lines given by

$12x^2+7xy-12y^2=0$ and $12x^2+7xy-12y^2-x+7y-1=0$

lie along the sides of a square. (*Roorkee Entrance 1986*)

42. Two vertices of a triangle are (4, —3) and (—2, 5). If the orthocentre of the triangle is at (1, 2), find the coordinates of the third vertex. (*Roorkee Entrance 1987*)

43. A circle of diameter 13 *m* with centre 'O' coinciding with the origin of co-ordinate axes has diameter AB on the x-axis. If the length of the chord AC be 5m, find the following :

(*a*) Equation of the pair of lines BC.

(*b*) The area of the smaller portion bounded between the circle and the chord AC. (*Roorkee Entrance 1983*)

44. If the line $x \cos \alpha+y \sin \alpha=p$ cuts the circle $x^2+y^2=a^2$ in M and N, then show that the circle whose diameter is MN is

$$x^2+y^2-a^2=2p(x \cos \alpha+y \sin \alpha-p).$$

(*Roorkee Entrance 1967*)

45. Find the equation of a circle which is co-axial with the circles

$$2x^2+2y^2-2x+6y-3=0 \text{ and } x^2+y^2+4x+2y+1=0.$$

It is given that the centre of the circle to be determined lies on the radical axis of these two cirlces.

(*Roorkee Entrance 1984*)

46. Find the condition such that the four points in which the circles

$$x^2+y^2+ax+by+c=0 \text{ and } x^2+y^2+a'x+b'y+c'=0$$

are intersected by the straight lines

$$AX+BY+C=0 \text{ and } A'x+B'y+C'=0$$

respectively, lie on another circle. (*Roorkee Entrance 1986*)

47. Obtain the equations of the straight lines passing through the point A(2, 0) and making 45° angle with the tangent at A to the circle $(x+2)^2+(y-3)^2=25$. Find the equations of the circles each of radius 3 whose centres are on these straight lines at a distance of $5\sqrt{2}$ from A.

(*Roorkee Entrance 1987*)

48. A circle has radius 3 units and its centre lies on the line $y=x-1$. Find the equation of this circle if it passes through (7, 3). (*Roorkee Entrance 1988*)

49. Find the equation of the circle having the lines

$x^2+2xy+3x+6y=0$ as its normals and having size just sufficient to contain the circle $x(x-4)+y(y-3)=0$.

(*Roorkee Entrance 1990*)

50. Observations made to estimate the radius of the moon have shown that the semi-vertical angle of the tangential cone, drawn with vertex at the observer O and touching the rim of the moon is $\frac{1}{4}$ of a degree. Use this information to determine the radius of the moon, given that the distance of the centre of the moon from the observer O is 3,84,000 km and for small angles, $\sin\theta$ is same as angle θ measured in radians.

(*Roorkee Entrance 1984*)

51. If $x+y+z=xyz$, prove that

$$\frac{2x}{1-x^2}+\frac{2y}{1-y^2}+\frac{2z}{1-z^2}=\frac{2x}{1-x^2}\cdot\frac{2y}{1-y^2}\cdot\frac{2z}{1-z^2}.$$

(*Roorkee Entrance 1983*)

52. If $\cos\alpha+\cos\beta+\cos\gamma=0$ and $\sin\alpha+\sin\beta+\sin\gamma=0$, then prove that

$$\cos 3\alpha+\cos 3\beta+\cos 3\gamma=3\cos(\alpha+\beta+\gamma)$$

(*Roorkee Entrance 1985*)

53. If a, b, c are the sides of a triangle ABC and $3a=b+c$, show that

$$\text{Cot}\,\frac{B}{2}\,\text{Cot}\,\frac{C}{2}=2.$$

(*Roorkee Entrance 1986*)

54. A man observes that when he moves up a distance c metres on a slope, the angle of depression of a point on the horizontal plane from the base of the slope is 30° ; and when he moves up further a distance c metres, the angle of depression of the point is 45°. Obtain the angle of inclination of the slope with the horizontal. (*Roorkee Entrance 1986*)

55. If in a triangle,

$$\frac{a^2-b^2}{a^2+b^2}=\frac{\sin(A-B)}{\sin(A+B)},$$

prove that it is either a right angled triangle or an isoceles triangle. *(Roorkee Entrance 1987)*

56. In any $\triangle$ ABC, show that

$(\cot(A/2)+\cot(B/2))(a\sin^2(B/2)+b\sin^2(A/2))=c\cot(C/2)$
(Roorkee Entrance 1988)

57. If the radii of the externally inscribed circles of a triangle are in arithmetic progression, then prove that $(a-b)(s-c)=(b-c)(s-a)$, where a, b, c are the sides of the triangle and $s=(a+b+c)/2$. *(Roorkee Entrance 1989)*

58. If x, y, z are the perpendicular distances of the vertices of a triangle ABC from the opposite sides and $\triangle$ the area of the triangle, then prove that

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{\triangle}(\text{Cot A}+\text{Cot B}+\text{Cot C}).$$

(Roorkee Entrance 1990)

59. In a right angled triangle ABC, the hypotenuse BC of length a is divided into n equal parts (n an odd positive integer). Let α be the acute angle subtending from A by that segment which contains the mid-point of the hypotenuse. Let h be the altitude to the hypotenuse of the triangle, prove that

$$\text{Tan }\alpha=4nh/[(n^2-1)a].$$

(Roorkee Entrance 1983)

60. Angle of elevation of a cloud from a point h meters above the surface of a lake is α and the angle of depression of its reflection in the lake is β. Prove that the height of the cloud above the surface of the lake is

$$\frac{h\sin(\alpha+\beta)}{\sin(\beta-\alpha)}.$$

(Roorkee Entrance 1983)

61. A person observes the top P of a vertical tower of height h from a station S_1 and finds that β_1 is the angle of elevation. He moves in a horizontal plane to second station S_2 and finds that $\angle PS_2S_1$ is γ_1 and the angle subtended by S_2S_1 at P is δ_1 and the angle of elevation is β_2. He moves again to a third station S_3 such that $S_3S_2=S_2S_1$, $\angle PS_3S_2=\gamma_2$ and the angle subtended by S_3S_2 at P is δ_2. Show that

$$\frac{\sin\gamma_1\sin\beta_1}{\sin\delta_1}\quad\frac{\sin\gamma_2\sin\beta_2}{\sin\delta_2}=\frac{h}{S_1S_2}.$$

(Roorkee Entrance 1985)

62. A man moves along a bank of a canal and observes a tower on the other bank. He finds that the angle of elevation of the top of the tower from each of the two points A and B, at a distance $6d$ apart, is α. From a third point C, between A and B, at a distance $2d$ from A, the angle of elevation is found to be β. Find the height of the tower and the width of the canal. *(Roorkee Entrance 1987)*

63. AB is a vertical flag whose end A is on a horizontal plane through A. C is the mid-point of AB and P is a point on horizontal plane through A. The portion CB subtends an angle β at P. If $n\text{AB}=\text{AP}$, then show that $\tan \beta = n/(2n^2+1)$. *(Roorkee Entrance 1989)*

64. A train is moving at a constant speed at an angle θ east of north. Observations of the train are from a fixed point. It is due north at some instant. Ten minutes earlier its bearing was α_1 west of north, whereas ten minutes afterwards its bearing is α_2 east of north. Find $\tan \theta$. *(Roorkee Entrance 1990)*

65. Find all the angles θ between $-\pi$ and π that satisfy the equation

$$5 \cos 2\theta + 2 \cos^2 \frac{\theta}{2} + 1 = 0. \qquad \textit{(Roorkee Entrance 1984)}$$

66. If $\cos^{-1} \dfrac{x}{2} + \cos^{-1} \dfrac{y}{3} = \theta$, then prove that

$$9x^2 - 12xy \cos \theta + 4y^2 = 36 \sin^2 \theta. \quad \textit{(Roorkee Entrance 1984)}$$

67. Evaluate :

$$\tan \left[\frac{1}{2} \cos^{-1} \frac{\sqrt{5}}{3} \right]. \qquad \textit{(Roorkee Entrance 1986)}$$

68. Show that $x=0$ is the only solution satisfying the equation $1+\sin^2 ax = \cos x$, where a is irrational. *(Roorkee Entrance 1987)*

69. Find the general solution of the following equation :

$$2(\sin x - \cos 2x) - \sin 2x\,(1+2 \sin x) + 2 \cos x = 0.$$

(Roorkee Entrance 1987)

70. Solve for x and y :

$$x \cos^3 y + 3x \cos y \sin^2 y = 14$$
$$x \sin^3 y + 3x \cos^2 y \sin y = 13. \quad \textit{(Roorkee Entrance 1988)}$$

71. Find all the values of α for which the equation

$$\sin^4 x + \cos^4 x + \sin 2x + \alpha = 0$$

is valid. Also find the general solution of the equation. *(Roorkee Entrance 1990)*

In each of the following problems four alternatives are given out of which exactly one is correct. Put a tick mark (√) against the

correct alternative :

72. A point on the circumference of a rotating wheel of diameter 100 cm is moving at the rate of 50 cm a second. In one second, it turns through

(*a*) $\frac{1}{2}$ radian (*b*) 1 radian

(*c*) 50 radians (*d*) 100 radians.

73. In a circle of diameter 20 cm, a chord is of length 10 cm. The angle (in radians) subtended by the chord at the centre is

(*a*) π (*b*) $2\pi/3$ (*c*) $\pi/3$ (*d*) $\pi/6$.

[**Note** : Assume $\pi = \frac{22}{7}$ if needed.]

74. An arc of a circle of length 22 cm subtends an angle of 60° at the centre of the circle. The radius of the circle is

(*a*) 42 cm (*b*) 22 cm (*c*) 21 cm (*d*) $\frac{11}{30}$ cm.

75. If x is an angle in the first quadrant and $\sin x = \frac{a}{b}$, then $\tan x$ equals

(*a*) $\frac{a}{\sqrt{(b^2-a^2)}}$ (*b*) $\frac{a}{\sqrt{(b^2+a^2)}}$

(*c*) $\frac{b}{\sqrt{(b^2+a^2)}}$ (*d*) $\frac{b}{\sqrt{(b^2-a^2)}}$.

76. x is an acute angle of a right-angled triangle whose base is a cm and whose area is b cm². It is possible to determine $\sin x$

(*a*) only when the hypotenuse is also given.

(*b*) only when the value of x is given.

(*c*) only when the height is also given.

(*d*) without any more information.

77. The sides of a triangle are a, $\sqrt{3}a$ and $2a$. The smallest angle of the triangle is

(*a*) $2\pi/5$ (*b*) $\pi/3$ (*c*) $\pi/4$ (*d*) $\pi/6$.

78. $\tan 6x$ equals

(*a*) $\frac{6 \tan x - \tan^6 x}{1-6 \tan^3 x}$ (*b*) $\frac{2 [3 \tan x - \tan^3 x]}{1-3 \tan^2 x}$

(*c*) $\frac{3 \sin x - 4 \sin^3 x}{4 \cos^3 x - 3 \cos x}$ (*d*) $\frac{3 \sin 2x - 4 \sin^3 2x}{4 \cos^3 2x - 3 \cos 2x}$.

79. Which of the following could have the value 1·1 ?

(*a*) $2 \sec x$ (*b*) $\sin x$ (*c*) $\cos x$ (*d*) $2 \tan x$.

80. If $\sin x = \cos x$, then a possible value of x is

(*a*) 1 (*b*) $\frac{\pi}{4}$ (*c*) $\frac{\pi}{2}$ (*d*) π.

81. If $\sin^2 x = \frac{1}{4}$ and x lies in the second quadrant, then x equals
(a) $\frac{5\pi}{6}$ (b) $\frac{3\pi}{4}$ (c) $\frac{2\pi}{3}$ (d) $\frac{7\pi}{12}$.

82. Given that sin 34°10′=·5616 and sin 34°20′=·5640, the value of sin 34°14′ would be
(a) ·5616 (b) ·5626 (c) ·5628 (d) ·5630.

83. Given that cos 69°10′=·3557 and cos 69°20′=·3529, which of the following could possibly be cos 69°13′ ?
(a) ·3527 (b) ·3528 (c) ·3549 (d) ·3565.

84. Given that $0° < x° < y° < z° < 90°$, $\cot x = a$, $\cot z = b$ and correction for $\cot z = c$, what is $\cot y$?
(a) $a-c$ (b) $a+c$ (c) $b+c$ (d) $b-c$.

85. If $\tan x = \frac{\sqrt{5}}{2}$, then a possible value of $\cos x$ is
(a) $\frac{2}{3}$ (b) $\frac{3}{2}$ (c) $\frac{\sqrt{5}}{3}$ (d) $\frac{3}{\sqrt{5}}$.

86. $\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2$ equals
(a) $\sin x + 1$ (b) $\cos x + 1$
(c) $\cos^2\frac{x}{2} + \sin^2\frac{x}{2} + \sin x \cos x$
(d) $\cos x + \sin x + 2\sin\frac{x}{2}\cos\frac{x}{2}$.

87. If $\sin^2 x + \cos^2 x + \cot^2 x = \operatorname{cosec}^2 y$, then a possible value of x is
(a) -1 (b) $-y$ (c) $\frac{y}{2}$ (d) $2y$.

88. The period of cot is
(a) 2π (b) π (c) $\frac{\pi}{2}$ (d) 1.

89. Through what distance should the graph of cos be translated along the positive direction of the x-axis so that it coincides with the graph of sin ?
(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) π (d) 2π.

90. 2π is the period of
(a) sec (b) tan (c) cot (d) none of these.

91. $-\sin x$ can be written as
(a) $\sin(\pi - x)$ (b) $\sin(\pi + x)$

(*c*) $\sin(2\pi+x)$ (*d*) $\sin\left(\frac{1}{x}\right)$.

92. If $\cos x=\cos y$, then a possible value of $\cos(x+y)$ is
(*a*) 0 (*b*) 1 (*c*) $\sin 2y$ (*d*) $\sin y$.

93. $\sin(A+B)\sin(A-B)$ equals
(*a*) $\sin(A^2-B^2)$ (*b*) $\sin 2A$
(*c*) $\sin A^2-\sin B^2$ (*d*) $\sin^2 A-\sin^2 B$.

94. $\dfrac{1-\tan^2\frac{\pi}{8}}{1+\tan^2\frac{\pi}{8}}$ equals
(*a*) $\cos\frac{\pi}{8}$ (*b*) $\sin\frac{\pi}{8}$
(*c*) $\sin\frac{\pi}{4}$ (*d*) $\cos\frac{\pi}{4}$.

95. Given that $\sin x=\frac{1}{4}$, $\sin 3x$ is
(*a*) $-\frac{11}{16}$ (*b*) $\frac{11}{16}$ (*c*) $\frac{3}{4}$ (*d*) $-\frac{3}{4}$.

96. If $\sin x=\sin y$, then a possible value of y is
(*a*) $-x$ (*b*) $\pi+x$ (*c*) $\pi-x$ (*d*) $2\pi+x$.

97. If $\cos x=\cos y$, then a possible value of y is
(*a*) $-x$ (*b*) $\pi-x$ (*c*) $\pi+x$ (*d*) $\frac{3\pi}{2}+x$.

98. If $\tan x=\tan y$, then a possible value of y is
(*a*) $3\pi-x$ (*b*) $3\pi+x$ (*c*) $\frac{\pi}{2}-x$ (*d*) $\frac{\pi}{2}+x$.

99. A balloon B is directly above the one end A of a bridge AC. The angle of depression at the end C of the bridge from the balloon is 40°. ∠C
(*a*) is 40° (*b*) is 50° (*c*) is 120°
(*d*) cannot be determined from the given data.

100. A 10 m long ladder rests against a wall in such a position that the angle between the ladder and the ground is 30°. The height of the higher end of the ladder above the ground is
(*a*) 20 m (*b*) 10 m (*c*) 5 m (*d*) $5\sqrt{3}$ m.

101. A vertical flagstaff stands on a horizontal plane. From a point 50 m from its foot, the angle of elevation of the top is 30°. The height of the flagstaff is
(*a*) 25 m (*b*) $\frac{50}{\sqrt{3}}$ m (*c*) $50\sqrt{3}$ m (*d*) 100 m.

ANSWERS

उपदेशलवं शास्त्रं कुरुते धीमतो यतः ।
तत्तु प्राप्यैव विस्तारं स्वयमेवोपगच्छति ॥
जले तैलं खले गुह्यं पात्रे दानं मनागपि ।
प्राज्ञे शास्त्रं स्वयं याति विस्तारं वस्तुशक्तितः ॥

A little instruction and guidance in science is sufficient for the intelligent student, for this alone will help him to develop his knowlege of his own accord. Science instilled into the intelligent mind has sufficient vitality in it to grow and expand by its own force like a drop of oil on a sheet of water, a piece of secret confined to a villain or a little act of charity to the deserving person.

—Bhaskara

ANSWERS

[*For each of the answers marked with an asterisk* (*), *an example different from the one given below could also have served the purpose*].

Exercise 1 (*a*)

1. *a*.
2. *a*.
3. (*a*) $A=\{6, 7, 8, 9\}$; (*b*) $B=\{-3, 8\}$; (*c*) $C=\{5, 10, 15, 20, \ldots\}$; (*d*) $D=\{1, 2, 3\}$.
4. (*a*) $A=\{x : x$ is an even natural number less than $12\}$;
   (*b*) $B=\{x : x$ is a natural number between 4 and $6\}$;
   (*c*) $C=\{x : x$ is an integer such that $100 < x < 1000\}$.

Exercise 1 (*b*)

1. A and C.
2. None.
4. (*a*) True ; (*b*) False ; (*c*) False.
5.* (*a*) $S=\{1, 2, 3\}$; (*b*) $S=\{a, b, c, d\}$.
6. (*a*), (*c*), (*d*), (*e*), (*g*).

Exercise 1 (*c*)

1. (*a*) $\{a, b, c, d, e\}$; (*b*) $\{c, d\}$;
   (*c*) $\{a, c, d, e\}$; (*d*) $\{e\}$;
   (*e*) $\{a, b, c, d, e\}$; (*f*) $\{a\}$.
2. (*a*) $\{3, 4\}$; (*b*) $\{1, 2, 3, 4, 5\}$;
   (*c*) $\{1, 2, 3, 4, 5\}$; (*d*) $\{1, 2, 3, 4, 5\}$,
   (*e*) $\{1, 2, 3, 4\}$; (*f*) $\{1, 2, 3, 4, 5\}$.
3. (*a*) $\{4, 6\}$; (*b*) $\{2, 3, 4, 5, 6, 7, 8\}$;
   (*c*) $\{3, 6, 7\}$; (*d*) ϕ ;
   (*e*) $\{3, 4, 6, 7, 8\}$; (*f*) ϕ.
4. (*a*), (*b*), (*c*), (*d*).
6. (*a*), (*d*), (*e*), (*h*).

9*. $A=\{a, b, c\}$, $B=\{a, d, e\}$, $C=\{b, c\}$.

12. (*b*) Read $A\sim(B\cup C)$ for $A\sim(B\sim C)$.
    (*c*) Read $A\sim(B\cap C)$ for $A\sim(B\cup C)$.

Exercise 1 (*d*)

1. (*a*) 7, 4 ; (*b*) 3, 0.
2. 21.
3. 7.
4. 23.
5. Add '10 take all the three subjects'. 100.
6. (*i*) 56 ; (*ii*) A.
7. (*i*) 15 ; (*ii*) 14.
8. (*i*) 10% ; (*ii*) 0% ; (*iii*) 40%.
10. At least 10%.

Exercise 1 (*e*)

1. $\{(a, *), (b, *), (c, *)\}$.
2. $\{(0, 5), (1, 5), (0, 6), (1, 6), (0, 7), (1, 7)\}$.
3. $\{(5, 0), (5, 1), (6, 0), (6, 1), (7, 0), (7, 1)\}$.
5. $A=B$.
6. $A\times(B\times C)=\{(x, (m, *)), (x, (n, *)), (y, (m, *)), (y, (n, *)), (z, (m, *)), (z, (n, *))\}$.
   $(A\times B)\times C=\{((x, m), *), ((x, n), *), ((y, m), *), ((y, n), *), ((z, m), *), ((z, n), *)\}$
   $A\times(B\times C)\neq(A\times B)\times C$.

10.* $A=\{0, 1\}$, $B=\{5, 6, 7\}$, $C=\{a\}$, $D=\{c, d\}$.

Exercise 1 (*f*)

1. $P(A)$, where $A=\{(1, 5), (1, 6), (2, 5), (2, 6)\}$.
2. $P(A)$, where $A=\{(a, a), (a, b), (b, a), (b, b)\}$.
3. (*a*) No, (*b*) No, (*c*) Yes, (*d*) No.
4. (*a*) $R_1=\{(1, 1), (2, 2), (3, 3), (4, 4)\}$.
   (*b*) $R_2=\{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$.
   (*c*) $R_3=\{(4, 1), (4, 2), (4, 3), (3, 1), (3, 2), (2, 1)\}$.
5. $\{1, 2, 3\}$. 6. $\{5, 6, 7, 8, 9\}$.
7. (*a*) $8 \mathrel{R} 12$, $(-5) \mathrel{R} 7$, $2 \mathrel{R} 18$, $20 \mathrel{R} 4$, $7 \not\mathrel{R} 7$, $3 \mathrel{R} 11$.
   (*b*) $12 \not\mathrel{R} 14$, $13 \mathrel{R} 5$, $(-9) \mathrel{R} 7$, $4 \not\mathrel{R} 7$, $(-6) \mathrel{R} 2$, $3 \not\mathrel{R} 6$, $15 \not\mathrel{R} 4$, $6 \not\mathrel{R} 9$.
   (*c*) $6 \mathrel{R} 11$, $(-4) \mathrel{R} 7$, $(-3) \mathrel{R} (-4)$, $2 \mathrel{R} (-4)$, $2 \mathrel{R} 7$, $5 \mathrel{R} 15$.

Exercise 1 (*g*)

1. (*a*), (*d*).
2. (*a*) Neither reflexive nor symmetric nor transitive.
   (*b*) Neither reflexive nor symmetric nor transitive.
   (*c*) Transitive but neither reflexive nor symmetric.
   (*d*) Transitive but neither reflexive nor symmetric.
   (*e*) Transitive but neither reflexive nor symmetric.
   (*f*) Transitive but neither reflexive nor symmetric.
   (*g*) Reflexive as well as symmetric as well as transitive.
   (*h*) Reflexive as well as symmetric as well as transitive.
   (*i*) Reflexive as well as symmetric as well as transitive.
   (*j*) Neither reflexive nor transitive but symmetric.
3. (*a*)* $\{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\}$.
   (*b*)* $\{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 5), (5, 1), (5, 2), (2, 5)\}$.
   (*c*)* $\{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 5)\}$.

(*d*)* $\{(1, 2), (2, 1), (1, 1)\}$.

(*e*)* $\{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 2), (2, 3)\}$.

(*f*)* $\{(1, 2), (2, 1), (1, 3), (3, 1)\}$.

(*g*)* $\{(1, 2), (2, 3), (1, 3)\}$.

4. (*a*) Neither reflexive nor transitive.
(*b*) Neither reflexive nor transitive.
(*c*) Neither reflexive nor transitive.
(*d*) Reflexive as well as transitive.
(*e*) Only transitive.
(*f*) Neither reflexive nor transitive.
(*g*) Neither reflexive nor transitive.
(*h*) Only transitive.
(*i*) Neither reflexive nor transitive.
(*j*) Neither reflexive nor transitive.
(*k*) Neither reflexive nor transitive.
(*l*) Reflexive as well as transitive.
(*m*) Neither reflexive nor transitive.
(*n*) Neither reflexive nor transitive.
(*o*) Neither reflexive nor transitive.
(*p*) Reflexive as well as transitive.
(*q*) Only transitive.
(*r*) Only transitive.
(*s*) Reflexive as well as transitive.
(*t*) Only reflexive.
(*u*) Neither reflexive nor transitive.

5. (*a*) Yes ; (*b*) No ; (*c*) Yes ; (*d*) Yes ; (*e*) Yes ; (*f*) No ; (*g*) No ; (*h*) No ; (*i*) Yes ; (*j*) Yes ; (*k*) Yes ; (*l*) Yes ; (*m*) Yes ; (*n*) Yes ; (*o*) Yes ; (*p*) No ; (*q*) Yes ; (*r*) Yes ; (*s*) No ; (*t*) Yes ; (*u*) Yes ; (*v*) No ; (*w*) Yes ; (*x*) Yes ; (*y*) No ; (*z*) Yes.

6. It is not necessary that
for each $a \in S \ \exists\ b \in S : a \sim b$.

9. (*a*) Only symmetric ; (*b*) Only symmetric ;
(*c*) Only symmetric.

10. (*a*).

Exercise 1 (h)

1. No relation is a function.
2. (a) Yes ; (b) Yes ; (c) Yes. 3. Yes.

Exercise 1 (i)

1. Yes. 2. (b) and (c). 3. (a) and (d) 4. (a) and (d).
5. (a)* $f : \mathbf{N} \to \mathbf{N}$ defined by $f(n)=n, \ \forall \ n \in \mathbf{N}$;
   (b)* $f : \mathbf{N} \to \mathbf{Z}$ defined by $f(n)=n, \ \forall \ n \in \mathbf{N}$;
   (c)* $f : \mathbf{R} \to \mathbf{R}^+$ defined by $f(x)=x^2, \ \forall \ x \in \mathbf{R}$;
   (d)* $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x)=x^2, \ \forall \ x \in \mathbf{R}$.

Exercise 1 (j)

1. $f(A)=]0, 1[\ ; f(B)=[0, 1] \ ; f(C)=]0, 1] \ ; f(D)=[0, \infty[$.
2. Read $D=]-\infty, 0[$ insteal of $D=]\infty, 0[$
   $f(A)=[1, 2[\ ; f(B)=[1, \infty[\ ; f(C)=[1, 10] \ ; f(D)=]1, \infty[$.
3. $f^{-1}(A)=]-\infty, \infty[\ ; f^{-1}(B)=]-2, 1] \ ; f^{-1}(C)=]\frac{1}{2}, \frac{3}{2}[$;
   $f^{-1}(D)=]-\infty, 0[$.
4. $f^{-1}(A)=\phi \ ; f^{-1}(B)=]-\infty, \infty[\sim\{0\} \ ; f^{-1}(C)=]-1, 1[$;
   $f^{-1}(D)=]-1, 1[\sim\{0\}$.
5.* Let $f : \{a, b, c\} \to \{x, y\}$ be defined by $f(a)=f(b)=x$ and $f(c)=y$.
   Let $A=\{a, c\}$ and $B=\{b\}$.

Exercse 1 (k)

1. (a) $fog\,(x)=2x^3+7$; $gof(x)=(2x+1)^3+3, \ \forall \ x \in \mathbf{R}$.
   (b) $fog(x)=(2x+5)^2$; $gof(x)=2x^2+5, \ \forall \ x \in \mathbf{R}$.
   (c) $fog(x)=(x+7)^3-2$; $gof(x)=x^3+5, \ \forall \ x \in \mathbf{R}$.
3. (a)* Let $f(x)=x, \ \forall \ x \in \mathbf{R}^+$ and $g(x)=x, \ \forall \ x \in \mathbf{R} \sim \mathbf{R}^+$.
   (b)* Let $f(x)=x, \ \forall \ x \in \mathbf{R}$ and $g(x)=x, \ \forall \ x \in \mathbf{R}^+$.
   (c)* Let $f(x)=0, g(x)=1, \ \forall \ x \in \mathbf{R}$.
   (d)* Let $f(x)=2x, g(x)=x/2, \ \forall \ x \in \mathbf{R}$.

Exercise 1 (l)

2. a and c.
3. (a) $g : \mathbf{R} \to \mathbf{R}$ defined by $g(x)=\frac{1}{3}(x-7), \ \forall \ x \in \mathbf{R}$.
   (b) $g : \mathbf{R} \to \mathbf{R}^+$ defined by $g(x)=\sqrt{\frac{x}{2}}, \ \forall \ x \in \mathbf{R}^+$.
   (c) $g : [0, 1] \to [0, 1]$ defined by $g(x)=x^{1/3} \ \forall \ x \in [0, 1]$.
4.* $f(n)=n+(-1)^{n-1}, \ \forall \ n \in \mathbf{N}$.

Exercise 1 (*m*)

1. (*a*) No ; (*b*) No ; (*c*) Yes ; (*d*) Yes ; (*e*) Yes ; (*f*) No.
2. Fig. 1.17. Missing entries : second row b, fourth row d, and fifth row d, c, c, d.

   Fig. 1·18. Missing entries : second row d, third row a, and fifth row c, b.
3. Missing entries : second row d, third row a, and fifth row c, b.
4. 2^4. 5. 3^9. 6. 10^{65}. 7. $n^{\frac{1}{2}n(n+1)}$.
8. (*a*) True (*b*) True (*c*) True (*d*) True (*e*) False (*f*) False (*g*) False.

Test Your Understanding I

1. (*a*) 2. (*b*) 3. (*a*) 4. (*a*) 5. (*c*)
6. (*d*) 7. (*b*) 8. (*c*) 9. (*d*) 10. (*d*).

Review Exercise I

1. Read 'of 5 elements' for 'of elements'.
2. $(a, b) \to a^b$ in **N**. 3. Subtraction in **Z**.
4. f is neither surjective nor invertible.
5. $g(x) = \frac{1}{3}(x+5)$.
6. $(f \circ g)(x) = (2x-5)^2$, $(g \circ f)(x) = 2x^2 - 5$ for all $x \in \mathbf{R}$.
8. commutative as well as associative.
9. commutative as well as associative.
10. No. 11. Report inconsistent. 12. 878 ; 504.

Exercise 2 (*a*)

19. No. 20. (*a*) False (*b*) False (*c*) False.

Exercise 3 (*a*)

2. $7+5i$.

Exercise 3 (*b*)

1. (*a*) $5-i$ (*b*) $7-3i$ (*c*) $6+2i$
2. (*a*) $1-3i$ (*b*) $3-10i$ (*c*) $-1-2i$ (*d*) $3a-2bi$. (*d*) $-2+10i$.

1. Read 'consistidg of 5' insteal of 'consisting of'.

2.* The operation of subtraction on the set **Z** of integers is not commutative.

3.* The operation of subtraction on the set **Z** of integers is not associative.

3. (*a*) $18-i$ (*b*) $39-23i$ (*c*) 25 (*d*) $3+4i$ (*e*) $2+11i$ (*f*) $4i$.
4. (*a*) $\frac{2}{5}+\frac{1}{5}i$ (*b*) $\frac{1}{10}+\frac{3}{10}i$ (*c*) $\frac{3}{25}-\frac{4}{25}i$ (*d*) $\frac{5}{169}-\frac{12}{169}i$.

5. $\frac{3}{13}+\frac{2}{13}i,\ \frac{4}{25}-\frac{3}{25}i,\ \frac{6}{25}-\frac{17}{25}i,\ \frac{6}{13}+\frac{17}{13}i.$

6. $\frac{1}{2}-\frac{1}{2}i,\ 2i,\ \frac{-1}{4}-\frac{1}{4}i.$

Exercise 3 (*c*)

1. (*a*) 7, 1 (*b*) 5, −12.
2. (*a*) $x=8, y=1$ (*b*) $x=7, y=1$.
3. (*a*) $\frac{3}{2}-\frac{3}{2}i$ (*b*) $\frac{9}{25}-\frac{13}{25}i$ (*c*) $\frac{-3}{17}+\frac{12}{17}i.$
   (*d*) $0+i.$
4. (*a*) $x=\frac{7}{2},\ y=\frac{-1}{2}$ (*b*) $x=\frac{-3}{2},\ y=\frac{-7}{2}.$
5. $\frac{6}{5}+\frac{12}{5}i.$
6. $p=2+i,\ q=2-i.$

Exercise 3 (*d*)

1. (*a*) $\sqrt{2}, -\frac{\pi}{4}$ (*b*) $\sqrt{2}, \frac{3\pi}{4}$ (*c*) $\sqrt{2}, \frac{-3\pi}{4}$
   (*d*) $3, \pi$ (*e*) $2, \frac{\pi}{2}$ (*f*) $4, -\frac{\pi}{2}.$
2. (*a*) $2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$ (*b*) $2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$
   (*c*) $2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$
   (*d*) $2\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$
   (*e*) $\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$
   (*f*) $1\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right).$
3. (*a*) $2\sqrt{10}$ (*b*) 10 (*c*) $\frac{\sqrt{10}}{5}.$
4. (*a*) $\sqrt{97}$ (*b*) $\frac{\sqrt{2}}{2}$

Exercise 3 (*e*)

1. $\pm(3+i)$ 2. $\pm(2\sqrt{2}+\sqrt{2}i)$ 3. $\pm(2-2i)$
4. $\pm(\sqrt{3}+\sqrt{2}i)$ 5. $\pm[(a+b)-(a-b)i].$
6. $\pm(5-2\sqrt{3}i).$ 7. $\pm(\sqrt{7}+2i).$
8. $\pm\frac{1}{\sqrt{2}}\left(\sqrt{x^2+x+1}+\sqrt{x^2-x+1}\ i\right).$

Exercise 3 (f)

9. $3\omega^2$.

Exercise 3 (h)

1. $\frac{1}{7}(52+54i)$.
2. $\frac{1}{2}(-69+71i)$.
3. $-3+4i,\ 21-8i$.
4. $\frac{1}{2}(-3+11i),\ \frac{-7}{8}(11+3i)$.
5. $5,\ 15+5i$.
6. Externally in the ratio 2 : 3.
8. Modulus$=\sqrt{2}$, argument$=\pi/4$.
10. $(az_1+bz_2+cz_3)/(a+b+c)$;
$(z_1a\cos A+z_2b\cos B+z_3c\cos C)/(4R\sin A\sin B\sin C)$,
where R is the circum-radius.

Exercise 3 (i)

1. $\sqrt{13}$.
2. $3+4i,\ 9+12i,\ 1+18i,\ -5+10i$.
3. $-1+10i,\ -7+2i,\ 1-4i,\ 7+4i$.
4. $-6+9i$.
5. $-7+2i,\ -5+6i,\ -4+3i$.
6. $7-6i,\ 11,\ 5+4i$.
10. $\frac{1}{2}(1+i)(1\pm\sqrt{3})$. 20. 5.

Test Your Understanding III

1. (c). 2. (b). 3. (d). 4. (c). 5. (a).
6. (c). 7. (c). 8. (c). 9. (d). 10. (a).

Review Exercise III

2. $\frac{63}{25}-\frac{16}{25}i$. 3. $\frac{x(x^2+y^2+1)}{x^2+y^2},\ \frac{y(x^2+y^2-1)}{x^2+y^2}$.

4. 8. 5. $p=4,\ q=-5$. 6. $2,\ -\frac{\pi}{6}$

8. $2\pm2\sqrt{2}i$.

9. $\pm\frac{3}{\sqrt{2}}(1+i)$. 10. 1.

Exercise 4 (a)

1. (i) and (iii) are identities.
2. (i) 1, 2. (ii) 1, 3. (iii) 3, -5. (iv) $\frac{1}{2},\ \frac{1}{3}$.

3. (*i*) 2, 4. (*ii*) 3, 5. (*iii*) $\frac{1}{4}$, $\frac{5}{4}$. (*iv*) $\frac{9\pm\sqrt{65}}{2}$

4. (*i*) 1, 5. (*ii*) $\frac{1}{2}$, 1. (*iii*) $\frac{5\pm\sqrt{13}}{6}$ (*iv*) $\frac{7\pm\sqrt{-159}}{8}$.

5. 1, 8. 6. -2, 7. 7. -3, $\frac{8}{3}$. 8. a, b.

9. 1, $\frac{c-a}{a-b}$ 10. 1, $\frac{c(a-b)}{a(b-c)}$. 11. $\frac{-5}{2}(\sqrt{3}+1)$, $\frac{3}{2}(\sqrt{3}+1)$

12. $-(2+\sqrt{3})$, $2(2+\sqrt{3})$.

Exercise 4 (*b*)

1. $\pm 2, \pm\sqrt{3}$. 2. $\frac{1}{2}$, $-\frac{1}{2}$. 3. 1. 4. $\pm\frac{1}{\sqrt{2}}$.

5. $\frac{1}{2}(-5\pm\sqrt{5})$, $\frac{1}{2}(-5\pm\sqrt{5})$. 6. 2, -4, $-1\pm i\sqrt{8}$.

7. $-\frac{3}{2}$, $\frac{-3}{2}$, $\frac{1}{2}(-3\pm\sqrt{10})$. 8. 2, $-\frac{19}{3}$, $\frac{1}{6}(-13\pm i\sqrt{599})$.

9. $\frac{1}{2}(3\pm\sqrt{5})$, $\frac{1}{2}(5\pm\sqrt{21})$ 10. $\frac{1}{2}(-7\pm 3\sqrt{5})$, $\frac{1}{2}(1\pm i\sqrt{3})$

11. $\frac{1}{2}(3\pm\sqrt{5})$, $\frac{1}{2}(-7\pm\sqrt{33})$. 12. 3, $-\frac{1}{2}$, $\frac{1}{4}(5\pm i\sqrt{47})$

13. $\frac{1}{3}$, $-\frac{16}{3}$. 14. 5, $\frac{1}{3}$.

15. 5. 16. 2. 17. 3, 10.

18. 0, 3. 19. 4, $\frac{-10}{3}$. 20. $\frac{-7\pm\sqrt{33}}{4}$, 3.

Exercise 4 (*c*)

1. $m=0$. 2. $2b^2=9ac$. 3. $4b^2=25ac$. 4. 12. 5. $l=n$.

6. $rp^2+pr^2-3pqr+q^3=0$.

Exercise 4 (*d*)

1. (*i*) Real, irrational and unequal. (*ii*) Imaginary and unequal. (*iii*) Real, rational and equal. (*iv*) Real, rational and unequal.

2. (*a*) $m=3$ or -2. (*b*) $m=5$ or 3. 3. $m=2$ or $-\frac{10}{9}$.

Exercise 4 (*e*)

1. $x^2+x-6=0$. 2. $5x^2-8x-48=0$. 3. $x^2-4x+1=0$.

4. $x^2-10x+1=0$. 5. $x^2-8x+23=0$.

6. $x^2-[4+\sqrt{3}+i\sqrt{3}]\,x+4+2\sqrt{3}+i[2\sqrt{3}+3]=0$.

7. $21x^2-22x+21=0$. **8.** $x^2-2ax+a^2+b^2=0$.
9. $x^2+a^2-2ab+b^2=0$. **10.** $x^2(a^2+b^2)+4aibx-(a^2+b^2)=0$.
11. $x^2-2a^2x+a^4-b^4=0$ **12.** $x^2-4x-1=0$.
13. $x^2+4x-1=0$ **14.** $x^2-4x+7=0$.
15. $7x^2+4x-1=0$ **16.** 1. **17.** 2. **18.** -2.

Exercise 4 (*f*)

1. (*i*) 7. (*ii*) 18. (*iii*) 7. (*iv*) 18. (*v*) 5.

2. p^3-3pq. **3.** $\frac{1}{p^2r^2}(q^2-2pr)$. **4.** $\frac{1}{a^4}(b^4+3a^2c^2-4ab^2c)$

5. $\frac{b^2}{a^2c^2}(b^2-4ac)$ **6.** $\frac{2}{11}$ **7.** -27.

8. (*i*) $-\frac{1}{q^2-2r}$. (*ii*) $r(q^2-2r)$.

10. $x^2+x+1=0$. **11.** $x^2+6x+25=0$. **12.** $3x^2-2x+1=0$.
13. $4y^2-49y+118=0$. **14.** $x^2+2x+1=0$. **15.** $121x^2+66x+40=0$.
16. $a^4x^2-2a^2(b^2-2ac)x+b^2(b^2-4ac)=0$.
17. $x^2-px+9q-2p^2=0$. **18.** $x^2-2x+3=0$.
19. $abx^2-x(a+b)+1=0$. **20.** $5x^2-10x+1=0$.
21. $acy^2+y(ab+bc)+b^2+(a-c)^2=0$.
22. $ax^2-(2ap-b)x+(ap^2-bp+c)=0$.
23. $(a^2q+abp+b^2)x^2-(ap+2b)x+1=0$.

Test Your Understanding IV

1. (*b*) **2.** (*c*) **3.** (*d*) **4.** (*c*) **5.** (*c*)
6. (*c*) **7.** (*b*) **8.** (*a*) **9.** (*c*) **10.** (*d*).

Review Exercise IV

1. $1, \frac{(a+b-2c)}{b+c-2a}$. **2.** $3(\sqrt{2}+1), 4(\sqrt{2}+1)$.

3. $\frac{9}{5}, \frac{-4}{5}$. **4.** 1. **5.** $2\pm i\sqrt{8}, \frac{1}{2}(11\pm\sqrt{73})$.

6. $-1, 4\pm\sqrt{15}, 3\pm\sqrt{8}$. **7.** $\pm 1, \pm i, \frac{1\pm\sqrt{-3}}{2}$.

8. $1, 1, 1, \frac{-3\pm\sqrt{5}}{2}$. **9.** 4. **10.** 0, 5.

11. $\pm 2, \sqrt{2}$. **12.** $x^2-4x+1=0$. **13.** $x^2+ix-1=0$.
15. $x^2+x+1=0$. **16.** $4b^3+a^2c+16ac^2-12abc=0$.

20 $2, -\frac{10}{9}$.

Exercise 5 (*a*)

1. (*i*) 21, 45, $3(r-1)$. (*ii*) -28, -60, $-4\,(r-1)$.
(*iii*) $-\frac{23}{5}$, $-\frac{71}{5}$, $\frac{25-6r}{5}$. (*iv*) -19, -67, $29-6r$.

2. 11th, **3.** 16th **4.** 34. **5.** 76. **6.** $-16, -12, -8, -4, 0, 4, 8,$ $12, 16, 20, 24, 28$. **7.** 5th. **8.** 67. **9.** 64.

10. $-\frac{1}{2}, 2, \frac{9}{2}$; $\frac{5r-6}{2}$. **12.** No.

Exercise 5 (*b*)

5. 4, 6, 8. **6.** 6, 8, 10. **7.** 5, 10, 15.
8. 4, 6, 8, 10. **9.** 3, 4, 5, 6, 7. **10.** 2, 4, 6, 8.

Exercise 5 (*c*)

1. (*i*) 18, (*ii*) $3p$. (*iii*) a^2+b^2. **2.** 11, 17, 23.

3. 4, $3\frac{1}{2}$, 3, $2\frac{1}{2}$, 2. **4.** $4r-3s, 3r-2s, 2r-s, r, s, 2s-r,$ $3s-2r, 4s-3r,$ **5.** $n^2-n+1, n^2-2n+2, \ldots\ldots, n$.

6. 15. **7.** 5. **8.** 1.

Exercise 5 (*d*)

1. (*i*) $\frac{1075}{2}$ (*ii*) 1080 (*iii*) 0 (*iv*) $\frac{15}{7}$ (*v*) 80.

2. (*i*) $\frac{n^2-n+2}{2}$ (*ii*) $p(n-p-4)$ (*iii*) $\frac{n}{2}\left[\frac{n(2x-y)-y}{x+y}\right]$.

3. (*i*) 645 (*ii*) 520 (*iii*) $\frac{p}{2}(9p-13)$. **4.** 7.

5. 6, 13 **6.** 25. **7.** 64, 63. **8.** 950. **9.** 8729.

10. 68229 **11.** (*i*) $\frac{n}{2}(7n+17)$ (*ii*) $\frac{n}{2}[a(n+1)+2b]$
(*iii*) $\frac{n}{2}(3n-7)$ **12.** 9, 6 **13.** (*i*) 3, 5, 7, 9,......
(*ii*) 3, 7, 11, 15, ... **14.** 43 : 71. **15.** $(4p+1) : (6p+1)$.

Exercise 5 (*e*)

1. 78732 **2.** $\frac{8}{125}$ **3.** 729 **4.** $(-1)^{p-1}\left(\frac{2}{3}\right)^{p-2}$

5. $x^{(3n-1)/2n}$ **6.** ·000000000049152. **7.** $\frac{1}{5}, \frac{1}{25}$. **8.** 6th.

9. 8th **10.** 8th **11.** 6, 12, 24, 48, ...

12. $x\left(\frac{x}{y}\right)^{\frac{n-p}{p-q}}$ 13. $m\left(\frac{m}{n}\right)^{-\frac{1}{2}}, m\left(\frac{m}{n}\right)^{\frac{-p}{2q}}$.

Exercise 5 (*f*)

5. 6, 12, 24. 6. 4, 6, 9. 7. 4, 6, 9. 8. 2, 4, 8.

9. 4, 8, 16, 32.

Exercise 5 (*g*)

1. $\frac{2}{\sqrt{3}}, 1, \frac{\sqrt{3}}{2}$.

2. $-13, \frac{13}{2}, -\frac{13}{4}, \frac{13}{8}, -\frac{13}{16}, \frac{13}{32}, -\frac{13}{64}, \frac{13}{128}$.

3. $\frac{16}{3}$, 8, 12, 18, 27. 4. $-1, \frac{3}{2}, -\frac{9}{4}, \frac{27}{8}$. 5. 2, 8, 32.

8. $n=\frac{1}{2}$

Exercise 5 (*h*)

1. $2(3^{10}-1)$. 2. $\frac{1}{12}(3^8-1)$. 3. $\frac{3}{8}[1-(\cdot 2)^{15}]$.

4. $\frac{1}{2}(1-3^{-n})$. 5. $\frac{5}{6}\left[1-\left(-\frac{1}{5}\right)^{n+1}\right]$. 6. $\frac{81}{2}[1-3^{-10}]$.

7. 4 8. 6·4.

Exercise 5 (*i*)

1. $6\frac{2}{3}$. 2. $\frac{3}{4}$. 3. $6(6+\sqrt{30})$. 4. $\frac{1}{4}$. 5. $\frac{2}{1-2x}$.

6. $4+3+\frac{9}{4}+\frac{27}{16}+\ldots, 12+3+\frac{3}{4}+\frac{3}{16}+\ldots\ldots,$

7. $r=\frac{1}{3}$. 8. (*i*) $\frac{1}{3}$ (*ii*) $\frac{5}{18}$ (*iii*) $\frac{358}{495}$. 9. $\frac{2}{3}+\frac{1}{3}+\frac{1}{6}+\ldots\ldots$

Exercise 5 (*j*)

1. $\frac{n}{3}(4n^2-1)$. 2. $n^2(2n^2-1)$. 3. $\frac{n}{3}(4n^2+6n-1)$.

4. $n(3n^2+12n+13)$. 5. 2815. 6. 10455. 7. 8950.

8. $\frac{n}{6}(4n^2-3n+35)$. 9. $\frac{n}{4}(n+1)(n+2)(n+3)$.

10. $\frac{n}{6}(n+1)(n+2)$.

Exercise 5 (*k*)

1. $\frac{1-x^n}{(1-x)^2}-\frac{nx^n}{1-x}$. 2. $\frac{3(1-x^n)}{(1-x)^2}-\frac{1+(3n-1)x^n}{1-x}$.

3. $\dfrac{4\{1-(-x)^n\}}{(1+x)^2}-\dfrac{1+(4n-1)(-x)^n}{1+x}$.

4. $\dfrac{3(1-x^n)}{(1-x)^2}-\dfrac{(3n-2)x^n+2}{1-x}$; $\dfrac{1+2x}{(1-x)^2}$.

5. $\dfrac{1+x+x^2}{(1+x)^2}$; $\dfrac{1+x+x^2}{(1+x)^2}+\dfrac{(-1)^{n+1}nx^n}{1+x}+\dfrac{(-1)^n x^{n+1}}{(1+x)^2}$.

6. $\dfrac{1}{(1-x)^3}$. 7. $\dfrac{1-3x}{(1+x)^3}$.

8. $\dfrac{25}{16}$. 9. $\dfrac{2}{9}$.

Test Your Understanding V

1. (*c*) 2. (*c*) 3. (*d*) 4. (*b*) 5. (*d*)
6. (*a*) 7. (*b*) 8. (*b*) 9. (*a*) 10. (*d*)

Review Exercise V

5. 7650. 6. 6, 2 7. $8r-1$ 8. 38 : 53. 11. $\dfrac{(a+br)(1-r^{2n})}{1-r^2}$

12. 8 14. 2, 5, 8. 15. $n(3n^2+6n+1)$.

16. $\dfrac{1}{12}n(n+1)(n+2)(3n+5)$.

17. $\dfrac{n}{3}(n^2+2)$ 18. $\dfrac{1-(-x)^n}{(1+x)^2}-\dfrac{n(-x)^n}{1+x}$.

19. $\dfrac{4(1-x^n)}{(1-x)^2}-\dfrac{3+(4n-3)x^n}{(1-x)}$: $\dfrac{1+3x}{(1-x)^2}$. 20. 17.

Exercise 6 (*a*)

1. 36. 2. 28. 3. 6. 4. 375. 5. 80.
6. 72. 7. 48. 8. 18. 9. 120. 10. 1050.

Exercise 6 (*b*)

1. 24. 2. 120. 3. 720. 4. 720. 5. 132.

6. 210. 7. 220. 8. 252. 9. $\dfrac{6\,!}{3\,!}$. 10. $\dfrac{12\,!}{10\,!}$.

11. $\dfrac{15\,!}{13\,!\,2\,!}$ 12. $\dfrac{14\,!}{11\,!\,3\,!}$ 13. $(n-1)[(n-1)\,!]$.

14. $[(n^2+n+1)][(n-1)\,!]$ 15. $(n+1)^2[(n-1)\,!]$.

16. 90. 17. 2.

Exercise 6 (*c*)

1. (*i*) 840 (*ii*) 56 (*iii*) 720. 2. 8. 3. 14.
4. 41. 5. 7. 7. 120. 8. 24. 9. 120.
10. 60.

Exercise 6 (*d*)

1. 3024, 2016.
2. ${}^{n-3}P_r$, ${}^{r}P_3 \times {}^{n-3}P_{r-3}$.
3. 282240, 40320.
4. 480.
5. 1440.
6. 3600.
7. 4320.
8. 3110400.
9. 144, 144.
10. 126.
11. 86400.
12. $\dfrac{m\,!\,(m+1)\,!}{(m-n+1)\,!}$.
13. 86658.
14. 146652.

Exercise 6 (*e*)

1. $\dfrac{15\,!}{7\,!\,5\,!\,3\,!}$.
2. (*i*) 360, (*ii*) 120, (*iii*) 1260. (*iv*) 907200.
3. 7560 ; 60.
4. 900.
5. $\dfrac{20\,!}{5\,!\,4\,!}$.
6. 12600.
7. 360.
8. 10800.
9. 1024.
10. 9.
11. 243.
12. $n\,!$, n^n, n^n-n.
13. 1944810000
14. 3374.
15. 1000.
16. 31.
17. 32.

Exercise 6 (*f*)

1. 5040.
2. 360.
3. $\frac{1}{2}(9\,!)$.
4. 12960.
5. 11 !
6. $\frac{1}{2}(n-1)\,!$
7. 2(9) !
8. 3600.
9. $\dfrac{10\,!}{5\,!\,6\,!}$
10. 43200.
11. $\dfrac{1}{2}\cdot\dfrac{20\,!}{7\,!\,4\,!}$.

Exercise 6 (*g*)

1. 10626, 3876.
2. 230300.
3. 210.
4. 56, 70.
5. 35.
6. 36 ; 84.
7. 9690 hrs.
8. 185.
9. 56 ; 21.
10. (*i*) $\frac{1}{2}n(n-1)-\frac{1}{2}p(p-1)+1$,

    (*ii*) $\frac{1}{6}[n(n-1)(n-2)-p(p-1)(p-2)]$.
11. 10.
12. 6.
13. 10, 5.

14. 231. 15. 56. 16. $\dfrac{75\,!}{40\,!\,35\,!}$.

17. $\dfrac{52\,!}{4\,!\,(13\,!)^4}$, $\dfrac{52\,!}{(13\,!)^4}$. 18. $\dfrac{15\,!}{(5\,!)^3}$

19. 59. 20. 42. 21. 136, 2454.

22. 136, 2454.

Test Your Understanding VI

1. (*c*) 2. (*d*) 3. (*c*) 4. (*b*)
5. (*a*) 6. (*b*) 7. (*d*) 8. (*b*)
9. (*a*) 10. (*a*).

Review Exercise VI

1. 1440, 3600. 2. 144. 3. 325 ; 260.
4. 4 20. 5. 8(9) ! 6. 360, 240, 12.
7. 64. 8. 120, 325. 9. 256.
10. $\frac{1}{2}(n-1)\,!$ 11. 3. 12. 80640.
13. 60. 14. 240. 15. 243.
16. 63. 17. 210, 84. 18. 104.

Exercise 7 (*a*)

1. (*i*) $32x^5+240x^4y+720x^3y^2+1080x^2y^3+810xy^4+243y^5$.

(*ii*) $x^6y^3+4x^{11/2}y^{7/2}+\frac{20}{3}x^5y^4+\frac{160}{27}x^{9/2}y^{9/2}+\frac{80}{27}x^4y^5+\frac{64}{81}x^{7/2}y^{11/2}+\frac{64}{729}x^3y^6$.

(*iii*) $\frac{64}{729}x^6-\frac{32}{27}x^4+\frac{20}{3}x^2-20+\frac{135}{4x^2}-\frac{243}{8x^4}+\frac{729}{64x^6}$.

(*iv*) $x^5\left(243y^5-810y^3+1080y-\frac{720}{y}+\frac{240}{y^3}-\frac{32}{y^5}\right)$.

(*v*) $1+10x+25x^2-40x^3-190x^4+92x^5+570x^6-360x^7-675x^8+810x^9-243x^{10}$.

2. (*i*) $140\sqrt{2}$. (*ii*) $2x(16x^4-20x^2a^2+5a^4)$.

3. $\dfrac{10500}{x^3}$ 4. $-63x^4$. 5. ${}^{2n}C_{r-1}x^{r-1}$.

6. $210a^{16}b^6$. 7. -252. 9. 210.

10. 180. 12. 252. 14. $\frac{189}{8}a^{17}, -\frac{21}{16}a^{19}$.

17. (*i*) 96059601 (*ii*) 997002999 (*iii*) 6765201.

18. $(1+5)^7$.

20. $\dfrac{(2n)\,!\,2^n}{n\,!\,n\,!}, \dfrac{2^{2n-p+1}(2n)\,!\,x^{2n-2p+2}}{(p-1)\,!\,(2n-p+1)\,!}, \dfrac{2^{p-1}(2n)\,!}{(p-1)\,!\,(2n-p+1)\,!\,x^{2n-2p+2}}$.

Exercise 7 (b)

1. (*i*) 5th term ; 70.
(*ii*) 7th term ; 5580130·5. $(18)^{17}(12)^{11} \cdot {}^{28}C_{11}$.
(*iii*) 8th and 9th terms ; 165.2^{19} (*iv*) 12th term ; $3^5 \cdot 5^{10} \cdot {}^{15}C_{10}$.
(*v*) 5th term ; $8^7 \cdot 5^4 \cdot {}^{11}C_4$. (*vi*) 10th ; 11th terms ; $(18)^{17}(12)^{11} \cdot {}^{28}C_{11}$.

2. (*i*) ${}^{11}C_5 = {}^{11}C_6 = 462$. (*ii*) ${}^{12}C_6 = 924$.

3. 9th, $50688\, x^4 y^8$. 4. $x=2, y=2, n=5$. 5. 14 or 23.

6. 7. 8. 7 or 14. 9. 11.

10. $\left(1+\frac{1}{2}\right)^8$. 11. 5. 12. 8.

13. $\frac{36}{37} < x < \frac{37}{36}$ 14. 1.

Exercise 7 (c)

4. (*i*) $(n-1)$th. (*ii*) $(n-7)$th. (*iii*) $(n-r+2)$th.

5. $2^{12}-1$. 6. $2^{12}-1$. 7. 2^{16}.

Exercise 7 (d)

1. $(x^2+1)^n$. 2. $19.17.11.13.4.6^{10}$. 4. -1365.

5. $\frac{(-1)^n(4n)!}{2^n n!\,(3n)!}$ 6. 0.

Exercise 7 (e)

1. (*i*) $1+2x+3x^2+4x^3$. (*ii*) $1+x+2x^2+\frac{14}{3}x^3$.
(*iii*) $\frac{1}{2}+\frac{3}{16}x^2+\frac{27}{256}x^4+\frac{135}{2048}x^6$.
(*iv*) $2\sqrt{2}-\frac{9\sqrt{2}}{2}x+\frac{27\sqrt{2}}{16}x^2+\frac{27\sqrt{2}}{64}x^3$.

2. $-\frac{21}{16}x^6$.

3. $\frac{1155}{8}x^{12}$; expansion is valid when $|x| < 2^{-\frac{1}{3}}$

4. $\frac{16b^4}{243a^5}$ (*ii*) $-\frac{81a^4}{32b^5}$.

5. (*i*) $\frac{(r+1)(r+2)(r+3)}{1.2.3}x^r$. (*ii*) $\frac{1.3.5\ldots\ldots(2r-1)}{2^{3r+1}r!}3^r.x^{2r}$.
(*iii*) $\frac{(2r!)}{6^r(r!)^2}x^r$ (*iv*) $-\frac{(n-1)(2n-1)\ldots\ldots(nr-n-1)}{r!}x^r$
(*v*) $\frac{1.4.7\ldots\ldots(3r-2)}{3^{r-1}r!}\left(\frac{x^2}{a^3}\right)^r$. 6. 11th ; $-\frac{17!}{8!}\cdot\frac{x^{20}}{2^{18}}$.

7. 8th ; $-\frac{455.2^{22}}{3^{21}}x^{14}$.

Exercise 7 (f)

1. 3432. **2.** $-\frac{5}{1024}\frac{b^{18}}{a^{35/2}}$.

3. $\frac{2^{3/2}.\ 1.3.5\ldots\ldots(2r-5)}{r\,!}\ \frac{3^r}{2^{2r}}$ **4.** (i) 3. (ii) $3r^2+r+1$.

(iii) 2^r. **6.** $\frac{1.3.5\ldots\ldots(2p-1)}{2^p.p\,!}$ where $p=\frac{r}{2}$ or $\frac{1}{2}(r-1)$, according as r is even or odd. **8.** -20. **9.** 1.

Exercise 7 (g)

11. 5·01330 **13.** 9·9933. **14.** 1·001 ; ·09967

15. (i) 3·0025 (ii) 1·9978.

Test Your Understanding VII

1. (b) **2.** (c) **3.** (d) **4.** (c)

5. (a) **6.** (b) **7.** (c) **8.** (d)

9. (b) **10.** (a).

Review Exercise VII

1. $(x^2+1)^n$. **3.** 9th ; 7920. **4.** 0.

12. $221-\frac{9x}{4}+\frac{27}{32}x^2+\frac{27}{128}x^3$.

13. $\frac{1.4.7\ldots\ldots(3r-2)}{3^{r-1}r\,!}\left(\frac{x^2}{a^3}\right)^r$. **14.** first ; 8.

15. $2^{r-3}(r^2+7r+8)$. **18.** 4·9959952.

19. 1·00039928. **20.** $\frac{920}{921}$.

Exercise 8 (a)

1. 2·7. **4.** $2e-5$. **6.** $5e$.

7. $11e$. **8.** $\frac{1}{2}e-1$.

Exercise 8 (b)

1. $1+2x^2+2x^4+\frac{4}{3}x^6+\frac{2}{3}x^8$. **2.** $\frac{128}{15}$.

3. $2+x+\frac{5}{2}x^2+\frac{4}{6}x^3+\frac{17}{24}x^4+\ldots\ldots+\frac{2^n+(-1)^n}{n\,!}x^n+\ldots\ldots$

6. $e^{x^2}-e^{y^2}$ **10.** 7·4. **11.** $(-1)^n\,[a+(a+1)n-n^2]/n\,!$

Exercise 8 (c)

1. $3\left(x-\frac{3}{2}x^2+3x^3-\frac{27}{4}x^4\right)$; valid for $|\,x\,|<\frac{1}{3}$

2. $4x^2-8x^4+\frac{64}{3}x^6-64x^8+\frac{1024}{5}x^5$; valid for $|x|<\frac{1}{2}$.

3. $\frac{128}{7}$ 9. $2\log_e\left(\frac{3}{2}\right)$.

Test Your Understanding VIII

1. (c) 2. (c) 3. (b) 4. (a)
5. (a) 6. (b) 7. (d) 8. (c)
9. (b) 10. (d).

Review Exercise VIII

2. ·8187 3. $\frac{1}{2}e^4$. 7. $3\log_e 2-1$.

Exercise 9 (a)

1. 5. 2. $\sqrt{145}$ 3. $3\sqrt{89}$.
4. $\sqrt{(a^2+b^2)}$. 5. $a(t_1\sim t_2)\sqrt{[(t_1+t_2)^2+4]}$.
6. $2a\sin\frac{1}{2}(\beta\sim\alpha)$. 7. 16. 18. $(2, -1)$.
19. $\left(\frac{17}{18}, \frac{71}{18}\right)$ 20. $(2+2\sqrt{3}, 5)$; $(2-2\sqrt{3}, 5)$.
21. (5, 6) ; (3, 4). 22. $\left(\frac{1+\sqrt{3}}{2}, \frac{7-5\sqrt{3}}{2}\right)$ or $\left(\frac{1-\sqrt{3}}{2}, \frac{7+5\sqrt{3}}{2}\right)$

Exercise 9 (b)

1. 25. 2. 9. 3. 23·5.
4. 29. 5. a^2 6. $a^2\,|(p-q)(q-r)(r-p)|$.
7. $2a^2\left|\sin\frac{1}{2}(\alpha-\beta)\sin\frac{1}{2}(\beta-\gamma)\sin\frac{1}{2}(\gamma-\alpha)\right|$.
8. $\frac{1}{2}c^2\,|(q-r)(r-p)(p-q)/(pqr)|$.
12. 20·5. 13. 15. 14. 20·5.

Exercise 9 (c)

1. $\left(\frac{52}{7}, \frac{54}{7}\right)$. 2. $\left(\frac{69}{2}, \frac{71}{2}\right)$. 3. $(19, -14)$.
4. $\left(\frac{-2}{5}, \frac{19}{5}\right)$; $(14, -1)$ 5. $\frac{3}{2}\sqrt{2}$, 3, $\frac{3}{2}\sqrt{10}$.
6. (5, 0), (15, 5). 7. $(-25, -35/2)$; $(-15, -15)$; $(-5, -25/2)$.
9. 3 : 5 internally. 10. 2 : 3 externally.
11. 1 : 1 internally. 12. 3 : 7 externally.

13. 1 : 2 externally. 14. $\left(\frac{31}{3}, \frac{8}{3}\right)$.

15. $(-4, -11)$. 16. $(-1, 11)$.

17. $(2+2\sqrt{2}, -5-\sqrt{2})$.

Exercise 9 (*d*)

1. $x^2+y^2+12x-2y+28=0$.
2. $7x-13y+10=0$.
3. $8x^2+8y^2-8x+18y-7=0$.
4. $5x^2+9y^2=45$. 5. $3x-4y+6=0$.
6. $2x^2+2y^2-6x-6y+10=k^2$.
7. $16x^2+15y^2-64y+64=0$.
8. $5x^2+5y^2+26ax+5a^2=0$.

Test Your Understanding IX

1. (*b*). 2. (*c*). 3. (*c*). 4. (*d*). 5. (*b*).
6. (*b*). 7. (*a*). 8. (*a*). 9. (*a*) 10. (*b*).

Review Exercise IX

1. $\sqrt{(a^2+b^2)}$.

7. $(1, 1)$; $\left(\frac{81}{17}, \frac{1}{17}\right)$. 8. $\left(\frac{5}{3}, \frac{2}{3}\right)$; $\left(\frac{7}{3}, -\frac{2}{3}\right)$.

9. $\left(\frac{11}{2}, -\frac{3}{2}\right)$; $(2, 2)$; $\left(-\frac{3}{2}, \frac{11}{2}\right)$.

10. (*i*) 2 : 1 internally. (*ii*) 2 : 1 externally.

12. $(-3, 6)$; $(-6, 10)$. 13. $(3x-x_1-x_2, 3y-y_1-y_2)$.

17. $\frac{1}{2} a^2 \mid (q-r)(r-p)(p-q) \mid$.

19. 24·5.

Exercise 10 (*a*)

1. (*i*) $x-2=0$ (*ii*) $x+7=0$ (*iii*) $x-a=0$.
2. (*i*) $y-4=0$ (*ii*) $y+5=0$ (*iii*) $y-b=0$.
3. (*i*) $y-4=0$ (*ii*) $x+2=0$.
4. (*i*) $x-a=0$ (*ii*) $y-b=0$.

Exercise 10 (*b*)

1. (*i*) $y=\frac{x}{\sqrt{3}}$ (*ii*) $y=\sqrt{3}x$
 (*iii*) $y+\sqrt{3}x=0$ (*iv*) $y+x=0$
 (*v*) $y+\frac{x}{\sqrt{3}}=0$ (*vi*) $y-\frac{x}{\sqrt{3}}=0$.

2. (*i*) 45° (*ii*) 60° (*iii*) 150°.

4. $y = \pm x$.

5. $y = \pm \frac{x}{\sqrt{3}}$ and $y = \pm \sqrt{3}x$.

Exercise 10 (*c*)

1. $y = x + 2$.
2. $y = -x + 3$.
3. $y = -x - 1$.

Exercise 10 (*d*)

1. $y = x + 1$.
2. $y = \sqrt{3}x + (3 + 2\sqrt{3})$.
3. $y = -\sqrt{3}x + (3\sqrt{3} - 2)$.
4. $x + \sqrt{3}y + 4 = 0$.
5. $x + y + 1 = 0$.
6. $(y - q) + (x - p) \cot \alpha = 0$.

Exercise 10 (*e*)

1. $x + y = 0$.
2. $x - y + 1 = 0$.
3. $x - y - 1 = 0$.
4. $3x - 2y - 19 = 0$.
5. $\frac{x}{a} + \frac{y}{b} = 1$.
6. $ax - by + ab = 0$.
7. $2x - y(m_1 + m_2) + 2am_1m_2 = 0$.
8. $x + m_1m_2y - a(m_1 + m_2) = 0$.
9. $x \cos \frac{1}{2}(\theta_1 + \theta_2) + y \sin \frac{1}{2}(\theta_1 + \theta_2) = a \cos \frac{1}{2}(\theta_1 - \theta_2)$.
10. $\frac{x}{a} \cos \frac{1}{2}(\theta_1 + \theta_2) + \frac{y}{b} \sin \frac{1}{2}(\theta_1 + \theta_2) = \cos \frac{1}{2}(\theta_1 - \theta_2)$.
11. $7x + 12y = 23$, $x + y = 4$, $3x + 8y = 27$.
12. $3y - 2x = 12$, $8x - 11y = -42$, $3x - 4y = -16$.
13. $5x - 14y = -3$, $x + 3y = -18$, $7x - 8y = 19$.
14. $x - y = -1$. $x + 5y = -7$, $2x + y = 4$.
15. $x - y = 1$, $y + 5x = 3$, $13x - 7y = 11$.
16, $29x + 4y + 5 = 0$, $8x - 5y - 21 = 0$, $13x + 14y + 47 = 0$.
17. $2x + y = 3$.
18. $3x - y = 2$.
19. $2x + 3y = 27$.
20. $1 : 1$.

Exercise 10 (*f*)

1. $\frac{x+2}{1/2} = \frac{y+1}{\sqrt{3}/2} = r$.
2. $\frac{x+4}{-1/\sqrt{2}} = \frac{y}{1/\sqrt{2}} = r$.
3. $\frac{x}{\sqrt{3}/2} = \frac{y+2}{1/2} = r$.
4. $\left(\frac{-1}{2}, \frac{3\sqrt{3}-2}{2}\right), \left(\frac{-7}{2}, \frac{-3\sqrt{3}-2}{2}\right)$.
5. $(3, -2)$, $(-1, 2)$.

Exercise 10 (*g*)

1. $5x+4y-20=0$.
2. $2x-3y+6=0$.
3. $3x+y+6=0$.
4. $x-y-a=0$.
5. $x+y+3=0$.
6. $x-y+5=0$.

Exercise 10 (*h*)

1. $x+y-3\sqrt{2}=0$.
2. $x-y+4\sqrt{2}=0$.
3. $x-\sqrt{3}y+4=0$.
4. $\sqrt{3}x-y+6=0$.

Exercise 10 (*i*)

1. $y=\frac{-2}{3}x+2$.
2. $y=\frac{1}{2}x+\frac{5}{2}$.
3. $y=\frac{4}{5}x+\frac{1}{5}$.
4. $y=\left(\frac{-3}{7}\right)x+\frac{4}{7}$.
5. $y=-8x+15$.
6. $\frac{x}{3}+\frac{y}{-9}=1$.
7. $\frac{x}{-4/3}+\frac{y}{4}=1$.
8. $\frac{x}{1}+\frac{y}{-1/2}=1$.
9. $\frac{x}{-7/2}+\frac{y}{7}=1$.
10. $\frac{x}{a}+\frac{y}{b}=1$.
11. 2, 3.
12. $\frac{x}{\sqrt{3}-4}+\frac{y}{4\sqrt{3}-3}=1$.
13. $x\cos 4\pi/3+y\sin 4\pi/3=\frac{5}{2}$.
14. $x\cos 11\pi/6+y\sin 11\pi/6=\frac{1}{2}$.
15. $x\cos\alpha+y\sin\alpha=\frac{6}{13}$, where $\cos\alpha=\frac{5}{13}$, $\sin\alpha=\frac{12}{13}$; $\frac{6}{13}$.

Exercise 10 (*j*)

1. $\left(\frac{-8}{5}, \frac{4}{5}\right)$.
2. $(-5, 2)$.
3. $\left(\frac{13}{6}, \frac{13}{6}\right)$.
4. $\left(\frac{a}{m_1m_2}, \frac{a(m_1+m_2)}{m_1m_2}\right)$.
5. $\left(\frac{a\cos\frac{\theta_1+\theta_2}{2}}{\cos\frac{\theta_1-\theta_2}{2}}, \frac{a\sin\frac{\theta_1+\theta_2}{2}}{\cos\frac{\theta_1-\theta_2}{2}}\right)$.
6. $(at_1t_2, a(t_1+t_2))$.
7. $\left(\frac{ab}{a+b}, \frac{ab}{a+b}\right)$.
8. $(5, 1)$; $(-1, -1)$; $(2, 4)$).

9. $(-3, 5)$; $(7, 11)$: $(-1, -1)$.

12. $k=\frac{19}{4}$.

13. $a=1$.

14. $x-2y+2=0$.

15. $4x-13y=0$.

16. $41x-112y-70=0$, $16x-59y-120=0$, $25x-53y+50=0$.

17. $3x-y-3=0$.

Exercise 10 (*k*)

1. $90°$.

2. $\tan^{-1}\frac{4}{3}$.

3. $\tan^{-1}\frac{23}{10}$.

4. $90°$.

5. $\frac{\pi}{3}$.

7. $x+2y=11$.

8. $3y-4x=4$

9. $3x-2y-6=0$.

10. $7x+5y=34$.

11. $22x-11y-53=0$.

12. $4x+8y-1=0$.

13. $2x+5y=0$.

14. $3x+2y-11=0$.

15. $\left(\frac{5}{2}, \frac{5}{2}\right)$.

16. $5x-y=0$; $x-5y=0$; $x+y-3=0$.

Exercise 10 (*l*)

1. $5\sqrt{2}$. 2. $\frac{3}{5}$. 3. 5. 4. $\frac{1}{\sqrt{2}}$. 5. 2.

7. $\frac{36}{\sqrt{10}}, \frac{18}{\sqrt{13}}, \frac{36}{\sqrt{34}}$. 8. $\frac{5}{\sqrt{13}}, \frac{5}{\sqrt{13}}, \frac{5}{\sqrt{2}}$. 9. 1. 10. 3.

Exercise 10 (*m*)

1. (*i*) $x-2y=0$, $x+2y=0$.
(*ii*) $x+y=0$, $x-4y=0$.
(*iii*) $x-y=0$, $4x-y=0$.
(*iv*) $x+5y=0$, $3x+4y=0$.

2. (*i*) $\tan^{-1}\frac{1}{3}$ (*ii*) $0°$ (*iii*) $90°$ (*iv*) 0.

3. 27.

5. $x^2-3xy+2y^2=0$.

Exercise 10 (*n*)

1. (*i*) Read the last term as -88.
(*ii*) Read the last three terms as $-4x-2y+1$.

2. (*i*) -1. (*ii*) 2.
(*iii*) $\frac{15}{2}$ or $\frac{-5}{2}$. (*iv*) -5 or $\frac{-35}{4}$.

3. $x+3y+9=0$; $3x-y+2=0$; $\left(-\frac{3}{2}, -\frac{5}{2}\right)$, $90°$.

4. $2x+5y-1=0,\ 3x-4y+2=0$; $\left(\frac{-6}{23}, \frac{7}{23}\right)$; $\tan^{-1}\frac{23}{14}$.

6. (2, 1).

7. $(-1, 3)$; $\tan^{-1}\frac{\sqrt{5}}{2}$.

8. $(1, -2)$; $\tan^{-1}\frac{2}{11}$.

9. $3x+2y-3=0,\ 2x-5y+2=0$; $\frac{19}{4}$.

10. $\lambda=\frac{5}{2}$ or $\frac{10}{3}$; $2x^2+5xy+2y^2=0,\ 3x^2+10xy+3y^2=0$.

13. Distance $=\frac{4}{5}$.

14. Distance $=\frac{6}{\sqrt{29}}$.

Test Your Understanding X

1. (*c*)	2. (*b*)	3. (*c*)	4. (*b*)	5. (*d*).
6. (*c*)	7. (*a*)	8. (*c*)	9. (*a*)	10. (*a*).

Review Exercise X

1. $6x+11y-8=0$; $3x+20y-33=0$; $9x+2y+17=0$.
2. $3x+y-17=0$.
3. 1 : 1 internally.
4. $4x-3y-1=0$.
5. $x-\sqrt{3}y-4-\sqrt{3}=0$.
6. $x-y-7=0$.
7. 3 : 4.
8. $3x+4y=1$.
9. 10·5.
10. 8.
11. $17x+20y=0$.
12. $\tan^{-1} 2$.
13. $5x+3y+8=0$.
14. $18x+5y+1=0$; $6x+11y-5=0$.
15. $\left(\frac{9}{7}, \frac{-4}{7}\right)$.
16. $\left(\frac{113}{13}, \frac{6}{13}\right)$.
17. 1.
18. $\sqrt{(a^2+b^2)}$.
19. $x=2$; $x=3y+\frac{1}{2}$, $x=2y-3$.

Exercise 11 (*a*)

1. (*i*) $x^2+y^2-6x-4y-12=0$.
   (*ii*) $x^2+y^2-10x+24y=0$.
   (*iii*) $x^2+y^2-2ax-2by=0$.
   (*iv*) $2x^2+2y^2-6x+10y+15=0$.
2. (*i*) Centre (0, 0), radius 4. (*ii*) Centre (0, 3), radius 3.
   (*iii*) Centre (2, 0), radius 2. (*iv*) Centre (4, 3), radius 5.

(*v*) Centre $(-1, -1)$, radius 1.

(*vi*) Centre $\left(\frac{2}{7}, \frac{1}{14}\right)$, radius $\frac{\sqrt{101}}{14}$.

3. Centre $(3, -2)$, radius 7.
4. $\left(\frac{b}{2a}, \frac{c}{2a}\right)$, $\frac{1}{2a}\sqrt{b^2+c^2}$.
5. A circle having centre at $(-3, 3)$ and radius 3.
6. $x^2+y^2-6x+4y+12=0$.
7. $2x^2+2y^2+4x+8y-23=0$.
8. $x^2+y^2+10x-14y+57=0$.
9. $2x^2+2y^2+14x-10y+5=0$.
10. $x^2+y^2-3x+4y=0$.

Exercise 11 (*b*)

1. (*i*) $x^2+y^2-2x-3y=0$.
   (*ii*) $3x^2+3y^2+2x-20y+17=0$.
   (*iii*) $x^2+y^2-13x-13y+52=0$.
   (*iv*) $x^2+y^2-22x-4y+25=0$.
2. $x^2+y^2-2ax-2by=0$.
3. $x^2+y^2-6x-8y+15=0$.
4. $(2, 1)$; $x^2+y^2-4x-2y-5=0$.
5. Centre $(4, 1)$, radius $\sqrt{5}$.
7. $x^2+y^2-5=0$.
8. $x^2+y^2+8x-9=0$, $x^2+y^2-8x-9=0$.
9. $x^2+y^2-2x-4y-95=0$, $x^2+y^2+2x-8y-83=0$.

Exercise 11 (*c*)

1. $x^2+y^2-x-2y+1=0$.
2. $x^2+y^2-5x+3y-22=0$.
3. $x^2+y^2-2x+2y-11=0$; centre $(1, -1)$; radius $\sqrt{13}$.
4. $x^2+y^2-2x-3y=0$.
5. $2x^2+2y^2-6x-18y+35=0$.

Exercise 11 (*d*)

1. $(-1, -1)$.
2. $(0, 2)$; $(1, 0)$.
3. $(-1, 0)$.
4. $x^2+y^2-x-y-2=0$.
5. $x^2+y^2-4y+3=0$.

Exercise 11 (*e*)

1. $5x-12y=169$
2. $x \cos\theta+y \sin\theta=a$.
3. $5x+12y=169$: $12x-5y=169$.
4. $x=4$.
5. $x+2y=\pm 2\sqrt{5}$.
6. $5x+12y+169=0$, $5x+12y-169=0$.
7. $x+y+3\sqrt{2}=0$, $x+y-3\sqrt{2}=0$.
8. $4x+3y+5=0$, $4x+3y-25=0$.

9. $x-\sqrt{3}y+10=0$, $x-\sqrt{3}y-10=0$.

10. $p=\pm a$. 11. $n^2=a^2(l^2+m^2)$.

12. 3. 13. 5.

Exercise 11 (*f*)

1. 16. 2. -15.

3. $x^2+y^2-6x+12y+12=0$. 4. $8x-13y-2=0$.

5. $7x-5y-2=0$. 6. $x-y-3=0$.

7. $2x-5y+4=0$. 8. $x=0$.

Exercise 11 (*g*)

1. $3x^2+3y^2-6x-8y=0$. 2. $x^2+y^2+4x+10y=0$.

3. $3x^2+3y^2-8x+2y=0$.

Exercise 11 (*h*)

3. $x^2+y^2-20x+20y+55=0$. 4. $x^2+y^2+6x-3y=0$.

5. $x^2+y^2-7x-9y+1=0$.

6. $31x^2+31y^2+90x-12y-109=0$.

7. $x^2+y^2+6x+4y-4=0$. 8. $x^2+y^2-16x-18y-4=0$.

9. $x^2+y^2+2x+2y+1=0$. 10. $x^2+y^2-6x-6y+9=0$.

Test Your Understanding XI

1. (*b*) 2. (*c*) 3. (*b*) 4. (*d*) 5. (*b*)

6. (*a*) 7. (*b*) 8. (*d*) 9. (*b*) 10. (*b*)

Review Exercise XI

1. $15x^2+15y^2-94x+18y+55=0$.

4. $x-4=0$. 5. $x^2+y^2-6x-8y=0$.

6. $x^2+y^2-17x-19y+50=0$. 7. $3y=2x+13$.

8. $2x^2+2y^2+2x+6y+1=0$. 10. $p=-2, q=-1$.

Exercise 12 (*a*)

1. $y^2-4x-16=0$. 2. $(-3/2, 0)$; $2x-3=0$.

3. $(-2, -5)$; $x+4=0$. 4. 9 ; 13/4.

5. $(-2, -4)$: $(-3/4, -4)$; 5.

6. $(1/3, 0)$; 4/3. 7. $x^2=4(1-y)$.

8. $(6, \pm 4\sqrt{3})$.

10. (*i*) $(-a, 0)$; $x-a=0$. (*ii*) $(0, a)$, $y+a=0$.

Exercise 12 (*b*)

1. $16x-8y+3=0$; $\left(\frac{3}{16}, \frac{3}{4}\right)$.

2. $98x+70y+75=0$; $\left(\frac{75}{98}, -\frac{15}{7}\right)$.

3. $3x+2y+1=0,\ x+2y+3=0.$
4. $2x-y+2=0,\ (1, 4)\ ;\ x+2y+16=0,\ (16, -16).$
5. $\left(\frac{1}{3}, \frac{4}{3}\right).$ 6. $(1, 1).$

Exercise 12 (*c*)

1. $16x^2+400y^2=25.$
2. $189x^2-96xy+161y^2-1230x-1640y+5525=0.$
3. $9x^2+12y^2=256.$ 4. $5x^2+9y^2=180.$
5. $\left(0, \pm\frac{1}{2\sqrt{5}}\right), y=\pm\frac{\sqrt{5}}{2}, \frac{1}{\sqrt{5}}, \frac{4}{5}.$
6. $\frac{1}{\sqrt{3}}, \left(\pm\frac{1}{\sqrt{6}}, 0\right), x=\pm\sqrt{\frac{3}{2}}, \frac{2\sqrt{2}}{3}$
7. $\left(\pm\frac{2}{5}, \pm\frac{1}{2\sqrt{5}}\right), \left(\pm\frac{1}{\sqrt{6}}, \pm\frac{\sqrt{2}}{3}\right).$
8. $(5, 3), \sqrt{\frac{3}{4}}, (5\pm2\sqrt{3}, 3), 2, x=\frac{15\pm8\sqrt{3}}{3}.$
9. $\frac{1}{2}, \left(1, -1\pm\frac{\sqrt{6}}{12}\right), \frac{\sqrt{6}}{4}.$
10. $\frac{1}{\sqrt{3}}.$ 11. $12, 8\sqrt{2}.$
12. (*i*) Outside (*ii*) Inside (*iii*) On the ellipse.

Exercise 12 (*d*)

1. $y=x\pm5.$ 2. $\sqrt{3}x-y\pm7=0.$
3. $3x+y\pm2\sqrt{7}=0.$
4. Read $16x^2+9y^2=144$ for $4x^2+9y^2=36.$ $\left(\frac{3}{\sqrt{2}}, 2\sqrt{2}\right).$
5. $\pm\sqrt{\frac{5}{6}}.$

Exercise 12 (*e*)

1. $3x^2-8xy-3y^2-24x-38y+63=0.$
2. Read 'standard equation' for 'equation'. $x^2-y^2=32.$
3. Read 'standard equation' for equation and (6, 0) for (0, 0). $\frac{x^2}{16}-\frac{y^2}{20}=1.$
4. $\frac{9x^2}{16}-\frac{9y^2}{20}=1.$
5. Read 'standard equation' for 'equation' $16x^2-2y^2=1.$

6. $\frac{5}{3}$, $(\pm 5, 0)$, $5x=\pm 9$, $\frac{32}{3}$.

7. $\frac{\sqrt{41}}{5}$; $\left(\pm\sqrt{41}, 0\right)$; $\left(\sqrt{41}\ x=\pm 25\right)$; $\frac{32}{5}$;
$\left(\pm\sqrt{41}, \pm\frac{16}{5}\right)$.

8. 6, 4; $\frac{\sqrt{13}}{3}$; $(\pm\sqrt{13}, 0)$; $2\frac{2}{3}$.

9. 8, 6; $(-4, -1)$; $(1, -1)$, $(-9, -1)$; $\frac{9}{2}$, $\frac{5}{4}$; $5x-36=0$,
$5x+4=0$.

10. 6, 8; $(3, 4)$; $(8, 4)$, $(-2, 4)$; $\frac{32}{3}$; $\frac{5}{3}$; $5x-24=0$, $5x-6=0$.

Exercise 12 (*f*)

1. $(5, 4)$.
2. $30x-24y\pm\sqrt{161}=0$.
3. $4x-3y\pm 3\sqrt{39}=0$.
4. $\left(\frac{-5\sqrt{2}}{2}, \frac{-3\sqrt{2}}{2}\right)$
5. $(1, 1)$.

Test Your Understanding XII

1. (*d*) 2. (*c*) 3. (*d*) 4. (*b*)
5. Read $\sqrt{7}/4$ instead of $\sqrt{7}/3$ (*c*).
6. (*d*) 7. (*d*) 8. (*d*) 9. (*c*) 10. (*b*).

Review Exercise XII

1. $(5, 0)$, 20.
2. $y^2=16x$.
3. $y=3x+\frac{1}{2}$; $\left(\frac{1}{6}, 1\right)$.
4. $\frac{-1}{2}$.
5. $\frac{\sqrt{5}}{3}$.
6. $\frac{3}{5}$.
7. $\pm\sqrt{\frac{7}{24}}$.
8. $6x+3y\pm 2\sqrt{3}=0$
9. $\frac{5}{4}$.

Exercise 13 (*a*)

1. (*a*) 1 (*b*) 1 (*c*) 1 (*d*) 1
2. (*a*) $\frac{1}{2\sqrt{2}}(\sqrt{3}+1)$ (*b*) $\frac{1}{2\sqrt{2}}(1-\sqrt{3})$

Exercise 13 (*b*)

2. (*i*) $2\cos 4x\cos x$ (*ii*) $2\sin x\sin 3x$
(*iii*) $2\sin 5x\cos 2x$ (*iv*) $2\cos(2y+x)\sin(2y-x)$
(*v*) $2\cos 2x\sin y$ (*vi*) $2\sin 3x\sin 5y$.
3. (*i*) $\sin 5x-\sin x$ (*ii*) $\sin 5x+\sin x$
(*iii*) $\cos 10x+\cos 2x$ (*iv*) $\cos 2x-\cos 8x$.

Exercise 13 (*c*)

1. (*i*) $\tan\frac{2\pi}{5}$ (*ii*) $\cos\frac{2\pi}{9}$
(*iii*) $\cot\frac{3\pi}{7}$ (*iv*) $\sin\frac{\pi}{4}$.

Exercise 13 (*d*)

1. (*i*) $12°15'36''$ (*ii*) $37°37'12''$ (*iii*) $42°9'0''$
2. (*i*) $19°{\cdot}29$ (*ii*) $3°{\cdot}26$ (*iii*) $2°{\cdot}88$.
3. 4th, 2nd, 2nd, 1st, 4th. 4. 75°.
5. 60°, 30°, 45°, 9°.
$114°{\cdot}58$, $171°{\cdot}86$, $286°{\cdot}44$, $17°{\cdot}19$, $572°{\cdot}88$ (correct to two places of decimals).
6. 4·19, 1·05, ·79, ·52, ·31, 1·83 (correct to two places of decimals).
7. 10·47 *cm* (correct to two places of decimals).
8. 6π or 18·85 radians, correct to two places of decimals.
9. one. 10. 8·4
11. 1131428·57 *cm* 12. 5 : 4
13. $1°{\cdot}44$ or $\cdot025°$ per second
14. 30 radians per second 15. 37775 km.

Exercise 13 (*e*)

1. (*i*) $(\sqrt{3}+1)/2\sqrt{2}$ (*ii*) $(1-\sqrt{3})/2\sqrt{2}$
2. $\sin 330°=-\frac{1}{2}$, $\cos 330°=\frac{\sqrt{3}}{2}$, $\tan 330°=-\frac{1}{\sqrt{3}}$
$\cot 330°=-\sqrt{3}$, $\sec 330°=\frac{2}{\sqrt{3}}$, $\csc 330°=-2$.
5. (*i*) $\frac{1}{2}$ (*ii*) $\frac{\sqrt{3}}{2}$ (*iii*) $\frac{1}{\sqrt{2}}$ (*iv*) $\frac{1}{2}$.
6. (*i*) $\sin 26°$ (*ii*) $\cos 50°$ (*iii*) $\sin 45°$ (*iv*) $\cos 34°$.
7. (*i*) $-\frac{\sqrt{3}}{2}$ (*ii*) $-1/\sqrt{2}$ (*iii*) $-1/\sqrt{3}$
(*iv*) 1 (*v*) -2 (*vi*) -2
(*vii*) 0 (*viii*) $-\frac{3}{2}$.

8. (*i*) $-\cos\theta$ (*ii*) $-\sin\theta$ (*iii*) $-\tan\theta$ (*iv*) $\cot\theta$
(*v*) $-\sec\theta$ (*vi*) $\sec\theta$ (*viii*) $-\tan\theta$.

9. (*i*) $\cos 19°$ (*ii*) $-\cos 27°$ (*iii*) $\cot 25°$
(*iv*) $-\cot 42°$ (*v*) $-\csc 13°$ (*vi*) $\sec 24°$.

10. $30°, 330°$. **11.** $240°$.

12. (*i*) $\sin 90°+\sin 30°$ (*ii*) $\sin 100°-\sin 40°$
(*iii*) $\cos 60°+\cos 30°$ (*iv*) $\cos 20°-\cos 100°$.

13. (*i*) $2 \sin 36° \cos 12°$ (*ii*) $2 \cos 42° \sin 30°$.

Exercise 13 (*g*)

1. (*i*) ·4253 (*ii*) ·6648 (*iii*) ·6168
(*iv*) ·6371 (*v*) 1·241 (*vi*) 1·167
(*vii*) ·9644 (*viii*) ·9605 (*ix*) 2·229 (*x*) 1·520

2. (*i*) 7°10′ (*ii*) 14°50′ (*iii*) 58°30′
(*iv*) 55°10′ (*v*) 67°20′ (*vi*) 54°30′
(*vii*) 67°50′ (*viii*) 69°20′ (*ix*) 26°30′
(*x*) 59°40′.

Exercise 13 (*h*)

1. (*i*) ·4754 (*ii*) ·6187 (*iii*) ·8371 (*iv*) ·8806
(*v*) 2·5 (*vi*) 1·020 (*vii*) ·7300 (*viii*) ·8313
(*ix*) 1·517 (*x*) 1·131 (*xi*) 1·037 (*xii*) 1·712.

2. (*i*) 3°36′ (*ii*) 57°27′ (*iii*) 23°24′ (*iv*) 31°26′
(*v*) 66°47′ (*vi*) 45°48′ (*vii*) 71°25′ (*viii*) 66°6′
(*ix*) 29°36′ (*x*) 67°24′ (*xi*) 56°44′ (*xii*) 27°15′.

Test Your Understanding XIII

1. (*c*) **2.** (*b*) **3.** (*c*) **4.** (*a*) **5.** (*a*)
6. (*a*) **7.** (*b*) **8.** (*d*) **9.** (*a*) **10.** (*c*)

Test Your Understanding XIV

1. (*a*) **2.** (*c*) **3.** (*b*) **4.** (*b*) **5.** (*d*)
6. (*a*) **7.** (*d*) **8.** (*a*) **9.** (*b*) **10.** (*d*).

Exercise 15 (*b*)

27. (*i*) 30 *sq. cm.* (*ii*) 336 *sq. cm.* (*iii*) 2430 *sq. cm.*
(*iv*) $\frac{1}{4}(3+\sqrt{3})$ *sq. cm.* (*v*) 30 *sq. cm.*

28. 5 cm.

Exercise 15 (*a*)

1. (*i*) $A=60°$. $B=45°$, $C=75°$.
(*ii*) $A=120°$, $B=45°$, $C=15°$.

	(*iii*) $A=49°$,	$B=59°$,	$C=72°$.
	(*iv*) $A=20°55'$,	$B=25°50'$,	$C=132°35'$
2.	$x>1$.		
3.	(*i*) $A=84°$,	$B=34°$,	$C=62°$.
	(*ii*) $A=43°$,	$B=61°$,	$C=76°$.
	(*iii*) $A=33°$,	$B=53°$,	$C=95°$.
5.	(*i*) $A=56°$,	$B=60°$,	$C=64°$.
	(*ii*) $A=60°$,	$B=43°20'$,	$C=76°40'$.
	(*iii*) $A=60°9'$,	$B=43°17'$,	$C=76°34'$.
	(*iv*) $A=38°56'$,	$B=57°58'$,	$C=83°6'$.

Exercise 15 (*c*)

1.	(*i*) $a=1$,	$B=30°$,	$C=120°$.
	(*ii*) $a=\sqrt{6}$,	$B=105°$,	$C=15°$.
	(*iii*) $c=13$,	$A=31°$,	$B=112°$.
	(*iv*) $b=15$,	$A=28°$,	$C=117°$.
3.	(*i*) $a=436$,	$B=78°15'$,	$C=49°35'$.
	(*ii*) $c=27{\cdot}29$,	$A=91°54'$,	$B=37°18'$.
	(*iii*) $b=199°1'$,	$A=83°7'$,	$C=42°17'$
	(*iv*) $c=226{\cdot}9$,	$A=39°45'$,	$B=73°35'$
4.	$A=94°40'$,	$B=25°20'$.	

Exercise 15 (*d*)

1.	(*i*) $A=106°$,	$b=15$,	$c=120$.
	(*ii*) $C=47°$,	$a=187$,	$c=139$.
	(*iii*) $B=97°50'$.	$a=169$,	$b=234$.
	(*iv*) $C=82°30'$,	$b=103$,	$c=172$.
3.	(*i*) $C=72°$,	$a=85$,	$c=82$.
	(*ii*) $A=78°10'$,	$a=144$,	$c=83$.
	(*iii*) $C=65°45'$,	$b=22{\cdot}66$,	$c=21{\cdot}63$.
	(*iv*) $A=33°49'$,	$a=16{\cdot}18$,	$c=25{\cdot}63$.

Exercise 15 (*e*)

1.	(*i*) $B=35°$,	$C=37°$,	$c=18$.
	(*ii*) $B=22°$,	$C=129°$,	$c=51$.
	(*iii*) $A_1=58°$,	$C_1=85°$,	$c_1=81$.
	and $A_2=122°$,	$C_2=21°$,	$c_2=29$.
	(*iv*) No solution		
	(*v*) No solution		
	(*vi*) $B_1=59°$,	$C_1=89°$,	$c_1=83$.
	and $B_2=121°$	$C_2=27°$,	$c_2=37$.
	(*vii*) $B=5°50'$,	$C=108°40'$,	$c=783$.
	(*viii*) $B=37°23'$,	$C=91°27'$,	$c=41{\cdot}34$.

(ix)	$B_1=67°20'$,	$C_1=60°10'$,	$c_1=247{\cdot}1$.
and	$B_2=112°40'$,	$C_2=23°50'$,	$c_2=106{\cdot}9$.
(x)	$B=50°10'$,	$C=28°30'$,	$c=11{\cdot}5$.
(xi)	$B_1=66°40'$,	$C_1=79°40'$,	$c_1=581{\cdot}2$.
and	$B_2=113°20'$,	$C_2=33°$,	$c_2=321$.
(xii)	No solution	(xiii) No solution	
(xiv)	$B=43°51'$,	$C=83°57'$,	$c=36{\cdot}95$.

Exercise 15 (*f*)

1. 2·12 metres.
2. 14°.
3. 29 metres.
4. 87·1 metres.
5. 444 metres
6. 430 metres.
7. height=35 metres, distance=20 metres.
8. 80 *m*.
9. height=60 *m*, distance=35 *m*.
10. The point is at a distance of 1205 m from one pole and the height of the pole is 21·7 metres.
11. 12 metres.
12. 15 minutes.
13. 33·3 metres.
14. 4·2 metres.
15. 65°.
16. 261 metres.

Exercise 15 (*g*)

1. 45 metres.
2. 7·53 *km* from P and 9·69 *km* from Q.
3. 43 *km*.
4. 54 metres
5. 620 metres from A, 440 metres from B, and 370 metres is the height of the balloon.
6. 8 *m*.
7. 8·8 *m*.
8. 62 *m*.
9. 11 metres.
10. 840 metres.
11. Elevation of the Sun=37° and angle made by hillside with the horizontal=2°.
12. $l \sin(\alpha-\gamma)\sin(\beta-\alpha)\csc(\beta+\gamma)\sec\alpha_1$, if the points are on the same side of the tree;

    $l \sin(\alpha-\gamma)\sin(\beta-\alpha)\csc(\beta-\gamma)\sec\alpha$, if the points are on opposite sides of the tree.
13. R can be either 290 *km* away or 140 *km* away from Q.
14. 13°.
15. 60 metres.

Test Your Understanding XV

1. (*a*)	2. (*c*)	3. (*c*)	4. (*b*)	5. (*b*)
6. (*d*)	7. (*b*)	8. (*c*)	9. (*a*)	10. (*c*).

Review Exercise XV

6. 30° 7. 120°.

Exercise 16 (a)

1. (i) $-\frac{\pi}{2}$, (ii) $\frac{\pi}{6}$, (iii) $\frac{\pi}{4}$, (iv) $-\frac{\pi}{3}$,
(v) π, (vi) $-\frac{\pi}{4}$, (vii) $\frac{5\pi}{6}$, (viii) $\frac{2\pi}{3}$.

2. $\frac{12}{13}$. 3. $\frac{4}{5}$. 4. $\frac{3}{5}$. 5. $\frac{15}{8}$.

6. $-\frac{24}{7}$. 7. $\frac{13}{5}$. 8. $-\frac{63}{65}$.

9. $\frac{140}{171}$. 10. $\sqrt{3}$. 13. Read $\sin^{-1}\frac{11}{13}$ for $\cos^{-1}\frac{11}{13}$.

Exercise 16 (b)

1. (i) $-\frac{\pi}{4}$, (ii) $-\frac{\pi}{3}$, (iii) $\frac{\pi}{6}$, (iv) $\frac{\pi}{4}$.

2. $\frac{24}{25}$. 3. $\frac{5}{\sqrt{11}}$. 4. $\frac{-5}{\sqrt{74}}$. 5. $-\frac{\sqrt{96}}{5}$.

6. $-\frac{7}{24}$. 7. $-\frac{8}{15}$. 13. $\frac{36}{85}$. 14. $-\frac{240}{161}$.

15. $\frac{63}{65}$.

Exercise 16 (c)

1. (i) π, (ii) $\frac{\pi}{3}$, (iii) $\frac{3\pi}{4}$, (iv) $\frac{\pi}{6}$,
(v) $\frac{\pi}{2}$, (vi) $-\frac{\pi}{6}$, (vii) $-\frac{\pi}{3}$, (viii) $\frac{\pi}{4}$.

2. $\frac{5}{4}$. 3. $\frac{17}{8}$. 4. $\frac{13}{12}$. 5. $\frac{13}{12}$.

6. $\frac{17}{15}$. 7. $\frac{25}{24}$. 8. $\frac{25}{24}$. 9. $-\frac{17}{8}$.

10. $\frac{16}{65}$.

Exercise 16 (d)

1. $x \in \{n\pi : n \in \mathbf{Z}\}$. 2. $x \in \left\{\frac{1}{2}(2n+1)\pi : n \in \mathbf{Z}\right\}$.

3. (i) $x \in \left\{n\pi+(-1)^n \frac{\pi}{6} : n \in \mathbf{Z}\right\}$.

(ii) $x \in \left\{n\pi-(-1)^n \frac{\pi}{3} : n \in \mathbf{Z}\right\}$.

(*iii*) $x \in \{2n\pi : n \in \mathbf{Z}\}$.

(*iv*) $x \in \left\{2n\pi \pm \frac{2}{3}\pi : n \in \mathbf{Z}\right\}$.

Exercise 16 (*e*)

1. (*i*) $\left\{n\pi + (-1)^n \frac{\pi}{3} : n \in \mathbf{Z}\right\}$,

(*ii*) $\left\{2n\pi \pm \frac{2}{3}\pi : n \in \mathbf{Z}\right\}$,

(*iii*) $\left\{n\pi + \frac{\pi}{4} : n \in \mathbf{Z}\right\}$,

(*iv*) $\left\{n\pi - \frac{\pi}{6} : n \in \mathbf{Z}\right\}$,

(*v*) $\left\{2n\pi \pm \frac{\pi}{3} : n \in \mathbf{Z}\right\}$,

(*vi*) $\left\{n\pi - (-1)^n \frac{\pi}{6} : n \in \mathbf{Z}\right\}$.

2. (*i*) $\left\{\frac{n\pi}{2} + \frac{\pi}{12} : n \in \mathbf{Z}\right\}$.

(*ii*) $\left\{\frac{n\pi}{3} - \frac{1}{3}\cot^{-1} 2 : n \in \mathbf{Z}\right\}$.

3. (*i*) $x \in \left\{2n\pi \pm \frac{2\pi}{3} : n \in \mathbf{Z}\right\}$,

(*ii*) $x \in \left\{n\pi - (-1)^n \frac{\pi}{4} : n \in \mathbf{Z}\right\}$.

Exercise 16 (*f*)

1. $x \in \left\{n\pi \pm \frac{\pi}{3} : n \in \mathbf{Z}\right\}$.

2. $x \in \left\{n\pi \pm \frac{\pi}{6} : n \in \mathbf{Z}\right\}$.

3. $x \in \left\{n\pi \pm \frac{\pi}{3} : n \in \mathbf{Z}\right\}$.

4. $x \in \left\{n\pi - (-1)^n \frac{\pi}{6} : n \in \mathbf{Z}\right\}$

5. $x \in \left[\{(2n+1)\pi : n \in \mathbf{Z}\} \cup \left\{2n\pi \pm \frac{2}{3}\pi : n \in \mathbf{Z}\right\}\right]$.

6. $x \in \left[\left\{n\pi + \frac{\pi}{4} : n \in \mathbf{Z}\right\} \cup \left\{n\pi - \tan^{-1}\frac{1}{3}\pi : n \in \mathbf{Z}\right\}\right]$

7. $x \in \left[\{2n\pi : n \in \mathbf{Z}\} \cup \left\{2n\pi \pm \cos^{-1}\left(-\frac{1}{4}\right) : n \in \mathbf{Z}\right\}\right]$.

8. $x \in \left\{ 2n\pi \pm \frac{\pi}{3} : n \in \mathbf{Z} \right\}$.

9. $x \in \left\{ 2n\pi \pm \frac{\pi}{3} : n \in \mathbf{Z} \right\}$.

10. $x \in \left\{ \frac{n\pi}{2} \pm \frac{\pi}{12} : n \in \mathbf{Z} \right\}$.

Exercise 16 (*g*)

1. $x \in \left\{ \frac{n\pi}{5} : n \in \mathbf{Z} \right\}$.

2. $x \in \left[\{n\pi : n \in \mathbf{Z}\} \cup \left\{ (2n+1)\frac{\pi}{8} : n \in \mathbf{Z} \right\} \right]$.

3. $x \in \{n\pi : n \in \mathbf{Z}\}$.

4. $x \in \left[\left\{ (4n-1)\frac{\pi}{2} : n \in \mathbf{Z} \right\} \cup \left\{ (4n-1)\frac{\pi}{10} : n \in \mathbf{Z} \right\} \right]$.

5. $x \in \left[\left\{ \frac{2n\pi}{p+q} : n \in \mathbf{Z} \right\} \cup \left\{ \frac{(2n+1)\pi}{p-q} : n \in \mathbf{Z} \right\} \right]$.

6. $x \in \left\{ \frac{2k+1}{m+n} \frac{\pi}{2} : k \in \mathbf{Z} \right\}$.

7. $x \in \left[\left\{ \frac{(2n+1)\pi}{4} : n \in \mathbf{Z} \right\} \cup \left\{ 2n\pi \pm \frac{\pi}{3} : n \in \mathbf{Z} \right\} \right]$.

8. $x \in \left[\left\{ \frac{n\pi}{3} : n \in \mathbf{Z} \right\} \cup \left\{ \frac{n\pi}{2} \pm \frac{\pi}{12} : n \in \mathbf{Z} \right\} \right]$.

9. $x \in \left[\left\{ (2n+1)\frac{\pi}{4} : n \in \mathbf{Z} \right\} \cup \left\{ 2n\pi \pm \frac{2\pi}{3} : n \in \mathbf{Z} \right\} \right]$.

10. $x \in \left[\left\{ \frac{n\pi}{3} : n \in \mathbf{Z} \right\} \cup \left\{ n\pi \pm \frac{\pi}{3} : n \in \mathbf{Z} \right\} \right]$.

11. $x \in \left[\left\{ \frac{2n\pi}{5} : n \in \mathbf{Z} \right\} \cup \left\{ \frac{(2n+1)\pi}{2} : n \in \mathbf{Z} \right\} \cup \{(2n+1)\pi : n \in \mathbf{Z}\} \right]$.

Exercise 16 (*h*)

1. $x \in \left\{ 2n\pi \pm \frac{\pi}{4} + \frac{\pi}{6} : n \in \mathbf{Z} \right\}$.

2. $x \in \left\{ 2n\pi \pm \frac{\pi}{3} - \frac{\pi}{3} : n \in \mathbf{Z} \right\}$.

3. $x \in \left\{ 2n\pi \pm \frac{\pi}{4} + \frac{\pi}{4} : n \in \mathbf{Z} \right\}$.

4. $x \in \left\{ n\pi + (-1)^n \frac{\pi}{2} + \frac{\pi}{4} : n \in \mathbf{Z} \right\}$.

5. $x \in \left\{ n\pi + (-1)^n \frac{\pi}{4} + \frac{\pi}{6} : n \in \mathbf{Z} \right\}$.

6. $x \in \left\{ 2n\pi \pm \frac{2\pi}{3} + \frac{\pi}{3} : n \in \mathbf{Z} \right\}$.

Exercise 16 (*i*)

1. $x \in \left\{ 2n\pi + \frac{2\pi}{3} : n \in \mathbf{Z} \right\}$.
2. $x \in \left\{ 2n\pi + \frac{5\pi}{6} : n \in \mathbf{Z} \right\}$.
3. $x \in \left\{ 2n\pi - \frac{\pi}{4} : n \in \mathbf{Z} \right\}$.
4. The solution-set is φ.
5. $x \in \left\{ 2n\pi + \frac{2\pi}{3} : n \in \mathbf{Z} \right\}$.
6. $x \in \left\{ 2n\pi - \frac{\pi}{6} : n \in \mathbf{Z} \right\}$.
7. $x \in \left\{ 2n\pi - \frac{\pi}{3} : n \in \mathbf{Z} \right\}$.

Test Your Understanding XVI

1. (*d*) 2. (*a*) 3. (*c*) 4. (*a*) 5. (*b*)
6. (*c*) 7. (*c*) 8. (*b*) 9. (*c*)
10. Read $\cos x - \sin x$ for $\cos x + \sin x$. (*a*).

Review Exercise XVI

1. $-\frac{1}{2}$.
2. $x = \frac{1}{3}$.
3. $\frac{140}{171}$.
4. $\frac{x^2}{1+x^2}$.
6. $x \in \left\{ (2n+1)\frac{\pi}{4} : n \in \mathbf{Z} \right\}$.
7. $x \in \left\{ n\pi + \frac{\pi}{4} \pm \frac{\pi}{6} : n \in \mathbf{Z} \right\}$.
8. $x \in \left\{ 2n\pi \pm \frac{\pi}{3} : n \in \mathbf{Z} \right\}$.
9. $x \in \left\{ n\frac{\pi}{2} + \frac{\pi}{8} : n \in \mathbf{Z} \right\}$.
10. $x \in \left\{ (2n+1)\frac{\pi}{4} : n \in \mathbf{Z} \right\}$.

Exercise 17 (*a*)

1. (*a*), (*b*), (*h*), (*i*), (*j*) and (*k*).
2. Weight as a variable and 2·5 *kg* as one of its values ; length as a variable and 110 *cm* as one of its values ; nationality as a variable and Christian as one of its values.
3. (*a*) A group of students. (*b*) A set of books. (*c*) A collection of flowers. (*d*) A collection of constituencies.

Exercise 17 (*b*)

4. (*a*) No ; Yes. (*b*) Yes.
5. (*a*) 800. (*b*) 1000. (*c*) 100. (*d*) 76. (*e*) 29·5. (*f*) 19.
9. Lower.

Exercise 17 (*c*)

1. 12, 20, 29, 30, 7.
2. The proportion of smokers in the age-groups '0—20' and '61 and above' is higher in village A. For the other age-groups, it is the other way round.
3. Except for the class 12—20, the proportion is higher in village A. The two villages are almost similar as regards buffaloes in the class 6—9.

Exercise 17 (*e*)

1. 4. **2.** Rs. 207 **3.** 7. **4.** A. **5.** Rs. 27·85.
6. 4. **7.** 19·21.
10. 4·24 hectares per farmer excluding those who own 9 hectares or more of land.
11. 34·4. **12.** 145·7. **13.** 27·85. **14.** 156 cm.
15. Rs. 1350 **16.** 153 cm.

Exercise 17 (*f*)

1. 130. **2.** 108. **3.** 7. **4.** 8. **5.** 35.
6. 13·5. **7.** 25. **8.** Rs. 46·72.
10. 59·35 marks. **11.** 10·89 marks. **12.** 26·74

Exercise 17 (*g*)

1. 34. **2.** 25. **3.** 28. **4.** 9·34. **5.** 20·57.

Exercise 17 (*h*)

1. 2·72 **2.** 4·85 **3.** 15·47 **4.** 2·24 **5.** 3·27
6. S.D.=Rs. 17·26 ; C.V.=15·86 **7.** 11·84 years
8. Player A is more consistent. His C.V. is less than that of player B.
9. Mean=35·2, S.D.=7·210 **10.** Mean=9·9, S.D.=·15.
11. Mean= 16·79 , S.D.= 6·75.
12. Mean=107·01 , S.D.=36·9.
13. Mean= 56·4 , S.D.=12·4.
14. Mean= 56·15 , S.D.=16·3.
15. Mean= 8·84 , S.D.= 2·62.
16. Mean= 54·91 , median=54·9, S.D.=11·86.
17. Mean=110·4 , S.D.= 1·75.

18. Mean= 40·16 , S.D.=19·75.
19. Mean= 33·104, S.D.=10·075.
20. S.D.=11·28.

Test Your Understanding XVII

1. (*d*)	2. (*c*)	3. (*d*)	4. (*b*)	5. (*b*)
6. (*b*)	7. (*a*)	8. (*c*)	9. (*c*)	10. (*b*).

Review Exercise XVII

1. (*a*) Many periods have been listed and counted more than once, *e.g.*, sleeping time is included in all the holidays as well. It is an incorrect tabulation of data and hence leads to fallacious conclusion.

   (*b*) It is the proportion which should be compared and not the actual number of accidents. Certainly railway travel was much more limited in 1929 as compared to that in 1979. Hence the *actual* number of accidents in 1929 should be less than that in 1979.

   (*c*) No conclusion can be drawn from this data unless similar information is available about the rest of the people.

   (*d*) Similar argument as in (*b*) above.

   (*e*) The wife must have banged the statistician. He chose the wrong statistics for comparison. He should have compared the maximum depth with the least height for safe crossing. It is such injudicious use and misleading advertisements as in (*c*) above that make people say that *statistics lie*.

2. Mean=3·55 per score ; median=4.

3. Mean burning time=480 minutes per candle ; median time=481 minutes.

4. Mean mass=1001·6 g per packet ; S.D.=1·43 g.

5. Mean=1·39 mistakes per page ; S.D.=1·28 mistakes.

6. 25, 39, 18, 11, 3, 2, 1. No. The values are only approximations to actual values.

9. *Class*

Class	...... and below : 1—3	3—5	5—7
Frequency :	41	37	42

Mean=3·52 per score ; meadian=4·03. Earlier values are better. Later values are only approximations.

11. Mean=26·5 ; median=26·54 ; mode=25·42.

12. Mean$=125{\cdot}65$; median$=128{\cdot}14$; variance$=156{\cdot}6$.
13. Mean$=154{\cdot}71$; median$=154{\cdot}86$; variance$=51{\cdot}04$.
14. Mean$=153{\cdot}45$; median$=153{\cdot}39$; S.D.$=6{\cdot}58$.
15. Mean$=60$; median$=65{\cdot}8$; S.D.$=14$.
16. Mean$=107{\cdot}01$; median$=108{\cdot}75$; S,D.$=36{\cdot}9$.
17. Mean$=54{\cdot}91$; median$=54{\cdot}9$: S.D.$=11{\cdot}86$.
18. Mean$=110$; median$=110$; S.D.$=43{\cdot}54$.
19. Mean$=68{\cdot}375$ kN ; S.D.$=4{\cdot}89$.
20. Mean deviation from the mean$=6$, S.D.$=7{\cdot}5$.

Exercise 18 (*b*)

1. $x=2{\cdot}5, y=35$; $P=37{\cdot}5$. 2. $x=3, y=2$; $C=1500$.
3. Minimize $C=5x+6y$ subject to $60x+67y\geqslant1250$, $300x+353y\geqslant3500$, $2x+y\geqslant60$ and $x\geqslant0$, $y\geqslant0$.
4. Maximize $7{\cdot}5x_1+3x_2$ subject to $x_1+3x_2\leqslant12$, $3x_1+x_2\leqslant12$ and $x_1\leqslant0$, $x_2\geqslant0$ 3 packets of each. Maximum daily profit $=$Rs. $31{\cdot}50$.
5. Maximize $2x_1+1{\cdot}5x_2$ subject to the constrains $2x_1+x_2\leqslant1000$, $x_1+x_2\leqslant800$, $x_1\leqslant400$, $x_2\leqslant700$ and $x_1\geqslant0$, $x_2\geqslant0$. 200 belts of type A and 600 of type B.

Test Your Understanding XVIII

1. (*d*)	2. (*b*)	3. (*a*)	4. (*d*)	5. (*c*).
6. (*d*)	7. (*c*)	8. (*a*)	9. (*a*)	10. (*c*).

Review Exercise XVIII

3. $x+y\geqslant5$, $y<0$ and $x\leqslant0$. 4. $x+y>2$.
5. $x<0$ and $y<0$.
8. Maximize $2x_1+3x_2$ subject to $x_1+x_2\geqslant10$ and $2x_1+2x_2<15$, $x_1\geqslant0$, $x_2\leqslant0$.
10. 1 jar of X and 5 jars of Y.

Assorted Problems

1. 44.
2. Mathematics alone 60, physics alone 35, chemistry alone 13. Twenty-two students did not offer any of the three subjects.

3. R is transitive, but is neither reflexive nor symmetric; S is symmetric but is neither reflexive nor transitive; T is reflexive but is neither symmetric nor transitive.

4. Domain$=\{1, 2, \ldots\ldots, 20\}$, Range$=\{1, 3, 5, \ldots\ldots, 39\}$. R is not reflexive, not symmetric, not transitive.

5. (*i*) Function (*ii*) Not a function.

6. Domain$=\mathbf{R}\sim[-1, 1]$; Range$=]0, \infty[$.

8. $x=2, y=3$; $x=-2, y=\frac{1}{3}$.

9. $\sqrt{(-1)(-1)}\neq\sqrt{(-1)}\times\sqrt{(-)}$.

10. If $a\leqslant\sqrt{2}-1$, $z=a+i[-1\pm\sqrt{1-2a-a^2}]$: if $a>\sqrt{2}-1$, there is no complex number satisfying the given equation.

11. $\mathbf{R}\sim]5, 9[$. 12. 4. 13. $2\pm i\sqrt{2}, 1\pm i$.

14. $3\pm\sqrt{17}$, $\frac{1}{2}(-3\pm i\sqrt{7})$, 15. $x=-\frac{1}{4}$.

16. 1540 balls. 17. $x=10^5, y=10$.

18. Either 2, 5, 8, ... ; 3, 6, 12, ...

or $\frac{25}{2}, \frac{79}{6}, \ldots\ldots\ldots, \frac{2}{3}, \frac{25}{3}, \ldots\ldots\ldots$

19. $\frac{1}{12}$. 20. $\frac{1}{x-1}-\frac{2^{n+1}}{x^{2n+2}-1}$. 21. $-\frac{2}{1+x^2}$.

22. 5. 23. $x^2-2x\cos n\theta+1=0$. 24. 81.

25. (*i*) 40 (*ii*) 116. 26. (*i*) 224, (*ii*) 896.

27. 0. 28. 7 or 14. 29. 7. 30. 32.

32. 1. 33. ${}^{1000}C_{50}\,x^{50}$. 34. $83x-35y+92=0$.

35. $-2\leqslant B\leqslant 7/2$. 36. $29x-2y-31=0$.

37. $\pi/3, \pi/3, \pi/3$.

38. $2x^2-7xy+3y^2+3x+y-2=0$; $\frac{7}{10}$.

39. $(1, 4), (3, -1)$.

40. $4x^2+4y^2+(5f+4g)x+(4f+5g)y=0$.

42. $(33, 26)$.

43. $100x^2-576y^2-1300x+4225=0$; $\frac{169}{4}\sin^{-1}(5/13)-15$.

45. $4x^2+4y^2+6x+10y-1=0$. 46. $\Sigma(BC'-B'C)(a-a')=0$.

47. $x-7y-2=0$; $7x+y-14=0$: $(x-9)^2+(y-1)^2=9$,
$(x+5)^2+(y+1)^2=9$; $(x-1)^2+(y-7)^2=9$;
$(x-3)^2+(y+7)^2=9$.

48. $x^2+y^2-14x-12y+76=0$ or $x^2+y^2-8x-6y+16=0$.

49. $x^2+y^2+6x-3y-45=0$. 50. 1676 km.

54. 105°. 62. $2\sqrt{2}d/\sqrt{(\cot^2\alpha-\cot^2\beta)}$;
$d\{(9\cot^2\beta-\cot^2\alpha)/(\cot^2\alpha-\cot^2\beta)\}^{1/2}$.

64. $2\sin\alpha_1\sin\alpha_2/\sin(\alpha_1-\alpha_2)$. 65. $\pm\pi/3$, $\pm\cos^{-1}(-3/5)$.

67. $\frac{1}{2}(3-\sqrt{5})$. 69. $2n\pi$, $n\pi+(-1)^n(-\pi/2)$,
$n\pi+(-1)^n\pi/3$.

70. $x=5\sqrt{5}$, $y=2n\pi+\tan^{-1}(1/2)$.

71. $-3/2\leqslant\alpha\leqslant\frac{1}{2}$. For each α satisfying these conditions,
$x=\frac{1}{2}\{n\pi+(-1)^n\sin^{-1}[1-\sqrt{2\alpha+3}]\}$, where n is any integer.

72. (*b*)	73. (*c*)	74. (*c*)	75. (*a*)
76. (*d*)	77. (*d*)	78. (*d*)	79. (*d*)
80. (*b*)	81. (*a*)	82. (*b*)	83. (*c*)
84. (*a*)	85. (*a*)	86. (*a*)	87. (*b*)
88. (*b*)	89. (*b*)	90. (*a*)	91. (*b*)
92. (*b*)	93. (*d*)	94. (*d*)	95. (*b*)
96. (*d*)	97. (*a*)	98. (*b*)	99. (*a*)
100. (*c*)	101. (*b*).		

□ □

TABLES

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घटाने पर शेष २२४ होगा। इस २२४ को प्रथम ज्यार्द्ध २२५ के साथ जोड़ देने से योगफल ४४९ होगा। यही द्वितीय ज्यार्द्ध है ॥१५॥ उक्त द्वितीय ज्यार्द्ध ४४९ को प्रथम ज्यार्द्ध से भाग करके भागफल २ लेकर यह २ इसके साथ पूर्व द्वितीय ज्यार्द्ध निष्कासन भाग फल से जो १ मिला है, जोड़ने से ३ होगा। इस ३ को उक्त भाजक २२५ से घटाने पर २२२ बचेगा, इसी २२२ को द्वितीय ज्यार्द्ध ४४९ के साथ जोड़ने से ६७१ होगा, यही तृतीय ज्यार्द्ध है। इसी प्रकार क्रमशः २४ ज्यार्द्ध गणना करनी होगी ।१६॥ किसी वृत्त के चतुर्थांश जिस का व्यासार्द्ध ३४३८ उस के ३४ अंश की ज्यार्द्ध निम्नलिखित होंगी ॥

	अंश	वा कला	ज्या		अंश	वा कला	ज्या
प्रथम कोण	३ ३/४	२२५	२२५	१३ वां कोण	४८।।।	२४७५	२२६७
द्वितीय ,,	७ १/२	४५०	४४९	१४ वां ,,	५२ १/२	२७७०	२४३१
तृतीय ,,	११ १/४	६७५	६७१	१५ वां ,,	५६ १/४	२९२५	२५८२
चतुर्थ ,,	१५	९००	८९०	१६ वां ,,	६०	३१५०	२७२८
पञ्चम ,,	१८ ३/४	११२५	११०५	१७ वां ,,	६३।।।	३३७५	२८५९
छठा ,,	२२ १/२	१३५०	१३१५	१८ वां ,,	६७ १/२	३६००	२९७८
सप्तम ,,	२६ १/४	१५७५	१५२०	१९ वां ,,	७१ १/४	३८२५	३०८४
अष्टम ,,	३०	१८००	१७१९	२० वां ,,	७५	४०५०	३१७७
नवम ,,	३३ ३/४	२०२५	१९१५	२१ वां ,,	७८।।।	४७२५	३३७२
दशम ,,	३७ १/२	२२५०	२०९३	२२ वां ,,	८२ १/२	४९५०	३४०९
एकादश ,,	४१ १/४	२४७५	२२६७	२३ वां ,,	८६ ३/४	५१७५	३४३१
द्वादश ,,	४५	२७००	२४३१	२४ वां ,,	९०	५४००	३४३८

पूर्वोक्त ज्यार्द्ध परिमाण सब को उलटे प्रकार से ३४३८ व्यासार्द्ध से पृथक् पृथक् घटाने पर जो अङ्क घटाने से बचेंगे उन को उत्क्रमज्या कहते हैं। प्रति ३४ अंश में इस प्रकार उत्क्रमज्या हो जाती हैं। १६-२२ श्लोक तक ॥

मुनयोरन्ध्रयमला रसषट्कामुनीश्वराः। द्व्यष्टैकारूपषड्दस्राः सागरार्थहुताशनाः ॥२३॥ खर्तुवेदा नवाद्रचर्था दिङ्नगास्त्र्यर्थकुञ्जराः। नगाम्बरवियच्चन्द्रारूपभूधरश-

FROM THE FIRST PRINTED EDITION OF THE SURYA SIDDHANTA
Printed at Meerut, India, *c.* 1867. This is the oldest Hiudu work on astronomy.

The earliest trigonometric tables are the ones in Sūrya Siddhānta. A page from Sūrya Siddhānta giving values of sines is reproduced above. The values given above are remarkably close to the modern values.

Values of Trigonometric Functions *(Contd.)*

Angle	sin	cos	tan	cot	sec	csc	Angle
4°00′	.0698	.9976	.0699	14.30	1.002	14.34	86°00′
10′	.0727	.9974	.0729	13.73	1.003	13.76	50′
20′	.0756	.9971	.0758	13.20	1.003	13.23	40′
30′	.0785	.9969	.0787	12.71	1.003	12.75	30′
40′	.0814	.9967	.0816	12.25	1.003	12.29	20′
50′	.0843	.9964	.0846	11.83	1.004	11.87	10′
5°00′	.0872	.9962	.0875	11.43	1.004	11.47	85°00′
10′	.0901	.9959	.0904	11.06	1.004	11.10	50′
20′	.0929	.9957	.0934	10.71	1.004	10.76	40′
30′	.0958	.9954	.0963	10.39	1.005	10.43	30′
40′	.0987	.9951	.0992	10.08	1.005	10.13	20′
50′	.1016	.9948	.1022	9.788	1.005	9.839	10′
6°00′	.1045	.9945	.1051	9.514	1.006	9.567	84°00′
10′	.1074	.9942	.1080	9.225	1.006	9.309	50′
20′	.1103	.9939	.1110	9.010	1.006	9.065	40′
30′	.1132	.9936	.1139	8.777	1.006	8.834	30′
40′	.1161	.9932	.1169	8.556	1.007	8.614	20′
50′	.1190	.9929	.1198	8.345	1.007	8.405	10′
7°00′	.1219	.9925	.1228	8.144	1.008	8.206	83°00′
10′	.1248	.9922	.1257	7.953	1.008	8.016	50′
20′	.1276	.9918	.1287	7.770	1.008	7.834	40′
30′	.1305	.9914	.1317	7.596	1.009	7.661	30′
40′	.1334	.9911	.1346	7.429	1.009	7.496	20′
50′	.1363	.9907	.1376	7.269	1.009	7.337	10′
Angle	**cos**	**sin**	**cot**	**tan**	**csc**	**sec**	**Angle**

Values of Trigonometric Functions (*Contd.*)

Angle	sin	cos	tan	cot	sec	csc	Angle
8°00′	·1392	·9903	·1405	7·115	1·010	7·185	82°00′
10′	·1421	·9899	·1435	6·968	1·010	7·040	50′
20′	·1449	·9894	·1465	6·827	1·011	6·900	40′
30′	·1478	·9890	·1495	6·691	1·011	6·765	30′
40′	·1507	·9886	·1524	6·561	1·012	6·636	20′
50′	·1536	·9881	·1554	6·435	1·012	6·512	10′
9°00′	·1564	·9877	·1584	6·314	1·012	6·392	81°00′
10′	·1593	·9872	·1614	6·197	1·013	6 277	50′
20′	·1622	·9868	·1644	6·084	1·013	6·166	40′
30′	·1650	·9863	·1673	5·976	1·014	6·059	30′
40′	·1679	·9858	·1703	5·871	1·014	5·955	20′
50′	·1708	·9853	·1733	5·769	1·015	5·855	10′
10°00′	·1736	·9848	·1763	5·671	1·015	5·759	80°00′
10′	.1765	·9843	·1793	5·576	1·016	5·665	50′
20′	.1794	·9838	·1823	5·485	1·016	5·575	40′
30′	.1822	·9833	·1853	5·396	1·017	5·487	30′
40′	.1851	·9827	·1883	5·309	1·018	5·403	20′
50′	.1880	·9822	·1914	5·226	1·018	5·320	10′
11°00′	·1908	·9816	·1944	5·145	1·019	5·241	79°00′
10′	·1937	·9811	·1974	5·066	1·019	5·164	50′
20′	·1965	·9805	·2004	4·989	1·020	5·089	40′
30′	·1994	·9799	·2035	4·915	1·020	5·016	30′
40′	·2022	·9793	·2065	4·843	1·021	4·945	20′
50′	·2051	·9787	·2095	4·773	1·022	4·876	10′
Angle	cos	sin	cot	tan	csc	sec	Angle

Values of Trigonometric Functions *(Contd.)*

Angle	sin	cos	tan	cot	sec	csc	Angle
12°00′	·2079	·9781	·2126	4·705	1·022	4·810	78°00′
10′	·2108	·9775	·2156	4·638	1·023	4·745	50′
20′	·2136	·9769	·2186	4·574	1·024	4·682	40′
30′	·2164	·9763	·2217	4·511	1·024	4·620	30′
40′	·2193	·9757	·2247	4·449	1·025	4·560	20′
50′	·2221	·9750	·2278	4·390	1·026	4·502	10′
13°00′	·2250	·9744	·2309	4·331	1·026	4·445	77°00′
10′	·2278	·9737	·2339	4·275	1·027	4·390	50′
20′	·2306	·9730	·2370	4·219	1·028	4·336	40′
30′	·2334	·9724	·2401	4·165	1·028	4·284	30′
40′	·2363	·9717	·2432	4·113	1·029	4·232	20′
50′	·2391	·9710	·2462	4·061	1·030	4·182	10′
14°00′	·2419	·9703	·2493	4·011	1·030	4·134	76°00′
10′	·2447	·9696	·2524	3·962	1·031	4·086	50′
20′	·2476	·9689	·2555	3·914	1·032	4·039	40′
30′	·2504	·9681	·2586	3·867	1·033	3·994	30′
40′	·2532	·9674	·2617	3·821	1·034	3·950	20′
50′	·2560	·9667	·2648	3·776	1·034	3·906	10′
15°00′	·2588	·9659	·2679	3·732	1·035	3·864	75°00′
10′	·2616	·9652	·2711	3·689	1·036	3·822	50′
20′	·2644	·9644	·2742	3·647	1·037	3·782	40′
30′	·2672	·9636	·2773	3·606	1·038	3·742	30′
40′	·2700	·9628	·2805	3·566	1·039	3·703	20′
50′	·2728	·9621	·2836	3·526	1·039	3·665	10′
Angle	**cos**	**sin**	**cot**	**tan**	**csc**	**sec**	**Angle**

Values of Trigonometric Functions (*Contd.*)

Angle	sin	cos	tan	cot	sec	csc	Angle
16°00′	·2756	·9613	·2867	3·487	1·040	3·628	74°00′
10′	·2784	·9605	·2899	3·450	1·041	3·592	50′
20′	·2812	·9596	·2931	3·412	1·041	3·556	40′
30′	·2840	·9588	·2962	3·376	1·043	3·521	30′
40′	·2868	·9580	·2994	3·340	1·044	3·487	20′
50′	·2896	·9572	·3026	3·305	1·045	3·453	10′
17°00′	·2924	·9563	·3057	3·271	1·046	3·420	73°00′
10′	·2952	·9555	·3089	3·237	1·047	3·388	50′
20′	·2979	·9546	·3121	3·204	1·048	3·356	40′
30′	·3007	·9537	·3153	3·172	1·049	3·326	30′
40′	·3035	·9528	·3185	3·140	1·049	3·295	20′
50′	·3062	·9520	·3217	3·108	1·050	3·265	10′
18°00′	·3090	·9511	·3249	3·078	1·051	3·236	72°00′
10′	·3118	·9502	·3281	3·047	1·052	3·207	50′
20′	·3145	·9492	·3314	3·018	1·053	3·179	40′
30′	·3173	·9483	·3346	2·989	1·054	3·152	30′
40′	·3201	·9474	·3378	2·960	1·056	3·124	20′
50′	·3228	·9465	·3411	2·932	1·057	3·098	10′
19°00′	·3256	·9455	·3443	2·904	1·058	3·072	71°00′
10′	·3283	·9446	·3476	2·877	1·059	3·046	50′
20′	·3311	·9436	·3508	2·850	1·060	3·021	40′
30′	·3338	·9426	·3541	2·824	1·061	2·996	30′
40′	·3365	·9417	·3574	2·798	1·062	2·971	20′
50′	·3393	·9407	·3607	2·773	1·063	2·947	10′
Angle	**cos**	**sin**	**cot**	**tan**	**csc**	**sec**	**Angle**

Values of Trigonometric Functions (*Contd.*)

Angle	sin	cos	tan	cot	sec	csc	Angle
20°00′	·3420	·9397	·3640	2·747	1·064	2·924	70°00′
10′	·3448	·9387	·3673	2·723	1·065	2·901	50′
20′	·3475	·9377	·3706	2·699	1·066	2·878	40′
30′	·3502	·9367	·3739	2·675	1·068	2·855	30′
40′	·3529	·9356	·3772	2·651	1·069	2·833	20′
50′	·3557	·9346	·3805	2·628	1·070	2·812	10′
21°00′	·3584	·9336	·3839	2·605	1·071	2·790	69°00′
10′	·3611	·9325	·3872	2·583	1·072	2·769	50′
20′	·3638	·9315	·3906	2·560	1·074	2·749	40′
30′	·3665	·9304	·3939	2·539	1·075	2·729	30′
40′	·3692	·9293	·3973	2·517	1·076	2·709	20′
50′	·3719	·9283	·4006	2·496	1·077	2·689	10′
22°00′	·3746	·9272	·4040	2·475	1·079	2·669	68°00′
10′	·3773	·9261	·4074	2·455	1·080	2·650	50′
20′	·3800	·9250	·4108	2·434	1·081	2·632	40′
30′	·3827	·9239	·4142	2·414	1·082	2·613	30′
40′	·3854	·9228	·4176	2·394	1·084	2·595	20′
50′	·3881	·9216	·4210	2·375	1·085	2·577	10′
23°00′	·3907	·9205	·4245	2·356	1·086	2·559	67°00′
10′	·3934	·9194	·4279	2·337	1·088	2·542	50′
20′	·3961	·9182	·4314	2·318	1·089	2·525	40′
30′	·3987	·9171	·4348	2·300	1·090	2·508	30′
40′	·4014	·9159	·4383	2·282	1·092	2·491	20′
50′	·4041	·9147	·4417	2·264	1·093	2·475	10′
Angle	cos	sin	cot	tan	csc	sec	Angle

Values of Trigonometric Functions *(Contd.)*

Angle	sin	cos	tan	cot	sec	csc	Angle
24°00′	·4067	·9135	·4452	2·246	1·095	2·459	66°00′
10′	·4094	·9124	·4487	2·229	1·096	2·443	50′
20′	·4120	·9112	·4522	2·211	1·097	2·427	40′
30′	·4147	·9100	·4557	2·194	1·099	2·411	30′
40′	·4173	·9088	·4592	2·177	1·100	2·396	20′
50′	·4200	·9075	·4628	2·161	1·102	2·381	10′
25°00′	·4226	·9063	·4663	2·145	1·103	2·366	65°00′
10′	·4253	·9051	·4699	2·128	1·105	2·352	50′
20′	·4279	·9038	·4734	2·112	1·106	2·337	40′
30′	·4305	·9026	·4770	2·097	1·108	2·323	30′
40′	·4331	·9013	·4806	2·081	1·109	2·309	20′
50′	·4358	·9001	·4841	2·066	1·111	2·295	10′
26°00′	·4384	·8988	·4877	2·050	1·113	2·281	64°00′
10′	·4410	·8975	·4913	2·035	1·114	2·268	50′
20′	·4436	·8962	·4950	2·020	1·116	2·254	40′
30′	·4462	·8949	·4986	2·006	1·117	2·241	30′
40′	·4488	·8936	·5022	1·991	1·119	2·228	20′
50′	·4514	·8923	·5059	1·977	1·121	2·215	10′
27°00′	·4540	·8910	·5095	1·963	1·122	2·203	63°00′
10′	·4566	·8897	·5132	1·949	1·124	2·190	50′
20′	·4592	·8884	·5169	1·935	1·126	2·178	40′
30′	·4617	·8870	·5206	1·921	1·127	2·166	30′
40′	·4643	·8857	·5243	1·907	1·129	2·154	20′
50′	·4669	·8843	·5280	1·894	1·131	2·142	10′
Angle	**cos**	**sin**	**cot**	**tan**	**csc**	**sec**	**Angle**

Values of Trigonometric Fnnctions *(Contd.)*

Angle	sin	cos	tan	cot	sec	csc	Angle
28°00′	·4695	·8829	·5317	1·881	1·133	2·130	62°00′
10′	·4720	·8816	·5354	1·868	1·134	2·118	50′
20′	·4746	·8802	·5392	1·855	1·136	2·107	40′
30′	·4772	·8788	·5430	1·842	1·138	2·096	30′
40′	·4797	·8774	·5467	1·829	1·140	2·085	20′
50′	·4823	·8760	·5505	1·816	1·142	2·074	10′
29°00′	·4848	·8746	·5543	1·804	1·143	2·063	61°00′
10′	·4847	·8732	·5581	1·792	1·145	2·052	50′
20′	·4899	·8718	·5619	1·780	1·147	2·041	40′
30′	·4924	·8704	·5658	1·767	1·149	2·031	30′
40′	·4950	·8689	·5696	1·756	1·151	2·020	20′
50′	·4975	·8675	·5735	1·744	1·153	2·010	10′
30°00′	·5000	·8660	·5774	1·732	1·155	2·000	60°00′
10′	·5025	·8646	·5812	1·720	1·157	1·990	50′
20′	·5050	·8631	·5851	1·709	1·159	1·980	40′
30′	·5075	·8616	·5890	1·698	1·161	1·970	30′
40′	·5100	·8601	·5930	1·686	1·163	1·961	20′
50′	·5125	·8587	·5969	1·675	1·165	1·952	10′
31°00′	·5150	·8572	·6009	1·664	1·167	1·942	59°00′
10′	·5175	·8557	·6048	1·653	1·169	1·932	50′
20′	·5200	·8542	·6088	1·643	1·171	1·923	40′
30′	·5225	·8526	·6128	1·632	1·173	1·914	30′
40′	·5250	·8511	·6168	1·621	1·175	1·905	20′
50′	·5275	·8496	·6208	1·611	1·177	1·896	10′
Angle	cos	sin	cot	tan	csc	sec	Angle

Values of Trigonometric Functions (Contd)

Angle	sin	cos	tan	cot	sec	csc	Angle
32°00′	·5299	·8480	·6249	1·600	1·179	1·887	58°00′
10′	·5324	·8465	·6289	1·590	1·181	1·878	50′
20′	·5348	·8450	·6330	1 580	1·184	1·870	40′
30′	·5373	8434	·6371	1·570	1·186	1·861	30′
40′	·5398	·8418	6412	1·560	1·188	1·853	20′
50′	·5422	·8403	·6453	1·550	1·190	1·844	10′
33°00′	·5446	·8387	·6494	1 540	1·192	1·836	57°00′
10′	·5471	·8371	·6536	1·530	1·195	1·828	50′
20′	·5495	8355	·6577	1·520	1·197	1·820	40′
30′	·5519	·8339	·6619	1·511	1·199	1·812	30′
40′	·5544	·8323	·6661	1·501	1·202	1·804	20′
50′	·5568	·8307	·6703	1·492	1·204	1·796	10′
34°00′	·5592	·8290	·6745	1·483	1 206	1·788	56°00′
10′	·5616	·8274	·6787	1·473	1 209	1·781	50′
20′	·5640	·8258	·6830	1·464	1 211	1·773	40′
30′	·5664	·8241	·6873	1·455	1·213	1·766	30′
40′	·5688	·8225	·6916	1·446	1·216	1·758	20′
50′	·5712	·8208	·6059	1·437	1·218	1·751	10′
35°00′	·5736	·8192	·7002	1·428	1·221	1·743	55°00′
10′	·5760	·8175	·7046	1·419	1·223	1·736	50′
20′	·5783	·8158	·7089	1 411	1·226	1·729	40′
30′	·5807	·8141	·7133	1·402	1·228	1·722	30′
40′	·5831	·8124	·7177	1·393	1·231	7·715	20′
50′	·5854	·8107	·7221	1·383	1·233	1·708	10′
Angle	**cos**	**sin**	**cot**	**tan**	**csc**	**sec**	**Angle**

Values of Trigonometric Functions *(Contd.)*

Angle	sin	cos	tan	cot	sec	csc	Angle
36°00′	·5878	·8090	·7265	1·376	1·236	1·701	54°00′
10′	·5901	·8073	·7310	1·368	1·239	1·695	50′
20′	·5925	·8056	·7355	1·360	1·241	1·688	40′
30′	·5948	·8039	·7400	1·351	1·244	1·681	30′
40′	·5972	·8021	·7045	1·343	1·247	1·675	20′
50′	·5995	·8004	·7490	1·335	1·249	1·668	10′
37°00′	·6018	·7986	·7536	1·327	1·252	1·662	53°00′
10′	·6041	·7969	·7581	1·319	1·255	1·655	50′
20′	·6065	·7951	·7627	1·311	1·258	1·649	40′
30′	·6088	·7934	·7673	1·303	1·260	1·643	30′
40′	·6111	·7916	·7720	1·295	1·263	1·636	20′
50′	·6134	·7898	·7766	1·288	1·266	1·630	10′
38°00′	·6157	·7880	·7813	1·280	1·269	1·624	52°00′
10′	.6180	·7862	·7860	1·272	1·272	1·618	50′
20′	·6202	·7844	·7907	1·265	1·275	1·612	40′
30′	·6225	·7826	·7954	1·257	1·278	1·606	30′
40′	·6248	·7808	·8002	1·250	1·281	1·601	20′
50′	·6271	·7790	·8050	1·242	1·284	1·595	10′
39°00′	·6293	·7771	·8098	1·235	1·287	1·589	51°00′
10′	·6316	·7753	·8146	1·228	1·290	1·583	50′
20′	·6338	·7735	·8195	1·220	1·293	1·578	40′
30′	·6361	·7716	·8243	1·213	1·296	1·572	30′
40′	·6383	·7698	·8292	1·206	1·299	1·567	20′
50′	·6406	·7679	·8342	1·199	1·302	1·561	10′
Angle	cos	sin	cot	tan	csc	sec	Angle

Values of Trigonometric Functions *(Contd.)*

Angle	sin	cos	tan	cot	sec	csc	Angle
40°00′	·6428	·7660	·8391	1·192	1·305	1·556	50°00′
10′	·6450	·7642	·8441	1·185	1·309	1·550	50′
20′	·6472	·7623	·8491	1·178	1·312	1·545	40′
30′	·6494	·7604	·8541	1·171	1·315	1·540	30′
40′	·6517	·7585	·8591	1·164	1·318	1·535	20′
50′	·6539	·7566	·8642	1·157	1·322	1·529	10′
41°00′	·6561	·7547	·8693	1·150	1·325	1·524	49°00′
10′	·6583	·7528	·8744	1·144	1·328	1·519	50′
20′	·6604	·7509	·8796	1·137	1·332	1·514	40′
30′	·6626	·7490	·8847	1·130	1·335	1·509	30′
40′	·6648	·7470	·8899	1·124	1·339	1·504	20′
50′	·6670	·7451	·8952	1·117	1·342	1·499	10′
42°00′	·6691	·7431	·9004	1·111	1·346	1·494	48°00′
10′	·6713	·7412	·9057	1·104	1·349	1·490	50′
20′	·6734	·7392	·9110	1·098	1·353	1·485	40′
30′	·6756	·7373	·9163	1·091	1·356	1·480	30′
40′	·6777	·7353	·9217	1·085	1·360	1·476	20′
50′	·6799	·7333	·9271	1·079	1·364	1·471	10′
43°00′	·6820	·7314	·9325	1·072	1·367	1·466	47°00′
10′	·6841	·7294	·9380	1·066	1·371	1·462	50′
20′	·6862	·7274	·9435	1·060	1·375	1·457	40′
30′	·6884	·7254	·9490	1·054	1·379	1·453	30′
40′	·6905	·7234	·9545	1·048	1·382	1·448	20′
50′	·6926	·7214	·9601	1·042	1·386	1·444	10′
Angle	**cos**	**sin**	**cot**	**tan**	**csc**	**sec**	**Angle**

Values of Trigonometric Functions (*Contd.*)

Angle	sin	cos	tan	cot	sec	csc	Angle
44°00′	·6947	·7193	·9657	1·036	1·390	1·440	46°00′
10′	·6967	·7173	·9713	1·030	1·394	1·435	50′
20′	·6988	·7153	·9770	1·024	1·398	1·431	40′
30′	·7009	·7133	·9827	1·018	1·402	1·427	30′
40′	·7030	·7112	·9884	1·012	1·406	1·423	20′
50′	·7050	·7092	·9942	1·006	1·410	1·418	10′
45°00′	·7071	·7071	1·000	1·000	1·414	1·414	45°00′
Angle	cos	sin	cot	tan	csc	sec	Angle

TABLE 2

Common Logarithms

N	0	1	2	3	4	5	6	7	8	9
10	·0000	·0043	·0086	·0128	·0170	·0212	·0253	·0294	·0334	·0374
11	·0414	·0453	·0492	·0531	·0569	·0607	·0645	·0682	·0719	·0755
12	·0792	·0828	·0864	·0899	·0934	·0969	·1004	·1038	·1072	·1106
13	·1139	·1173	·1206	·1239	·1271	·1303	·1335	·1367	·1399	·1430
14	·1461	·1492	·1523	·1553	·1584	·1614	·1644	·1673	·1703	·1732
15	·1761	·1790	·1818	·1847	·1875	·1903	·1931	·1959	·1987	·2014
16	·2041	·2068	·2095	·2122	·2148	·2175	·2201	·2227	·2253	·2279
17	·2304	·2330	·2355	·2380	·2405	·2430	·2455	·2480	·2504	·2529
18	·2553	·2577	·2601	·2625	·2648	·2672	·2695	·2718	·2742	·2765
19	·2788	·2810	·2833	·2856	·2878	·2900	·2923	·2945	·2967	·2989
20	·3010	·3032	·3054	·3075	·3096	·3118	·3139	·3160	·3181	·3201
21	·3222	·3243	·3263	·3284	·3304	·3324	·3345	·3365	·3385	·3404
22	·3424	·3444	·3464	·3483	·3502	·3522	·3541	·3560	·3579	·3598
23	·3617	·3636	·3655	·3674	·3692	·3711	·3729	·3747	·3766	·3784
24	·3802	·3820	·3838	·3856	·3874	·3892	·3909	·3927	·3945	·3962
25	·3979	·3997	·4014	·4031	·4048	·4065	·4082	·4099	·4116	·4133
26	·4150	·4166	·4183	·4200	·4216	·4232	·4249	·4265	·4281	·4298
27	·4314	·4330	·4346	·4362	·4378	·4393	·4409	·4425	·4440	·4456
28	·4472	·4487	·4502	·4518	·4533	·4548	·4564	·4579	·4549	·4609
29	·4624	·4639	·4654	·4669	·4683	·4698	·4713	·4728	·4742	·4757
30	·4771	·4786	·4800	·4814	·4829	·4843	·4857	·4871	·4886	·4900
31	·4914	·4928	·4942	·4955	·4969	·4983	·4997	·5011	·5024	·5038
32	·5051	·5065	·5079	·5092	·5105	·5119	·5132	·5145	·5159	·5172
33	·5185	·5198	·5211	·5224	·5237	·5250	·5263	·5276	·5289	·5302
34	·5315	·5328	·5340	·5353	·5366	·5378	·5391	·5403	·5416	·5428

Common Logarithms (*Contd.*)

N	0	1	2	3	4	5	6	7	8	9
35	·5441	·5453	·5465	·5478	·5490	·5502	·5514	·5527	·5539	·5551
36	·5563	·5575	·5587	·5599	·5611	·5623	·5635	·5647	·5658	·5670
37	·5682	·5694	·5705	·5717	·5729	·5740	·5752	·5763	·5775	·5786
38	·5798	·5809	·5821	·5832	·5843	·5855	·5866	·5877	·5888	·5899
39	·5911	·5922	·5933	·5944	·5955	·5966	·5977	·5988	·5999	·6010
40	·6021	·6031	·6042	·6053	·6064	·6075	·6085	·6096	·6107	·6117
41	·6128	·6138	·6149	·6160	·6170	·6180	·6191	·6201	·6212	·6222
42	·6232	·6243	·6253	·6263	·6274	·6284	·6294	·6304	·6314	·6325
43	·6335	·6345	·6355	·6365	·6375	·6385	·6395	·6405	·6415	·6425
44	·6435	·6444	·6454	·6464	·6474	·6484	·6493	·6503	·6513	·6522
45	·6532	·6542	·6551	·6561	·6571	·6580	·6590	·6599	·6609	·6618
46	·6628	·6637	·6646	·6656	·6665	·6675	·6684	·6693	·6702	·6712
47	·6721	·6730	·6739	·6749	·6758	·6767	·6776	·6785	·6794	·6803
48	·6812	·6821	·6830	·6839	·6848	·6857	·6866	·6875	·6884	·6893
49	·6802	·6911	·6920	·6928	·6937	·6946	·6955	·6964	·6972	·6891
50	·6990	·6998	·7007	·7016	·7024	·7033	·7042	·7050	·7059	·7067
51	·7076	·7084	·7093	·7101	·7110	·7118	·7126	·7135	·7143	·7152
52	·7160	·7168	·7177	·7185	·7193	·7202	·7210	·7218	·7226	·7235
53	·7243	·7251	·7259	·7267	·7275	·7284	·7292	·7300	·7308	·7316
54	·7324	·7332	·7340	·7348	·7356	·7364	·7372	·7380	·7388	·7396
55	·7404	·7412	·7419	·7427	·7435	·7413	·7451	·7459	·7466	·7474
56	·7482	·7490	·7497	·7505	·7513	·7520	·7528	·7536	·7543	·7515
57	·7559	·7566	·7574	·7582	·7589	·7597	·7604	·7612	·7619	·7627
58	·7634	·7642	·7649	·7657	·7664	·7672	·7679	·7686	·7694	·7701
59	·7709	·7716	·7723	·7731	·7738	·7745	·7752	·7769	·7767	·7774

Common Logarithms

(Contd.)

N	0	1	2	3	4	5	6	7	8	9
60	·7782	·7789	·7796	·7803	·7810	·7818	·7825	·7832	·7839	·7846
61	·7853	·7860	·7868	·7875	·7882	·7889	·7896	·7903	·7910	·7917
62	·7924	·7931	·7938	·7945	·7952	·7959	·7966	·7973	·7980	·7987
63	·7993	·8000	·8007	·8014	·8021	·8028	·8035	·8041	·8048	·8055
64	·8062	·8069	·8075	·8032	·8089	·8096	·8102	·8109	·8106	·8122
65	·8129	·8136	·8142	·8149	·8156	·8162	·8169	·8176	·8182	·8189
66	·8195	·8202	·8209	·8215	·8222	·8228	·8235	·8241	·8248	·8254
67	·8261	·8267	·8274	·8280	·8287	·8293	·8290	·8306	·8312	·8319
68	·8325	·8331	·8338	·8344	·8351	·8357	·8363	·8370	·8376	·8332
69	·8388	·8395	·8401	·8407	·8414	·8420	·8426	·8432	·8439	·8444
70	·8451	·8457	·8463	·8470	·8476	·8482	·8488	·8494	·8500	·8505
71	·8513	·8519	·8525	·8531	·8537	·8543	·8549	·8555	·8561	·8567
72	·8573	·8579	·8585	·8591	·8597	·8603	·8609	·8615	·8621	·8627
73	·8633	·8639	·8645	·8651	·8657	·8663	·8669	·8675	·8681	·8686
74	·8692	·8698	·8704	·8710	·8716	·8722	·8727	·8733	·8739	·8745
75	·8751	·8756	·8762	·8768	·8774	·8779	·8785	·8791	·8797	·8802
76	·8808	·8814	·8820	·8825	·8831	·8837	·8842	·8848	·8854	·8859
77	·8865	·8871	·8876	·8882	·8837	·8893	·8399	·8904	·8910	·8915
78	·8921	·8927	·8932	·8938	·8943	·8949	·8954	·8960	·8956	·8971
79	·8976	·8982	·8987	·8993	·8998	·9004	·9009	·9015	·9020	·9025
80	·9031	·9036	·9042	·9047	·9053	·9058	·9063	·9069	·9074	·9079
81	·9085	·9090	·9096	·9101	·9106	·9112	·9117	·9122	·9128	·9133
82	·9138	·9143	·9149	·9154	·9159	·9165	·9170	·9175	·9180	·9186
83	·9191	·9196	·9201	·9206	·9212	·9217	·9222	·9227	·9232	·9238
84	·9243	·9248	·9253	·9258	·9263	·9269	·9274	·9279	·9284	·9289

Common Logarithms *(Contd)*

N	0	1	2	3	4	5	6	7	8	9
85	·9294	·9299	·9304	·9309	·9315	·9320	·9325	·9330	·9335	·9340
86	·9345	·9350	·9355	·9360	·9365	·9370	·9375	·9380	·9385	·9390
87	·9395	·9400	·9405	·9410	·9415	·9420	·9425	·9430	·9435	·9440
88	·9445	·9450	·9455	·9460	·9465	·9469	·9474	·9479	·9484	·9489
89	·9494	·9499	·9504	·9509	·9513	·9518	·9523	·9528	·9533	·9538
90	·9542	·9547	·9552	·9557	·9562	·9566	·9571	·9576	·9581	·9586
91	·9590	·9595	·9600	·9605	·9609	·9614	·9619	·9624	·9628	·9633
92	·9638	·9643	·9647	·9652	·9657	·9661	·9666	·9671	·9675	·9680
93	·9685	·9689	·9694	·9699	·9703	·9708	·9713	·9717	·9722	·9727
94	·9731	·9736	·9741	·9745	·9750	·9754	·9759	·9763	·9768	·9773
95	·9777	·9782	·9786	·9791	·9795	·9800	·9805	·9809	·9814	·9818
96	·9823	·9827	·9832	·9836	·9841	·9845	·9850	·9854	·9859	·9863
97	·9868	·9872	·9877	·9881	·9886	·9890	·9894	·9899	·9903	·9908
98	·9912	·9917	·9921	·9926	·9930	·9934	·9939	·9943	·9948	·9952
99	·9956	·9961	·9965	·9969	·9974	·9978	·9983	·9987	·9991	·9996

TABLE 3

Four-place Logarithms of Values of Trigonometric Functions*

Angle	L sin	L tan	L cot	L cos	Angle
0°00′	—	—	—	10 0000	90°00′
10′	7·4637	7·4637	12 5363	10 0000	50′
20′	7·7648	7·7648	12·2352	10·0000	40′
30′	7·9408	7·9409	12·0591	10·0000	30′
40′	8·0658	0·0658	11·9342	10·0000	20′
50′	8·1627	8·1627	11·8373	10·0000	10′
1°00′	8·2419	8·2419	11·7581	9·9999	89°00′
10′	8·3088	8 3089	11·6911	9·9999	50′
20′	8·3668	8·3669	11·6331	9·9999	40′
30′	8·4179	8·4181	11·5819	9·9999	30′
40′	8·4637	8·4638	11 5362	9·9998	20′
50′	8 5050	8 5053	11·4947	9·9998	10′
2°00′	8·5428	8·5431	11·4569	9·9997	88°00′
10′	8·5776	8·5779	11·4221	9·9997	50′
20′	8·6097	8·6101	11·3899	9·9996	40′
30′	8·6397	8 6401	11·3599	9·9996	30′
40′	8·6677	8·6682	11·3318	9·9995	20′
50′	8·6940	8·6945	11 3055	9·9995	10′
3°00′	8 7188	8·7194	11·2806	9·9994	87°00′
10′	8·7423	8·7429	11·2571	9·9993	50′
20′	8·7645	8·7652	11·2348	9·9993	40′
30′	8·7857	8·7865	11·2135	9·9992	30′
40′	8·8059	8·8067	11·1933	9·9991	20′
50′	8·8251	8·8261	11·1739	9 9990	10′
Angle	L cos	L cot	L tan	L sin	Angle

*This table g ves the logarithms increased by 10. Hence in each case 10 should by subtracted.

Four-place Logarithms of Values of Trigonometric Functions (*Contd.*)

Angle	L sin	L tan	L cot	L cos	Angle
4°00′	8·8436	8·8446	11·1554	9·9989	86°00′
10′	8·8613	8·8624	11·1376	9·9989	50′
20′	8·8783	8·8795	11·1205	9·9988	40′
30′	8·8946	8·8960	11·1040	9·9987	30′
40′	8·9104	8·9118	11·0882	9·9986	20′
50′	8·9256	8·9272	11·0728	9·9985	10′
5°00′	8·9403	8·9420	11·0580	9·9983	85°00′
10′	8·9545	8·9563	11·0437	9·9982	50′
20′	8·9682	8·9701	11·0299	9·9981	40′
30′	8·9816	8·9836	11·0164	9·9980	30′
40′	8·9945	8·9966	11·0034	9·9979	20′
50′	9·0070	9·0093	10·9907	9·9977	10′
6°00′	9·0192	9·0216	10·9784	9·9976	84°00′
10′	9·0311	9·0336	10·9664	9·9975	50′
20′	9·0426	9·0453	10·9547	9·9973	40′
30′	9·0539	9·0567	10·9433	9·9972	30′
40′	9·0648	9·0678	10·9322	9·9971	20′
50′	9·0755	9·0786	10·9214	9·9969	10′
7°00′	9·0859	9·0891	10·9109	9·9968	83°00′
10′	9·0961	9·0995	10·9005	9·9966	50′
20′	9·1060	9·1096	10·8904	9·9964	40′
30′	9·1157	9·1194	10·8806	9·9963	30′
40′	9·1252	9·1291	10·8709	9·9961	20′
50′	9·1345	9·1385	10·8615	9·9959	10′
Angle	L cos	L cot	L tan	L sin	Angle

Four-place Logarithms of Values of Trigonometric Functions (*Contd.*)

Angle	L sin	L tan	L cot	L cos	Angle
8°00′	9·1436	9·1478	10·8522	9·9958	82°00′
10′	9·1525	9·1569	10·8431	9·9956	50′
20′	9·1612	9·1658	10·8342	9·9954	40′
30′	9·1697	9·1745	10·8255	9·9952	30′
40′	9·1781	9·1831	10·8169	9·9950	20′
50′	9·1863	9·1915	10·8085	9·9948	10′
9°00′	9·1943	9·1997	10·8003	9·9946	81°00′
10′	9·2022	9·2078	10·7922	9·9944	50′
20′	9·2100	9·2158	10·7842	9·9942	40′
30′	9·2176	9·2236	10·7764	9·9940	30′
40′	9·2251	9·2313	10·7687	6·9938	20′
50′	9·2324	9·2389	10·7611	9·9936	10′
10°00′	9·2397	9·2463	10·7537	9·9934	80°00′
10′	9·2468	9·2536	10·7464	9·9931	50′
20′	9·2538	9·2609	10·7391	9·9929	40′
30′	9·2606	9·2680	10·7320	9·9927	30′
40′	9·2764	9·2750	10·7250	9·9924	20′
50′	9·2740	9·2819	10·7181	9·9922	10′
11°00′	9·2806	9·2887	10·7113	9·9919	79°00′
10′	9·2870	9·2953	10·7047	9·9917	50′
20′	9·2934	9·3020	10·6980	9·9914	40′
30′	9·2997	9·3085	10·6915	9·9912	30′
40′	9·3058	9·3149	10·6851	9·9909	20′
50′	9·3119	9·3112	10·6788	9·9907	10′
Angle	L cos	L cot	L tan	L sin	Angle

Four place Logarithms of Values of Trigonometric Functions *(Contd.)*

Angle	L sin	L tan	L cot	L cos	Angle
12°00′	9.3179	9.3275	10.6725	9.9904	78°00′
10′	9.9338	9.3336	10.6664	9.9901	50′
20′	9.3296	9.3397	10.6603	9.9899	40′
30′	9.3353	9.3458	10.6542	9.9896	30′
40′	9.3410	9.3517	10.6483	9.9893	20′
50′	9.3466	9.3576	10.6424	9.9890	10′
13°00′	9.3521	9.3634	10.6366	9.9887	77°00′
10′	9.3575	9.3691	10.6309	9.9884	50′
20′	9.3629	9.3748	10.6252	9.9881	40′
30′	9.3682	9.3804	10.6196	9.9878	30′
40′	9.3734	9.3859	10.6141	9.9875	20′
50′	9.3786	9.3914	10.6086	9.9872	10′
14°00′	9.3837	9.3968	10.6032	9.9869	76°00′
10′	9.3887	9.4021	10.5979	9.9866	50′
20′	9.3937	9.4074	10.5926	9.9863	40′
30′	9.3986	9.4127	10.5873	9.9859	30′
40′	9.4035	9.4178	10.5822	9.9856	20′
50′	9.4083	9.4230	10.5770	9.9853	10′
15°00′	9.4130	9.4281	10.5719	9.9849	75°00′
10′	9.4177	9.4331	10.5669	9.9846	50′
20′	9.4223	9.4381	10.5619	9.9843	40′
30′	9.4269	9.4430	10.5570	9.9839	30′
40′	9.4314	9.4479	10.5521	9.9836	20′
50′	9.4359	9.4527	10.5473	9.9832	10′
Angle	L cos	L cot	L tan	L sin	Angle

Four-place Logarithms of Values of Trigonometric Functions *(Contd.)*

Angle	L sin	L tan	L cot	L cos	Angle
16°00′	9·4403	9·4575	10·5425	9·9828	74°00′
10′	9·4447	9·4622	10·5378	9·9825	50′
20′	9·4491	9·4669	10·5331	9·9821	40′
30′	9·4533	9·4716	10·5284	9·9817	30′
40′	9·4576	9·4762	10·5238	9·9814	20′
50′	9·4618	9·4808	10·5192	9·9810	10′
17°00′	9·4659	9·4853	10·5147	9·9806	73°00′
10′	9·4700	9·4898	10·5102	9·9802	50′
20′	9·4741	9·4943	10·5057	9·9798	40′
30′	9·4781	9·4987	10·5013	9·9794	30′
40′	9·4821	9·5031	10·4969	9·9790	20′
50′	9·4861	9·5075	10·4925	9·9786	10′
18°00′	9·4900	9·5118	10·4882	9·9782	72°00′
10′	9·4939	9·5161	10·4839	9·9778	50′
20′	9·4977	9·5203	10·4797	9·9774	40′
30′	9·5015	9·5245	10·4755	9·9770	30′
40′	9·5052	9·5287	10·4713	9·9765	20′
50′	9·5090	9·5329	10·4671	9·9761	10′
19°00′	9·5126	9·5370	10·4630	9·9757	71°00′
10′	9·5163	9·5411	10·4589	9·9752	50′
20′	9·5199	9·5451	10·4549	9·9748	40′
30′	9·5235	9·5491	10·4509	9·9743	30′
40′	9·5270	9·5531	10·4469	9·9739	20′
50′	9·5306	9·5571	10·4429	9·9734	10′
Angle	L cos	L cot	L tan	L sin	Angle

TABLE 1
Values of Trigonometric Functions

Angle	sin	cos	tan	cot	sec	csc	Angle
0°00′	·0000	1·000	·0000	—	1·000	—	90°00′
10′	·0029	1·000	·0029	343·8	1·000	343·8	50′
20′	·0058	1·000	·0058	171·9	1·000	171·9	40′
30′	·0087	1·000	·0087	114·6	1·000	114·6	30′
40′	·0116	·9999	·0116	85·94	1·000	85·95	20′
50′	·0145	·9999	·0145	68·75	1·000	68·76	10′
1°00′	·0175	·9998	·0175	57·29	1·000	57·30	89°00′
10′	·0204	·9998	·0204	49·10	1·000	49·11	50′
20′	·0233	·9997	·0233	42·96	1·000	42·98	40′
30′	·0262	·9997	·0262	38·19	1·000	38·20	30′
40′	·0291	·9996	·0291	34·37	1·000	34·38	20′
50′	·0320	·9995	·0320	31·24	1·001	31·26	10′
2°00′	·0349	·9994	·0349	28·64	1·001	28·65	88°00′
10′	·0378	·9993	·0378	26·43	1·001	26·45	50′
20′	·0407	·9992	·0407	24·54	1·001	24·56	40′
30′	·0436	·9990	·0437	22·90	1·001	22·93	30′
40′	·0465	·9989	·0466	21·47	1·001	21·49	20′
50′	·0494	·9989	·0495	20·21	1·001	20·23	10′
3°00′	·0523	·9985	·0524	19·08	1·001	19·11	87°00′
10′	·0552	·9985	·0553	18·07	1·002	18·10	50′
20′	·0581	·9983	·0582	17·17	1·002	17·20	40′
30′	·0610	·9981	·0612	16·35	1·002	16·38	30′
40′	·0640	·9980	·0641	15·60	1·002	15·64	20′
50′	·0669	·9978	·0670	14·92	1·002	14·96	10′
Angle	**cos**	**sin**	**cot**	**tan**	**csc**	**sec**	**Angle**

Four-place Logarithms of Values of Trigonometric Functions (*Contd.*)

Angle	L sin	L tan	L cot	L cos	Angle
20°00′	9·5341	9·5611	10·4389	9·9730	70°00′
10′	9·5375	9·5650	10·4350	9·9725	50′
20′	9·5409	9·5689	10·4311	9·9721	40′
30′	9·5443	9·5727	10·4273	9·9716	30′
40′	9·5477	9·5766	10·4234	9·9711	20′
50′	9·5510	9·5804	10·4196	9·9706	10′
21°00′	9·5543	9·5842	10·4158	9·9702	69°00′
10′	9·5576	9·5879	10·4121	9·9697	50′
20′	9·5609	9·5917	10·4083	9·9692	40′
30′	9·5641	9.5954	10·4046	9·9687	30′
40′	9·5673	9·5991	10·4009	9·9682	20′
50′	9·5704	9·6028	10·3072	9·9677	10′
22°00′	9·5736	9·6064	10·3936	9·9672	68°00′
10′	9·5767	9·6100	10·3900	9·9667	50′
20′	9·5798	9·6136	10·3864	9·9661	40′
30′	9·5828	9·6172	10·3828	9·9656	30′
40′	9·5859	9·6208	10·3792	9·9651	20′
50′	9·5889	9·6243	10·3757	9·9646	10′
23°00′	9·5919	9·6279	10·3721	9·9640	67°00′
10′	9·5948	9·6314	10·3686	9·9635	50′
20′	9·5978	9·6348	10·3652	9·9629	40′
30′	9·6007	9·6383	10·3617	9·9624	30′
40′	9·6036	9·6417	10·3583	9·9618	20′
50′	9·6065	9·6452	10·3548	9·9613	10′
Angle	L cos	L cot	L tan	L sin	Angle

Four-place Logarithms of Values of Trigonometric Functions *(Contd.)*

Angle	L sin	L tan	L cot	L cos	Angle
24°00′	9·6093	9·6486	10·3514	9·9607	66°00′
10′	9·6121	9·6520	10·3480	9·9602	50′
20′	9·6149	9·6553	10·3447	9·9596	40′
30′	9·6177	9·6587	10·3413	9·9590	30′
40′	9·6205	9·6620	10·3380	9·9584	20′
50′	9·6232	9·6654	10·3446	9·9579	10′
25°00′	9·6259	9·6687	10·3313	9·9573	65°00′
10′	9·6286	9·6720	10·3280	9·9567	50′
20′	9·6313	9·6752	10·3248	9·9561	40′
30′	9·6340	9·6785	10·3215	9·9555	30′
40′	9·6366	9·6817	10·3183	9·9549	20′
50′	9·6392	9·6850	10·3150	9·9543	10′
26°00′	9·6418	9·6882	10·3118	9·9537	64°00′
10′	9·6444	9·6914	10·3086	9·9530	50′
20′	9·6470	9·6946	10·3054	9·9524	40′
30′	9·6495	9·6977	10·3023	9·9518	30′
40′	9·6521	9·7009	10·2991	9·9512	20′
50′	9·6546	9·7040	10·2960	9·9505	10′
27°00′	9·6570	9·7072	10·2928	9·9499	63°00′
10′	9·6595	9·7103	10·2897	9·9492	50′
20′	9·6620	9·7134	10·2866	9·9486	40′
30′	9·6644	6·7165	10·2835	9·9479	30′
40′	9·6668	9·7196	10·2804	9·9473	20′
50′	9·6692	9·7226	10·2774	9·9466	10′
Angle	L cos	L cot	L tan	L sin	Angle

Four-Place Logarithms of Values of Trigonometric Functions (*Contd.*)

Angle	L sin	L tan	L cot	L cos	Angle
28°00′	9·6716	9·7257	10·2743	9·9459	62°00′
10′	6·6740	9·7287	10·2713	9·9453	50′
20′	9·6763	9·7317	10·2683	9·9446	40′
30′	9·6787	9·7348	10·2652	9·9439	30′
40′	9·6810	9·7378	10·2622	9·9432	20′
50′	9·6833	9·7408	10·2592	9·9425	10′
29°00′	9·6856	9·7438	10·2562	9·9418	61°00′
10′	9·6878	9·7467	10·2533	9·9411	50′
20′	9·6901	9·7497	10·2503	9·9404	40′
30′	9·6923	9·7526	10·2474	9·9397	30′
40′	9·6946	9·7556	10·2444	9·9390	20′
50′	9·6968	9·7585	17·2415	9·9383	10′
30°00′	9·6990	9·7614	10·2386	9·9375	60°00′
10′	9·7012	9·7644	10·2356	9·9368	50′
20′	9·7033	9·7673	10·2327	9·9361	40′
30′	9·7055	9·7701	10·2299	9·9353	30′
40′	9·7076	9·7730	10·2270	9·9346	20′
50′	9·7097	9·7759	10·2241	9·9338	10′
31°00′	9·7118	9·7788	10·2212	9·9331	59°00′
10′	9·7139	9·7816	10·2184	9·9323	50′
20′	9·7160	9·7845	10·2155	9·9315	40′
30′	9·7181	9·7873	10·2127	9·9308	30′
40′	9·7201	9·7902	10·2098	9·9300	20′
50′	9·7222	9·7930	10·2070	9·9292	10′
Angle	**L cos**	**L cot**	**L tan**	**L sin**	**Angle**

Four-Place Logarithms of Values of Trigonometric Functions *(Contd.)*

Angle	L sin	L tan	L cot	L cos	Angle
32°00′	9·7242	9·7958	10·2042	9·9284	58°00′
10′	9·7262	9·7986	10·2014	9·9276	50′
20′	9·7282	9·8014	10·1986	9·9268	40′
30′	9·7302	9·8042	10·1958	9·9260	30′
40′	9·7322	9·8070	10·1930	9·9252	20′
50′	9·7342	9·8097	10·1903	9·9244	10′
33°00′	9·7361	9·8125	10·1875	9·9236	57°00′
10′	9·7380	9·8153	10·1847	9·9228	50′
20′	9·7400	9·8180	10·1820	9·9219	40′
30′	9·7419	9·8208	10·1792	9·9211	30′
40′	9·7438	9·8235	10·1765	9·9203	20′
50′	9·7457	9·8263	10·1737	9·9194	10′
34°00′	9 7476	9·8290	10·1710	9 9186	56°00′
10′	9·7494	9·8317	10·1683	9·9177	50′
20′	9·7513	9·8344	10·1656	9·9169	40′
30′	9·7531	9·8371	10·1629	9·9160	30′
40′	9·7550	9·8398	10·1602	9·9151	20′
50′	9 7568	9·8425	10·1575	9 9142	10′
35°00′	9·7586	9·8452	10·1548	9·9134	55°00′
10′	9·7604	9·8479	10·1521	9·9125	50′
20′	9·7622	9·8506	10·1494	9·9116	40′
30′	9·7640	9·8533	10·1467	9·9107	30′
40′	9·7657	9·8559	10·1441	9·9098	20′
50′	9·7675	9·8586	10·1414	9·9089	10′
Angle	L cos	L cot	L tan	L sin	Angle

Four Place Logarithms of Values of Trigonometric Functions (Contd.)

Angle	L sin	L tan	L cot	L cos	Angle
36°00′	9·7692	9·8613	10·1387	9·9080	54°00′
10′	9·7710	9·8639	10·1361	9·9070	50′
20′	9·7727	9·8666	10·1334	9·9061	40′
30′	9·7744	9·8692	10·1308	9·9052	30′
40′	9·7761	9·8718	10·1282	9·9042	20′
50′	9·7778	0·8745	10·1255	9·9033	10′
37°00′	9·7795	9·8771	10·1229	9·9023	53°00′
10′	9·7811	9·8797	10·1203	9·9014	50′
20′	9·7828	9·8824	10·1176	9·9004	40′
30′	9·7844	9·8850	10·1150	9·8995	30′
40′	9·7861	9·8876	10·1124	9·8985	20′
50′	9·7877	9·8902	10·1098	9·8975	10′
38°00′	9·7893	9·8928	10·1072	9·8965	52°00′
10′	9·7910	9·8945	10·1046	9·8955	50′
20′	9·7926	9·8980	10·1020	9·8945	40′
30′	9·7941	9·9006	10·0994	9·8935	30′
40′	9·7957	9·9032	10·0968	9·8925	20′
50′	9·7973	9·9058	10·0942	9·8915	10′
39°00′	9·7989	9·9084	10·0916	9·8905	51°00′
10′	9·8004	9·9110	10·0890	9·8895	50′
20′	9·8020	9·9135	10·0865	9·8884	40′
30′	9·8035	9·9161	10·0839	9·8874	30′
40′	9·8050	9·9187	10·0813	9·8864	20′
50′	9·8066	9·9212	10·0788	9·8853	10′
Angle	L cos	L cot	L tan	L sin	Angle

Four-place Logarithms of Values of Trigonometric Functions *(Contd.)*

Angle	L sin	L tan	L cot	L cos	Angle
40°00′	9·8081	9·9238	10·0762	9·8843	50°00′
10′	9·8096	9·9264	10·0736	9·8832	50′
20′	9·8111	9·9289	10·0711	9·8821	40′
30′	9·8125	9·9315	10·0685	9·8810	30′
40′	9·8140	9·9341	10·0659	9·8800	20′
50′	9·8155	9·9366	10·0634	9·8789	10′
41°00′	9·8169	9·9392	10·0608	9·8778	49°00′
10′	0·8184	9·9417	10·0583	9·8767	50′
20′	9·8198	9·9443	10·0557	9·8756	40′
30′	9·8213	9·9468	10·0532	9·8745	30′
40′	9·8227	9·9494	10·0506	9·8733	20′
50′	9·8241	9·9519	10·0481	9·8722	10′
42°00′	9·8255	9·9544	10·0456	9·8711	48°00′
10′	9·8269	9·9570	10·0430	9·8699	50′
20′	9·8283	9·9595	10·0405	9·8688	40′
30′	9·8297	9·9621	10·0379	9·8676	30′
40′	9·8311	9·9646	10·0354	9·8665	20′
50′	9·8324	9·9671	10·0329	9·8653	10′
43°00′	9·8338	9·9697	10·0303	9·8641	47°00′
10′	9 8351	9·9722	10·0278	9·8629	50′
20′	9·8365	9·9747	10·0253	9·8618	40′
30′	9·8378	9·9772	10·0228	9·8606	30′
40′	9·8391	9·9798	10·0202	9·8594	20′
50′	9·8405	9·9823	10·0177	9·8582	10′
Angle	L cos	L cot	L tan	L sin	Angle

Four-place Logarithms of Values of Trigonometric Functions (Contd)

Angle	L sin	L tan	L cot	L cos	Angle
44°00′	9 8418	9 9848	10 0152	9·8569	46°00′
10′	9 8431	9 9874	10·0126	9·8557	50′
20′	9·8444	9·9899	10·0101	9·8545	40′
30′	9·8457	9·9924	10·0076	9·8532	30′
40′	9 8469	9·9949	10·0051	9·8520	20′
50′	9·8482	9·9975	10·0025	9·8507	10′
45°00′	9·8495	10·0000	10·0000	9 8495	45°00′
Angle	L cos	L cot	L tan	L sin	Angle

TABLE 4

Squares and Square Roots

N	N²	√N	√10N	N	N²	√N	√10N
1·0	1·00	1·000	3·162	3·5	12·25	1·871	5·916
1·1	1·21	1·049	3·317	3·6	12·96	1·897	6·000
1·2	1·44	1·095	3·464	3·7	13·69	1·924	6·083
1·3	1·69	1·140	3·606	3·8	14·44	1·949	6·164
1·4	1·96	1·183	3·742	3·9	15·21	1·975	6·245
1·5	2·25	1·225	3·873	4·0	16·00	2·000	6·325
1·6	2·56	1·265	4·000	4·1	16·81	2·025	6·403
1·7	2·89	1·304	4·123	4·2	17·64	2·049	6·481
1·8	3·24	1·342	4·243	4·3	18·49	2·074	6·557
1·9	3·61	1·378	4·359	4·4	19·36	2·098	6·633
2·0	4·00	1·414	4·472	4·5	20·25	2·121	6·708
2·1	4·41	1·449	4·583	4·6	21·16	2·145	6·782
2·2	4·84	1·483	4·690	4·7	22·09	2·168	6·856
2·3	5·29	1·517	4·796	4·8	23·04	2·191	6·928
2·4	5·76	1·549	4·822	4·9	24·01	2·214	7·000
2·5	6·25	1·581	5·000	5·0	25·00	2·236	7·071
2·6	6·76	1·612	5·099	5·1	26·01	2·258	7·111
2·7	7·29	1·643	5·196	5·2	27·04	2·280	7·211
2·8	7·84	1·673	5·292	5·3	28·09	2·302	7·280
2·9	8·41	1·703	5·385	5·4	29·16	2·324	7·348
3·0	9·00	1·732	5·477	5·5	30·25	2·345	7·416
3·1	9·61	1·761	5·568	5·6	31·36	2·366	7·483
3·2	10·24	1·789	5·657	5·7	32·49	2·387	7·550
3·3	10·82	1·817	5·745	5·8	33·64	2·408	7·616
3·4	11·56	1·844	5·831	5·9	34·81	2·427	7·681

Square and Square Roots

(Contd.)

N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N	N^2	$\sqrt{N}$	$\sqrt{10N}$
6 0	36.00	2 449	7.746	8.1	65.61	2 846	9.000
6.1	37.21	2 470	7.810	8.2	67.24	2.864	9.055
6.2	38.44	2.490	7.870	8.3	68.89	2.881	9.110
6.3	39.69	2.510	7.937	8.4	70.56	2.898	9.165
6.4	40.96	2.530	8.000	8.5	72.25	2.915	9.220
6.5	42.25	2.550	8.062	8.6	73.96	2.933	9.274
6.6	43.56	2.569	8.124	8.7	75.69	2.950	9.327
6.7	44.39	2.588	8.185	8.8	77.44	2.966	9.381
6 8	46.24	2.608	8.246	8.9	79.21	2.983	9 434
6.9	47.61	2.627	8.307	9.0	81.00	3.000	9.487
7.0	49.00	2.646	8.367	9.1	82.81	3 017	9.539
7.1	50.41	2.665	8.426	9.2	84.64	3.033	9.592
7.2	51.84	2.683	8.485	9.3	86.49	3.050	9.644
7.3	53.29	2.702	8.544	9.4	88.36	3.066	9.695
7.4	54.76	2.720	8.602	9.5	90.25	3.082	9.747
7.5	56.25	2.739	8.660	9.6	92.16	3.098	9.798
7 6	57.76	2 757	8.718	9.7	94.09	3.114	9.849
7.7	59.29	2.775	8.775	9.8	96.04	3.130	9.899
7.8	60.84	2.793	8.832	9 9	98.01	3.146	9.950
7.6	92.41	2.811	8.888	10.0	100 00	3.162	10 00
8.0	64.00	2.828	8.944				

TABLE 5

Cubes and Cube Roots

N	N^3	$\sqrt[3]{N}$	$\sqrt[3]{10N}$	$\sqrt[3]{100N}$	N	N^3	$\sqrt[3]{N}$	$\sqrt[3]{10N}$	$\sqrt[3]{100N}$
1.0	1.000	1.000	2.154	4.642	3.5	42.875	1.518	3.271	7.047
1.1	1.331	1.032	2.224	4.791	3.6	46.656	1.533	3.302	7.114
1.2	1.728	1.063	2.289	4.932	3.7	50.653	1.547	3.332	7.179
1.3	2.197	1.091	2.351	5.066	3.8	54.872	1.560	3.362	7.243
1.4	2.744	1.119	2.410	5.192	3.9	59.319	1.574	3.391	7.306
1.5	3.375	1.145	2.466	5.313	4.0	64.000	1.587	3.420	7.368
1.6	4.096	1.170	2.520	5.429	4.1	68.921	1.601	3.448	7.429
1.7	4.913	1.193	2.571	5.540	4.2	74.088	1.613	3.476	7.489
1.8	5.832	1.216	2.621	5.646	4.3	79.507	1.626	3.503	7.548
1.9	6.859	1.239	2.668	5.749	4.4	85.184	1.639	3.530	7.606
2.0	8.000	1.260	2.714	5.848	4.5	91.125	1.651	3.557	7.663
2.1	9.261	1.281	2.759	5.944	4.6	97.336	1.663	3.583	7.719
2.2	10.648	1.301	2.802	6.037	4.7	103.823	1.675	3.609	7.775
2.3	12.167	1.320	2.844	6.127	4.8	110.592	1.687	3.634	7.830
2.4	13.824	1.339	2.884	6.214	4.9	117.649	1.698	3.659	7.884
2.5	15.625	1.357	2.924	6.300	5.0	125.000	1.710	3.684	7.937
2.6	17.576	1.375	2.962	6.383	5.1	132.651	1.721	3.708	7.990
2.7	19.683	1.392	3.000	6.463	5.2	140.608	1.732	3.733	8.041
2.8	21.952	1.409	3.037	6.542	5.3	148.877	1.744	3.756	8.093
2.9	24.386	1.426	3.072	6.619	5.4	157.464	1.754	3.780	8.143
3.0	27.000	1.442	3.107	6.694	5.5	166.375	1.765	3.803	8.193
3.1	29.791	1.458	3.141	6.768	5.6	175.616	1.776	3.826	8.243
3.2	32.768	1.404	3.175	6.840	5.7	185.193	1.786	3.849	8.291
3.3	35.937	1.489	3.208	6.910	5.8	195.112	1.797	3.871	8.340
3.4	39.304	1.504	3.240	6.980	5.9	205.372	1.807	3.893	8.387

N	N^3	$\sqrt[3]{N}$	$\sqrt[3]{10N}$	$\sqrt[3]{100N}$	N	N^3	$\sqrt[3]{N}$	$\sqrt[3]{10N}$	$\sqrt[3]{100N}$
6·0	216·000	1·817	3·915	1·434	8·1	531·441	2·008	4·327	9·322
6·1	226·981	1·827	3·936	8·481	8·2	551·368	2·017	4·344	9·360
6·2	234·328	1·837	3·958	8·527	8·3	521·787	2·025	4·362	9·398
6·3	250·047	1·847	3·979	8·573	8·4	592·704	2·033	4·380	9·435
6·4	262·144	1·857	4·000	8·618	8·5	614·125	2·041	4·397	9·473
6·5	274·625	1·866	4·021	8·662	8·6	636·056	2·049	4·414	9·510
6·6	287·496	1·876	4·041	8·707	8·7	658·503	2·057	4·431	9·546
6·7	300·763	1·885	4·062	8·750	8·8	681·472	2·065	4·448	9·583
6·8	314·432	1·895	4·082	8·794	8·9	704·969	2·072	4·465	9·619
6·9	328·509	1·904	4·102	8·837	9·0	729·000	2·080	4·481	9·655
7·0	343·000	1·913	4·121	8·879	9·1	753·571	2·088	4·498	9·691
7·1	357·911	1·922	4·141	8·921	9·2	778·688	2·095	4·514	9·726
7·2	373·248	1·931	4·160	8·963	9·3	804·357	2·103	4·531	9·761
7·3	389·017	1·940	4·179	9·004	9·4	830·584	2·110	4·547	9·796
7·4	405·224	1·949	4·198	9·045	9·5	857·375	2·118	4·563	9·830
7·5	421·875	1·957	4·217	9·086	9·6	884·736	2·125	4·579	9·865
7·6	438·976	1·966	4·236	9·126	9·7	912·673	2·133	4·595	9·899
7·7	456·533	1·975	4·254	9·166	9·8	941·192	2·140	4·610	9·933
7·8	474·552	1·983	4·273	9·205	9·9	970·299	2·147	4·626	9·967
7·9	493·039	1·992	4·291	9·244	10·0	1000·000	2·154	4·642	10·000
8·0	512·000	2·000	4·309	9·283					

SYMBOLS

$\sim$	negation
$\wedge$	conjunction (and)
$\vee$	disjunction (or)
$\Rightarrow$	implies
$\Leftrightarrow$	is equivalent to
$\{\ \}$	set
$\in$	is an element of
$\notin$	is not an element of
$:$	such that
$\subset$	is contained in (is a subset of)
$\supset$	contains (is a superset of)
$X \sim A$	complement of A with respect to X
$\cup$	union
$\cap$	intersection
ϕ	the empty set
$\exists$	there exists
$\forall$	for all
$\mathbf{N}$	the set of natural numbers
$\mathbf{Z}$	the set of integers
$\mathbf{Q}$	the set of rational numbers
$\mathbf{Q}^+$	the set of positive rational numbers
$\mathbf{R}$	the set of real numbers
$\mathbf{R}^+$	the set of positive real numbers
$\mathbf{C}$	the set of complex numbers
$\therefore$	therefore
$\because$	because

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